

Chapter II

Linear Time Invariant Systems

- Discrete-Time Linear Time-Invariant Systems
 - The convolution sum
- Continuous-Time Linear Time-Invariant Systems
 - The convolution integral
- Properties of Linear Time-Invariant Systems
- Causal Linear Time-Invariant Systems
Described by Differential & Difference Equations
- Singularity Functions

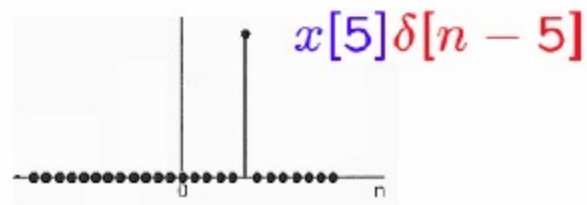
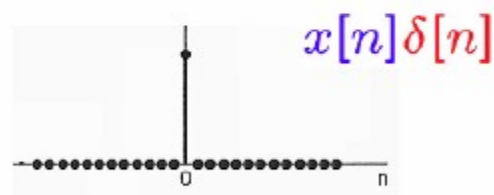
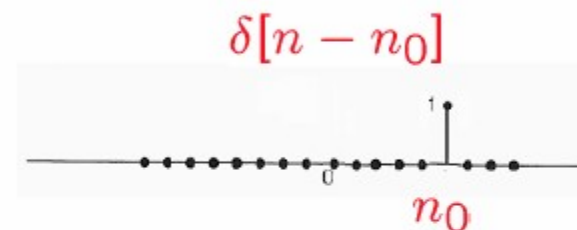
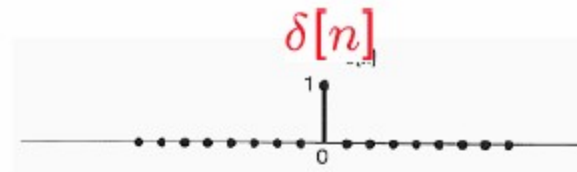
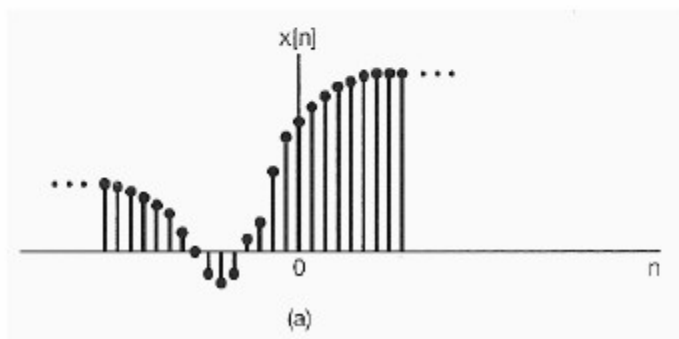
■ Sample by Unit Impulse

- For $x[n]$

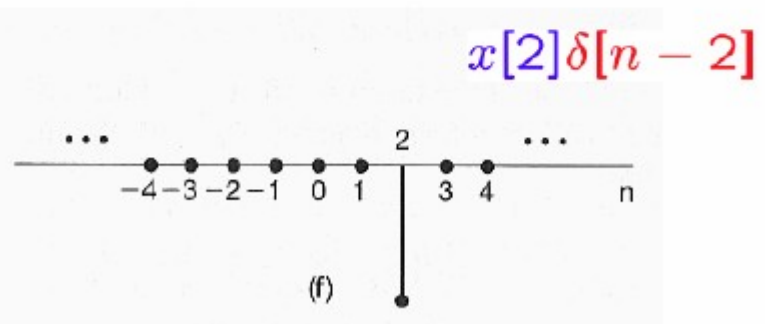
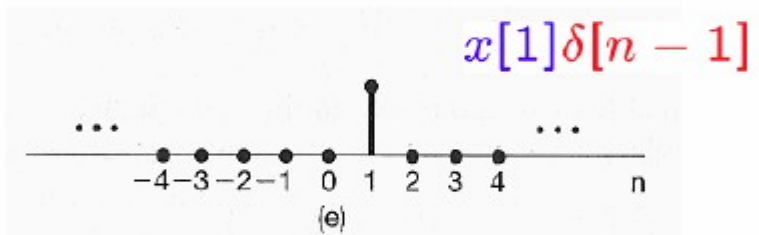
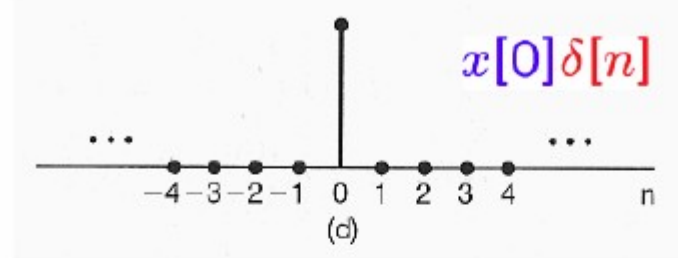
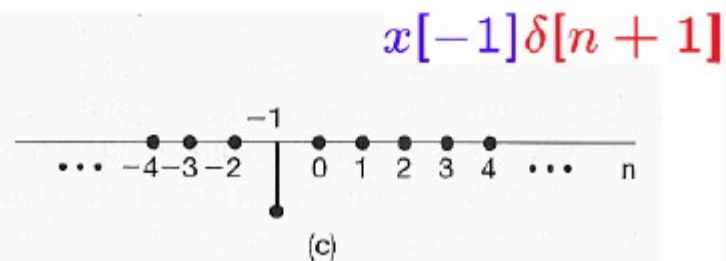
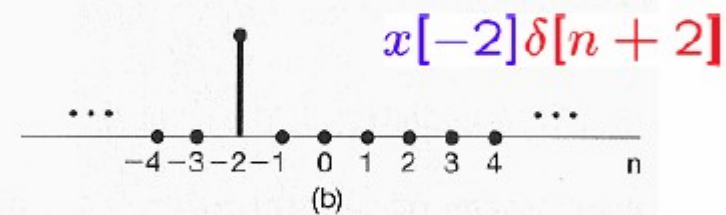
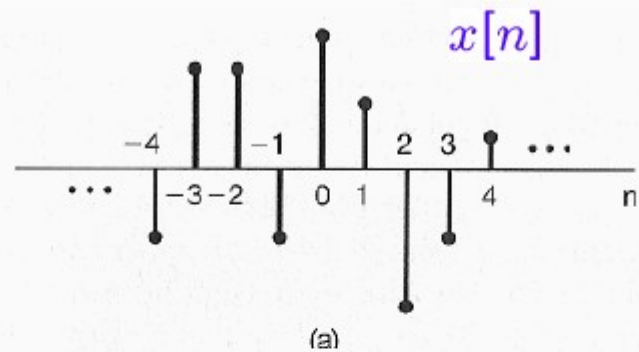
$$x[n]\delta[n] = x[0]\delta[n]$$

- More generally,

$$x[n]\delta[n - n_0] = x[n_0]\delta[n - n_0]$$



Representation of DT Signals by Impulses:



▪ Representation of DT Signals by Impulses:

- More generally,

$$\begin{aligned}x[n] &= \cdots + x[-3]\delta[n+3] + x[-2]\delta[n+2] \\&\quad + x[-1]\delta[n+1] + x[0]\delta[n] + x[1]\delta[n-1] \\&\quad + x[2]\delta[n-2] + x[3]\delta[n-3] + \cdots \\&= \sum_{k=-\infty}^{+\infty} x[k]\delta[n-k]\end{aligned}$$

- The **sifting property** of the DT unit impulse
- $x[n]$ = a **superposition** of **scaled** versions of **shifted** unit impulses $\delta[n-k]$

- DT Unit Impulse Response & Convolution Sum:

input $\longrightarrow$ Linear System $\longrightarrow$ output

$\delta[n] \longrightarrow$ Linear System $\longrightarrow h_0[n]$

$\delta[n - 1] \longrightarrow$ Linear System $\longrightarrow h_1[n]$

$\delta[n - 2] \longrightarrow$ Linear System $\longrightarrow h_2[n]$

$\vdots$

$\delta[n - k] \longrightarrow$ Linear System $\longrightarrow h_k[n]$

▪ DT Unit Impulse Response & Convolution Sum:

$$x[n] \longrightarrow \text{Linear System} \longrightarrow y[n]$$

$$x[0] \cdot \delta[n] \longrightarrow \text{Linear System} \longrightarrow h_0[n] \cdot x[0]$$

$$x[1] \cdot \delta[n - 1] \longrightarrow \text{Linear System} \longrightarrow h_1[n] \cdot x[1]$$

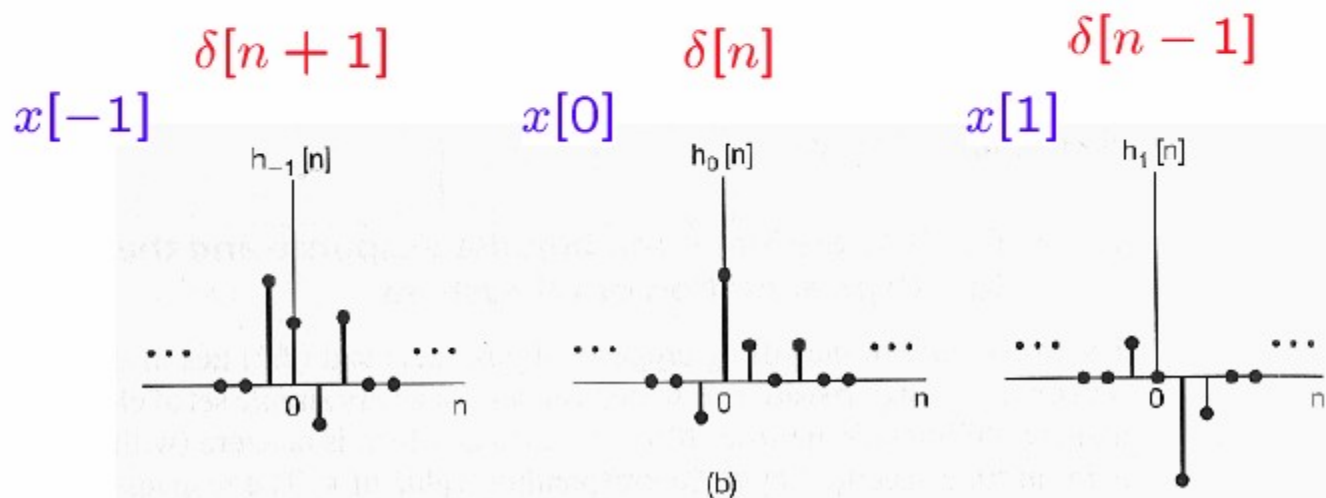
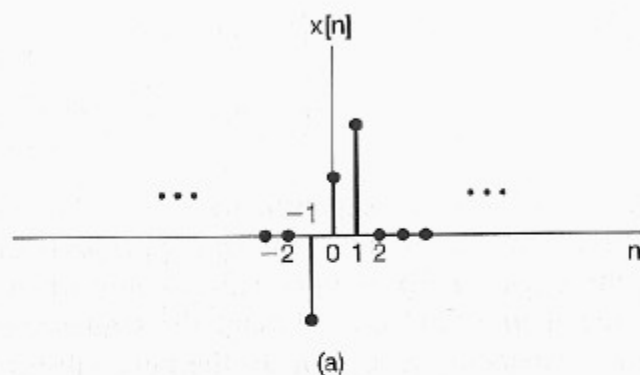
$$x[2] \cdot \delta[n - 2] \longrightarrow \text{Linear System} \longrightarrow h_2[n] \cdot x[2]$$

⋮

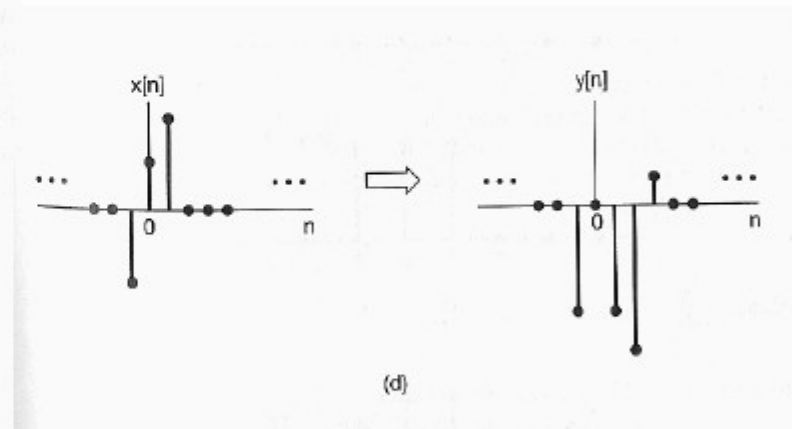
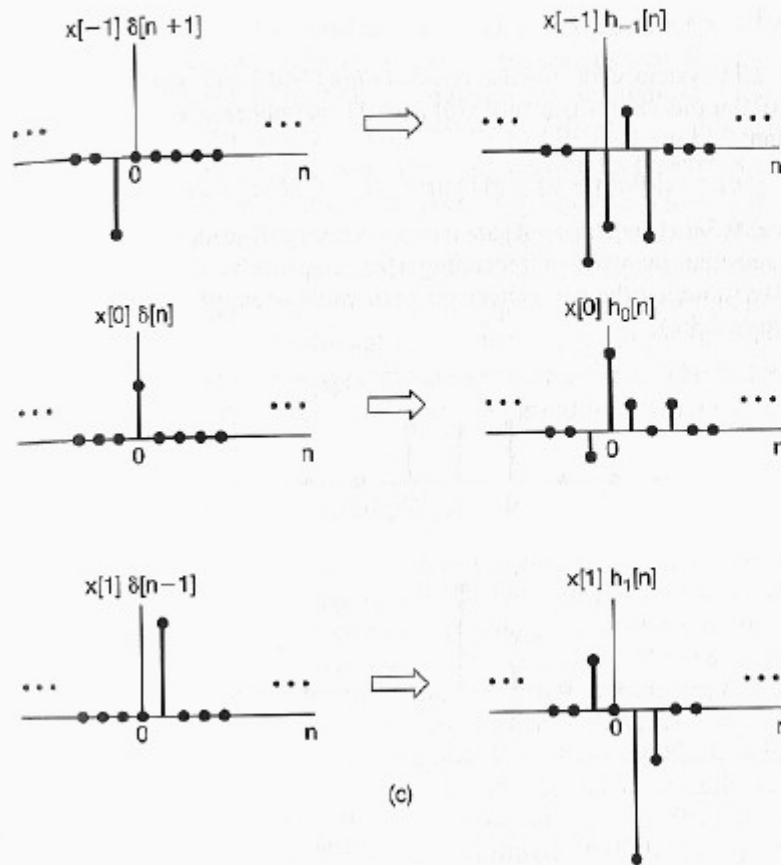
$$x[k] \cdot \delta[n - k] \longrightarrow \text{Linear System} \longrightarrow h_k[n] \cdot x[k]$$

$$x[n] = \sum_{k=-\infty}^{+\infty} x[k] \delta[n - k] \quad \Longrightarrow \quad y[n] = \sum_{k=-\infty}^{+\infty} x[k] h_k[n]$$

$$x[n] = \sum_{k=-\infty}^{+\infty} x[k] \delta[n-k] \quad \Rightarrow \quad y[n] = \sum_{k=-\infty}^{+\infty} x[k] h_k[n]$$



$$x[n] = \sum_{k=-\infty}^{+\infty} x[k] \delta[n-k] \quad \Rightarrow \quad y[n] = \sum_{k=-\infty}^{+\infty} x[k] h_k[n]$$



$x[n] \longrightarrow \text{Linear System} \longrightarrow y[n]$

- If the linear system is also time-invariant

- Then,

$$h_k[n] = h_0[n - k] = h[n - k]$$

- Hence, for an LTI system,

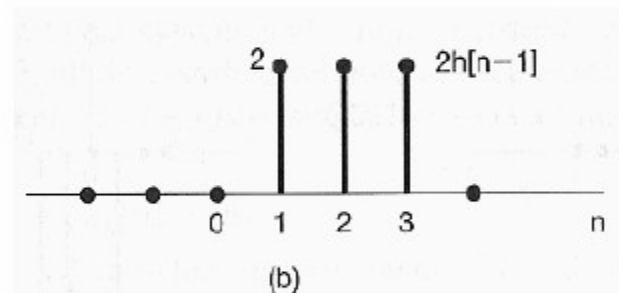
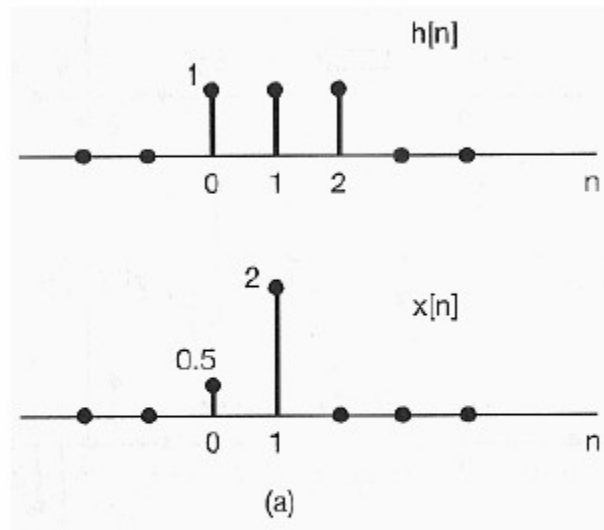
$$y[n] = \sum_{k=-\infty}^{+\infty} x[k]h[n - k] = \sum_{k=-\infty}^{+\infty} x[n - k]h[k]$$

- Known as the convolution of $x[n]$ & $h[n]$
 - Referred as the convolution sum or superposition sum

- Symbolically, $y[n] = x[n] * h[n] = h[n] * x[n]$

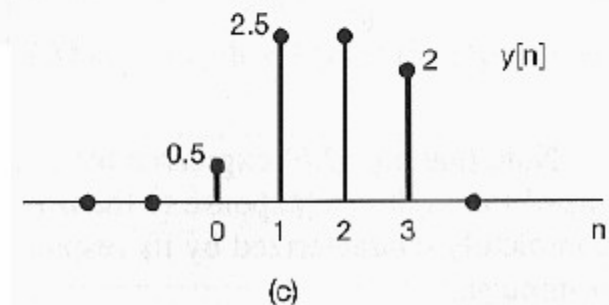
■ Example 2.1: $y[n] = \sum_{k=-\infty}^{+\infty} x[k]h[n-k]$ $x[n] \longrightarrow \boxed{h[n]} \longrightarrow y[n]$

$$= \dots + x[-1]h[n+1] + x[0]h[n] + x[1]h[n-1] + x[2]h[n-2] + \dots$$

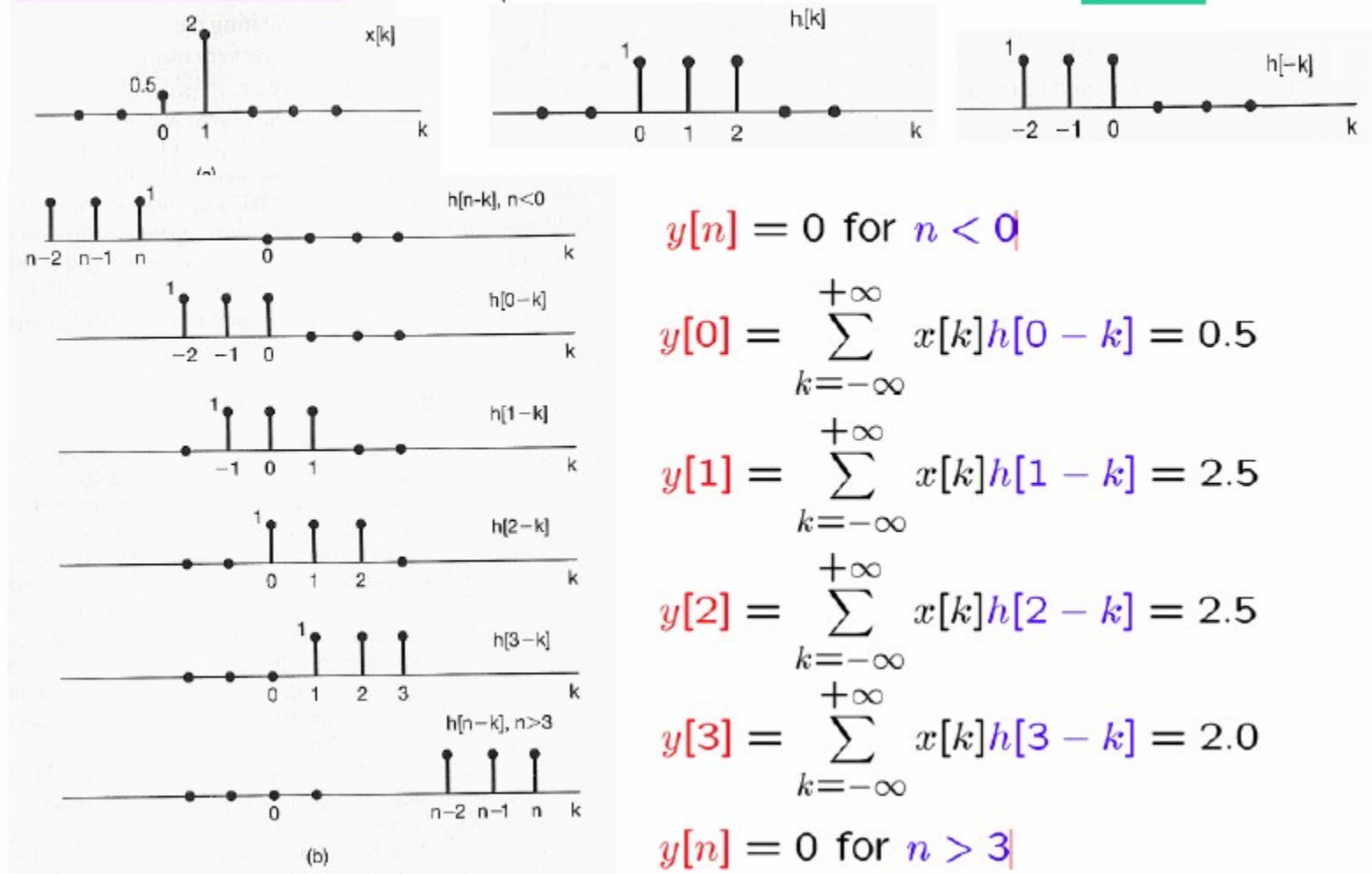


$$y[n] = x[0]h[n-0] + x[1]h[n-1]$$

$$= 0.5h[n] + 2h[n-1]$$

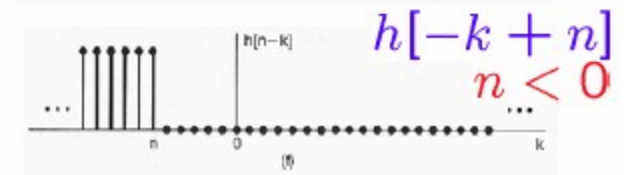
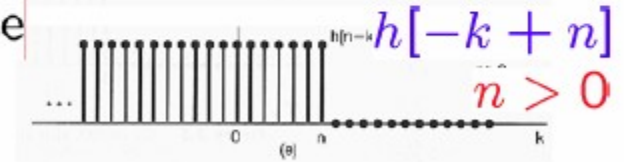
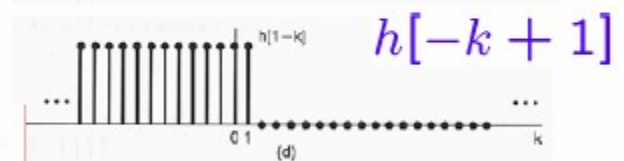
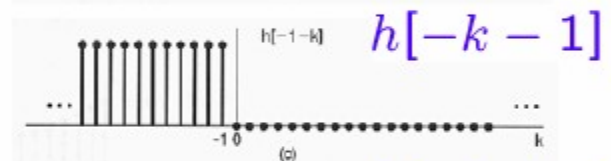
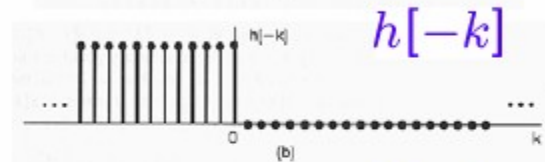
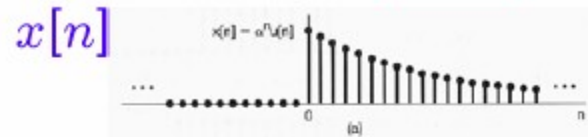
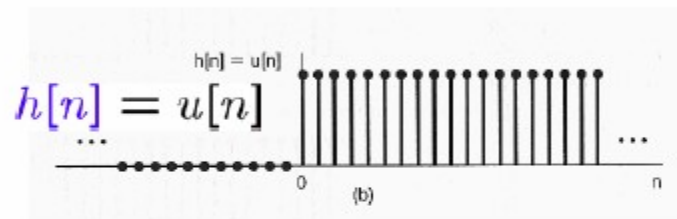
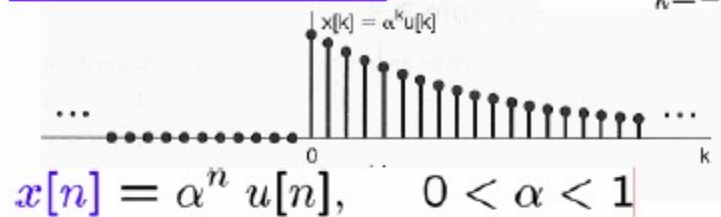


■ Example 2.2: $y[n] = \sum_{k=-\infty}^{+\infty} x[k]h[n-k]$ $x[n] \longrightarrow h[n] \longrightarrow y[n]$



■ Example 2.3:

$$y[n] = \sum_{k=-\infty}^{+\infty} x[k]h[n-k] \quad x[n] \longrightarrow \boxed{h[n]} \longrightarrow y[n]$$



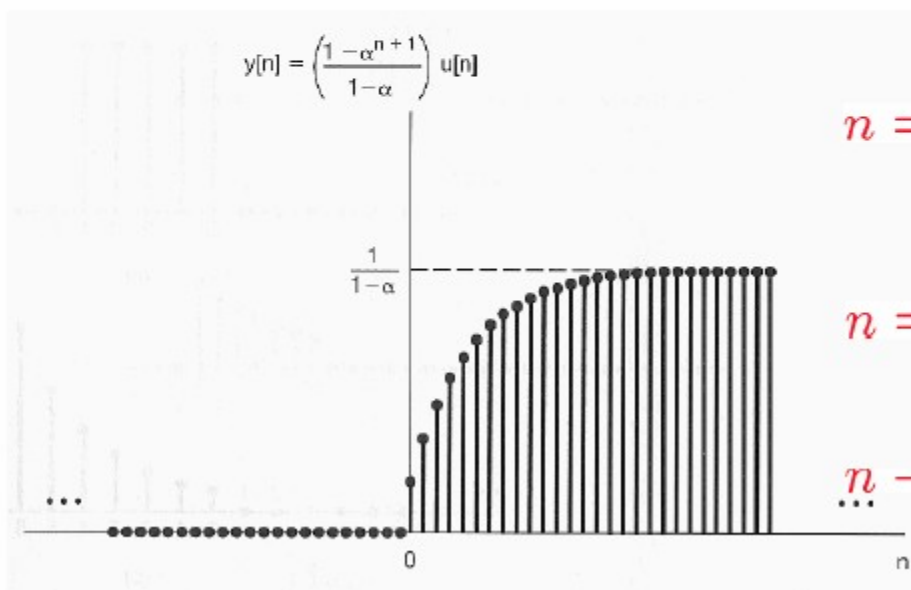
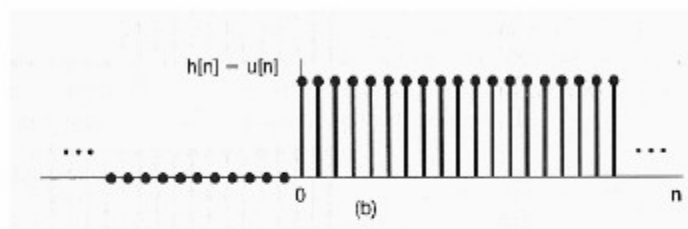
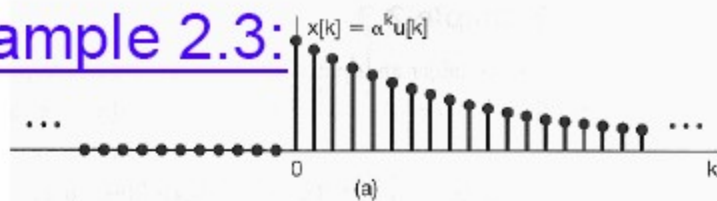
for $n < 0$, $x[k] h[n-k] = 0 \Rightarrow y[n] = 0$

for $n \geq 0$, $x[k] h[n-k] = \begin{cases} \alpha^k, & 0 \leq k \leq n \\ 0, & \text{otherwise} \end{cases}$

$$\Rightarrow y[n] = \sum_{k=0}^n \alpha^k = \frac{1 - \alpha^{n+1}}{1 - \alpha}$$

for all n , $y[n] = \left(\frac{1 - \alpha^{n+1}}{1 - \alpha} \right) u[n]$

■ Example 2.3:



$$x[n] \longrightarrow h[n] \longrightarrow y[n]$$

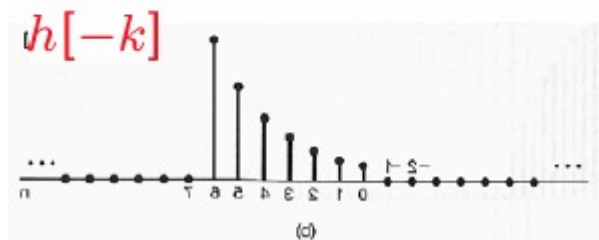
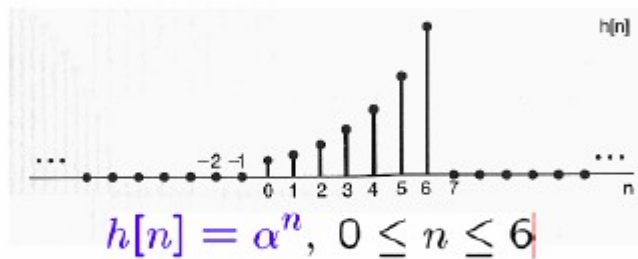
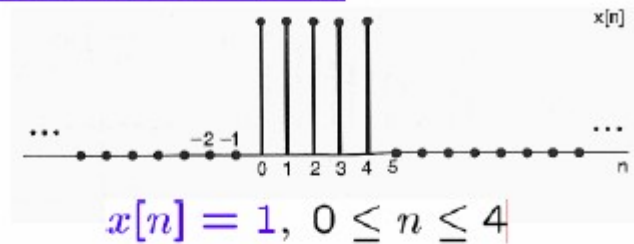
$$\alpha = \frac{7}{8}$$

$$n = 0 \quad y[0] = \frac{1 - \frac{7}{8}}{1 - \frac{7}{8}} = 1$$

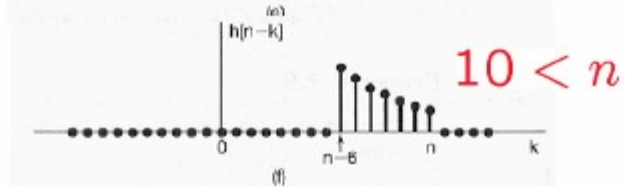
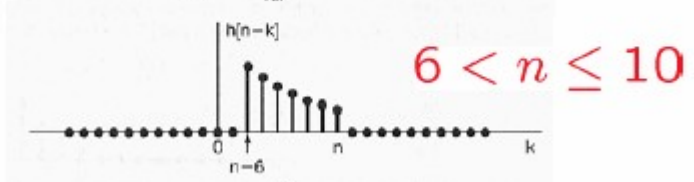
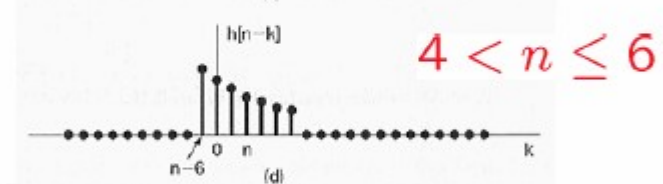
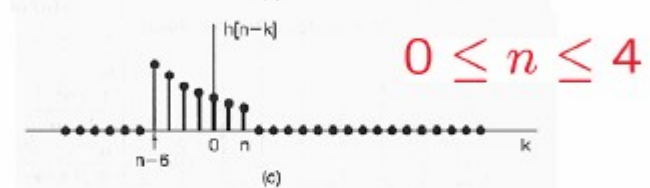
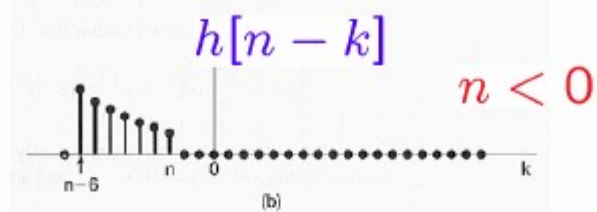
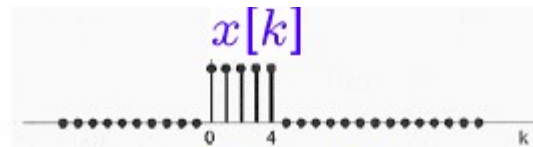
$$n = 1 \quad y[1] = \frac{1 - \left(\frac{7}{8}\right)^2}{1 - \frac{7}{8}} = \frac{15}{8}$$

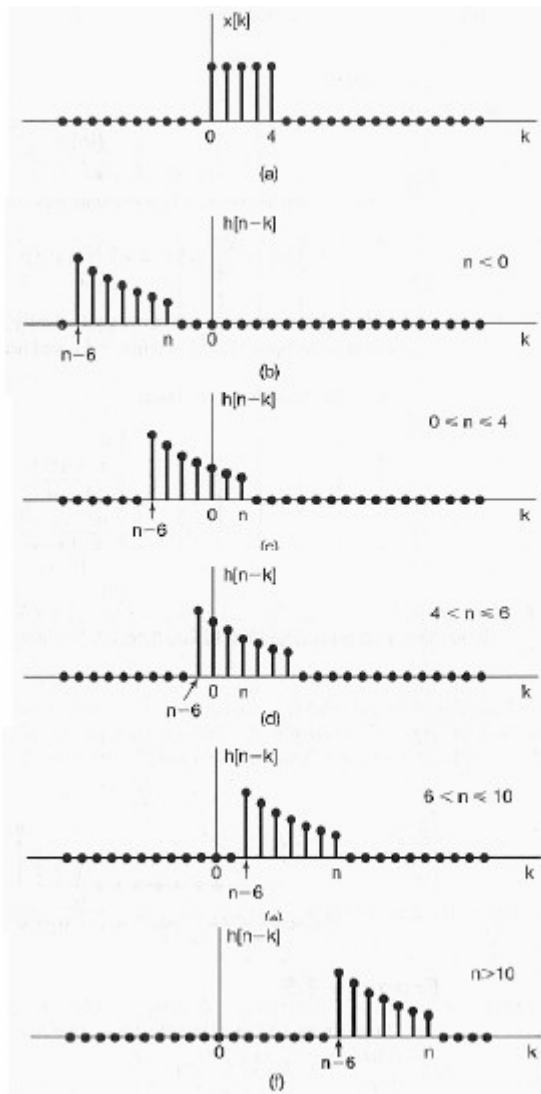
$$n \rightarrow \infty \quad y[n] = \frac{1 - 0}{1 - \frac{7}{8}} = 8$$

■ Example 2.4:



$$x[n] \longrightarrow h[n] \longrightarrow y[n]$$





$$x[n] \longrightarrow h[n] \longrightarrow y[n]$$

for $n < 0$, $x[k] h[n-k] = 0 \Rightarrow y[n] = 0$

for $0 \leq n \leq 4$, $x[k] h[n-k] = \begin{cases} \alpha^{n-k}, & 0 \leq k \leq n \\ 0, & \text{otherwise} \end{cases}$

$$\Rightarrow y[n] = \sum_{k=0}^n \alpha^{n-k} = \frac{1 - \alpha^{n+1}}{1 - \alpha}$$

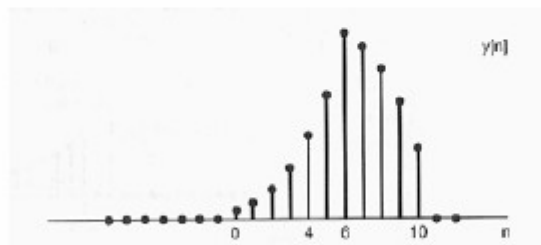
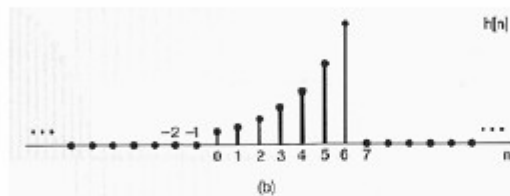
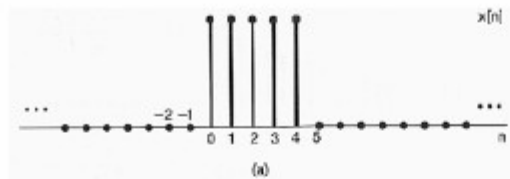
for $4 < n \leq 6$, $x[k] h[n-k] = \begin{cases} \alpha^{n-k}, & 0 \leq k \leq 4 \\ 0, & \text{otherwise} \end{cases}$

$$\Rightarrow y[n] = \sum_{k=0}^4 \alpha^{n-k} = \frac{\alpha^{n-4} - \alpha^{n+1}}{1 - \alpha}$$

for $6 < n \leq 10$, $x[k] h[n-k] = \begin{cases} \alpha^{n-k}, & (n-6) \leq k \leq 4 \\ 0, & \text{otherwise} \end{cases}$

$$\Rightarrow y[n] = \sum_{k=n-6}^4 \alpha^{n-k} = \frac{\alpha^{n-4} - \alpha^7}{1 - \alpha}$$

for $n > 10$, $y[n] = 0$



$$x[n] \longrightarrow h[n] \longrightarrow y[n]$$

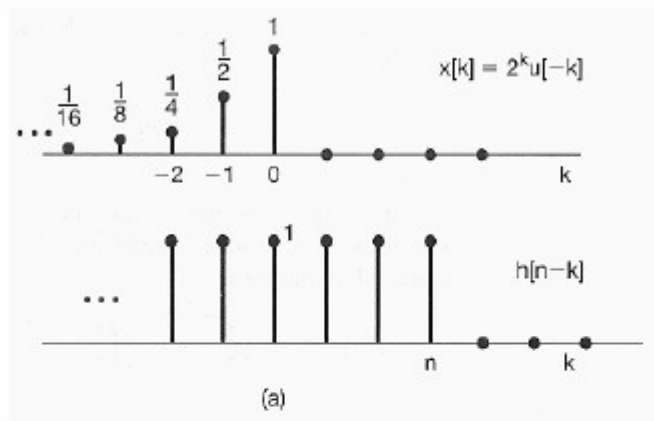
$$x[n] = 1, \quad 0 \leq n \leq 4$$

$$h[n] = \alpha^n, \quad 0 \leq n \leq 6$$

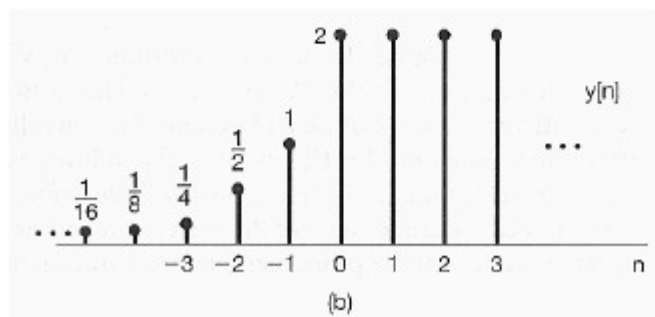
$$y[n] = \begin{cases} 0, & n < 0 \\ \frac{1 - \alpha^{n+1}}{1 - \alpha}, & 0 \leq n \leq 4 \\ \frac{\alpha^{n-4} - \alpha^{n+1}}{1 - \alpha}, & 4 < n \leq 6 \\ \frac{\alpha^{n-4} - \alpha^7}{1 - \alpha}, & 6 < n \leq 10 \\ 0, & 10 < n \end{cases}$$

- Example 2.5: $x[n] = 2^n u[-n]$
 $h[n] = u[n]$

$$x[n] \longrightarrow \boxed{h[n]} \longrightarrow y[n]$$



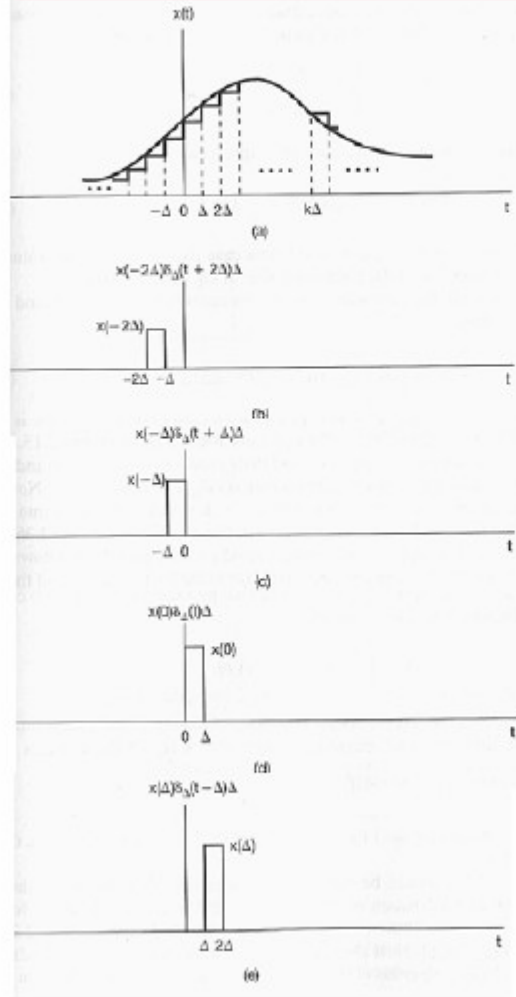
$$\begin{aligned} \text{for } n \geq 0, \quad y[n] &= \sum_{k=-\infty}^0 x[k] h[n-k] = \sum_{k=-\infty}^0 2^k \\ &= \sum_{r=0}^{\infty} \left(\frac{1}{2}\right)^r = \frac{1}{1 - (1/2)} = 2 \end{aligned}$$



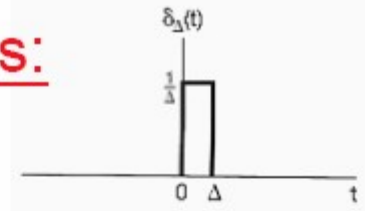
$$\begin{aligned} \text{for } n < 0, \quad y[n] &= \sum_{k=-\infty}^n x[k] h[n-k] = \sum_{k=-\infty}^n 2^k \\ &= \sum_{l=-n}^{\infty} \left(\frac{1}{2}\right)^l = \sum_{m=0}^{\infty} \left(\frac{1}{2}\right)^{m-n} \\ &= \left(\frac{1}{2}\right)^{-n} \sum_{m=0}^{\infty} \left(\frac{1}{2}\right)^m = 2^n \cdot 2 = 2^{n+1} \end{aligned}$$

- Discrete-Time Linear Time-Invariant Systems
 - The convolution sum
- Continuous-Time Linear Time-Invariant Systems
 - The convolution integral
- Properties of Linear Time-Invariant Systems
- Causal Linear Time-Invariant Systems
Described by Differential & Difference Equations
- Singularity Functions

Representation of CT Signals by Impulses:



$$\delta_{\Delta}(t) = \begin{cases} \frac{1}{\Delta}, & 0 \leq t < \Delta \\ 0, & \text{otherwise} \end{cases}$$



$$\hat{x}(t) = \sum_{k=-\infty}^{+\infty} x(k\Delta)\delta_{\Delta}(t - k\Delta)\Delta$$

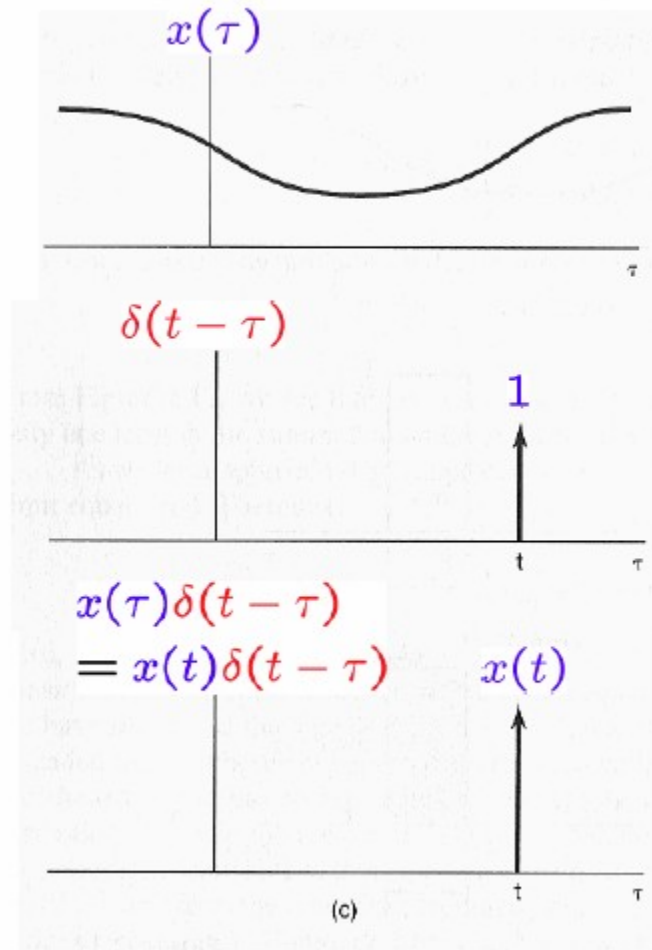
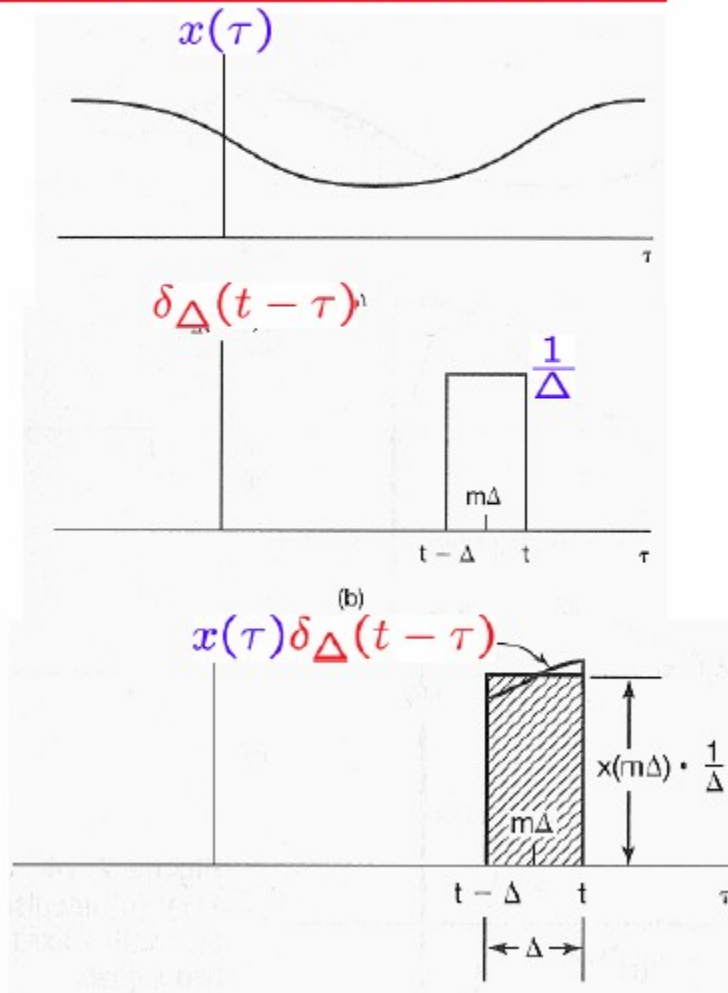
$$x(t) = \lim_{\Delta \rightarrow 0} \sum_{k=-\infty}^{+\infty} x(k\Delta)\delta_{\Delta}(t - k\Delta)\Delta$$

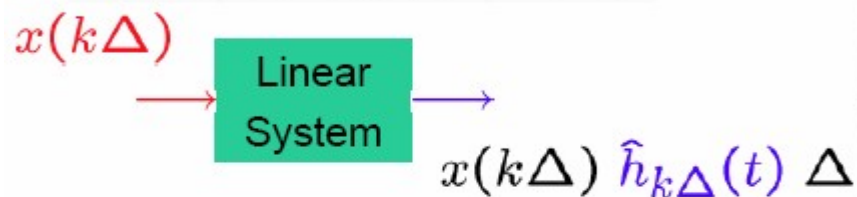
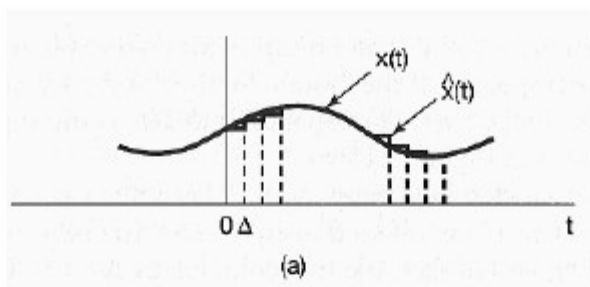
$$x(t) = \int_{-\infty}^{+\infty} x(\tau)\delta(t - \tau)d\tau$$

the **sifting property** of CT impulse

$x(t)$ = an integral of weighted,
shifted impulses

■ Graphical interpretation:

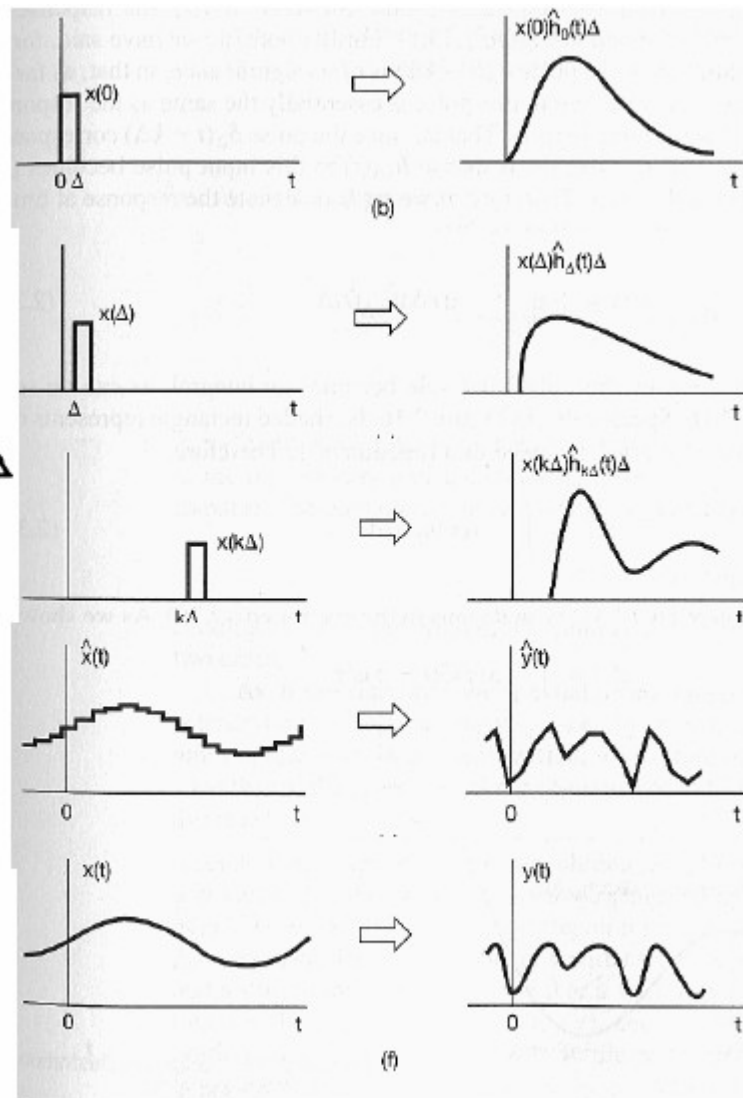




$$\hat{y}(t) = \sum_{k=-\infty}^{+\infty} x(k\Delta) \hat{h}_{k\Delta}(t) \Delta$$

$$y(t) = \lim_{\Delta \rightarrow 0} \sum_{k=-\infty}^{+\infty} x(k\Delta) \hat{h}_{k\Delta}(t) \Delta$$

$$y(t) = \int_{-\infty}^{+\infty} x(\tau) h_{\tau}(t) d\tau$$



- CT Unit Impulse Response & Convolution Integral:

$$\delta(t - \tau) \longrightarrow \text{Linear System} \longrightarrow h_{\tau}(t)$$

$$x(t) \longrightarrow \text{Linear System} \longrightarrow y(t)$$

$$x(t) = \int_{-\infty}^{+\infty} x(\tau) \delta(t - \tau) d\tau$$

$$\implies y(t) = \int_{-\infty}^{+\infty} x(\tau) h_{\tau}(t) d\tau$$

- If the linear system is also time-invariant $x(t) \longrightarrow h(t) \longrightarrow y(t)$

- Then,

$$h_{\tau}(t) = h_0(t - \tau) = h(t - \tau)$$

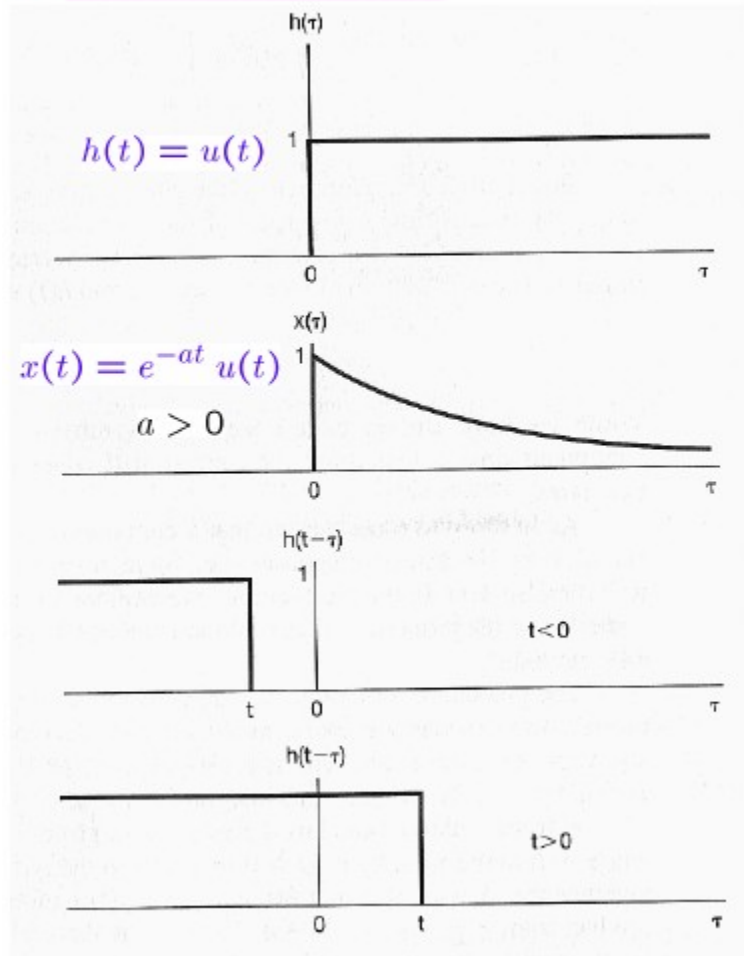
- Hence, for an LTI system,

$$y(t) = \int_{-\infty}^{+\infty} x(\tau)h(t - \tau)d\tau = \int_{-\infty}^{+\infty} h(\tau)x(t - \tau)d\tau$$

- Known as the convolution of $x(t)$ & $h(t)$
 - Referred as the convolution integral or the superposition integral

- Symbolically, $y(t) = x(t) * h(t) = h(t) * x(t)$

■ Example 2.6:



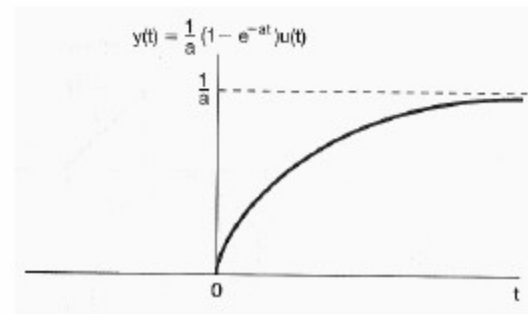
$$x(t) \longrightarrow \boxed{h(t)} \longrightarrow y(t)$$

for $t < 0$, $x(\tau) h(t - \tau) = 0$

$$\Rightarrow y(t) = 0$$

$$\text{for } t \geq 0, \quad x(\tau) h(t - \tau) = \begin{cases} e^{-a\tau}, & 0 \leq \tau \leq t \\ 0, & \text{otherwise} \end{cases}$$

$$\begin{aligned} \Rightarrow y(t) &= \int_0^t e^{-a\tau} d\tau \\ &= -\frac{1}{a} e^{-a\tau} \Big|_0^t \\ &= \frac{1}{a} (1 - e^{-at}) \end{aligned}$$

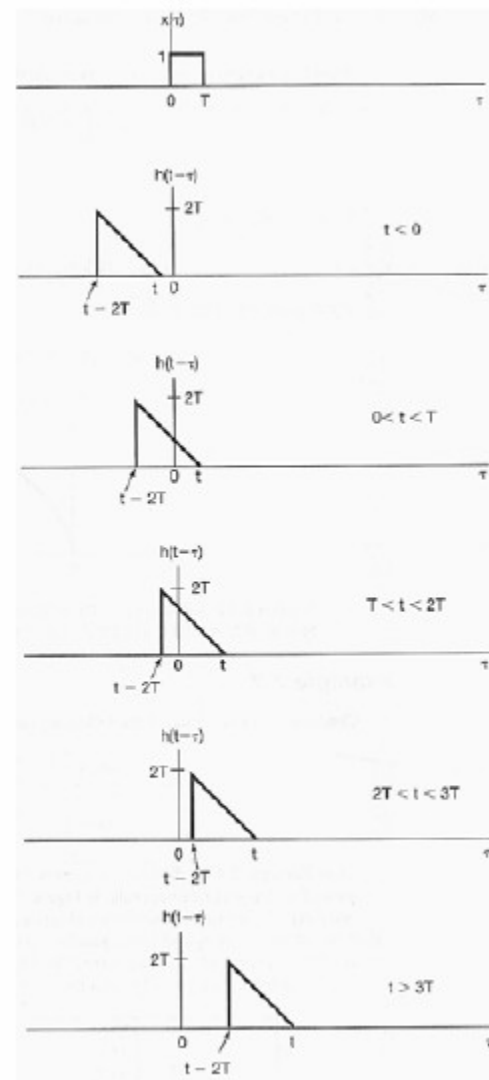
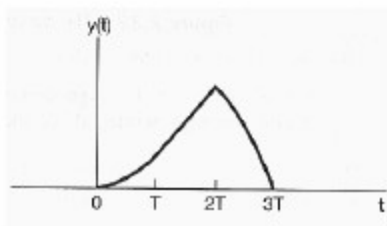


■ Example 2.7: $x(t) \longrightarrow h(t) \longrightarrow y(t)$

$$x(t) = \begin{cases} 1, & 0 < t < T \\ 0, & \text{otherwise} \end{cases}$$

$$h(t) = \begin{cases} t, & 0 < t < 2T \\ 0, & \text{otherwise} \end{cases}$$

$$y(t) = \begin{cases} 0, & t < 0 \\ \frac{1}{2}t^2, & 0 < t < T \\ Tt - \frac{1}{2}T^2, & T < t < 2T \\ -\frac{1}{2}t^2 + Tt + \frac{3}{2}T^2, & 2T < t < 3T \\ 0, & 3T < t \end{cases}$$

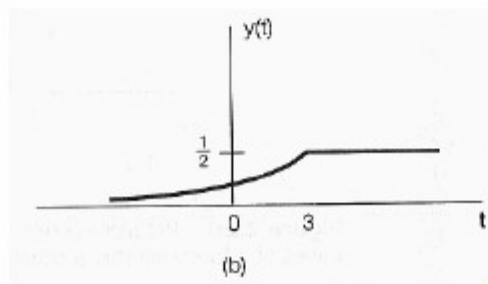
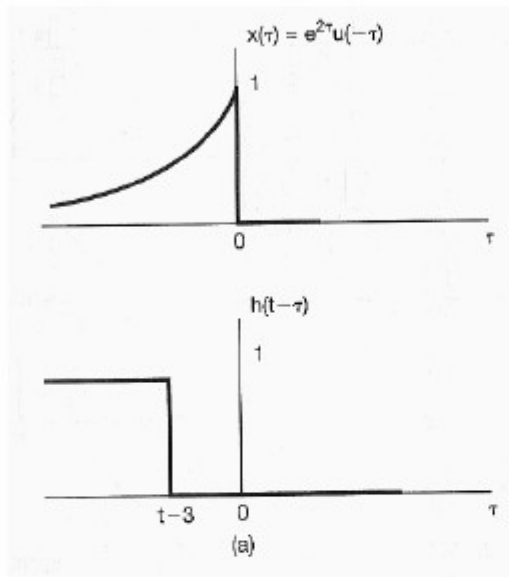


■ Example 2.8:

$$x(t) = e^{2t}u(-t)$$

$$h(t) = u(t - 3)$$

$$x(t) \longrightarrow h(t) \longrightarrow y(t)$$



$$\text{for } t - 3 \leq 0, \quad y(t) = \int_{-\infty}^{t-3} e^{2\tau} d\tau$$

$$= \frac{1}{2} e^{2(t-3)}$$

$$\text{for } t - 3 \geq 0, \quad y(t) = \int_{-\infty}^0 e^{2\tau} d\tau$$

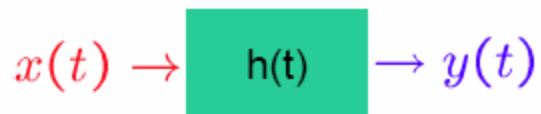
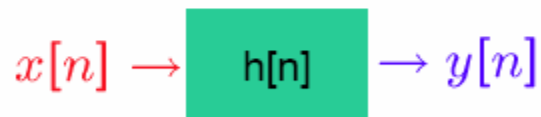
$$= \frac{1}{2}$$

- Discrete-Time Linear Time-Invariant Systems
 - The convolution sum
- Continuous-Time Linear Time-Invariant Systems
 - The convolution integral
- Properties of Linear Time-Invariant Systems
- Causal Linear Time-Invariant Systems
Described by Differential & Difference Equations
- Singularity Functions

- Convolution Sum & Integral of LTI Systems:

$$y[n] = \sum_{k=-\infty}^{+\infty} x[k]h[n-k] = x[k] * h[n]$$

$$y(t) = \int_{-\infty}^{+\infty} x(\tau)h(t-\tau)d\tau = x(t) * h(t)$$



■ Properties of LTI Systems

1. Commutative property
2. Distributive property
3. Associative property
4. With or without memory
5. Invertibility
6. Causality
7. Stability
8. Unit step response

- Commutative Property: $n - k = r$

$$x[n] * h[n] = \sum_{k=-\infty}^{+\infty} x[k] h[n - k] = \sum_{r=-\infty}^{+\infty} x[n - r] h[r]$$

$$= \sum_{r=-\infty}^{+\infty} h[r] x[n - r] = h[n] * x[n]$$

$$x(t) * h(t) = \int_{-\infty}^{+\infty} x(\tau) h(t - \tau) d\tau \quad \begin{array}{l} t - \tau = \sigma \\ -d\tau = d\sigma \end{array}$$

$$= \int_{+\infty}^{-\infty} x(t - \sigma) h(\sigma) (-d\sigma) = \int_{-\infty}^{+\infty} x(t - \sigma) h(\sigma) d\sigma$$

$$= \int_{-\infty}^{+\infty} h(\sigma) x(t - \sigma) d\sigma = h(t) * x(t)$$

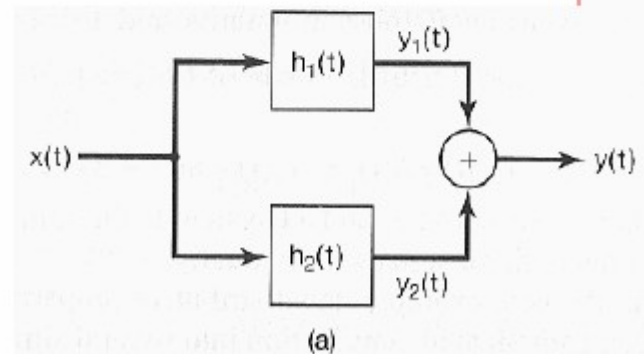
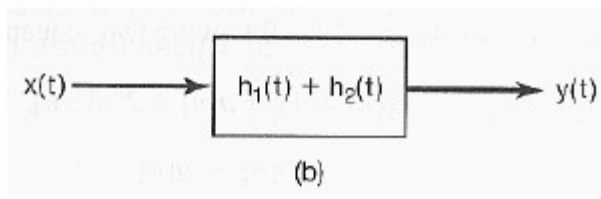
- Distributive Property:

$$x[n] * h[n] = \sum_{k=-\infty}^{+\infty} x[k]h[n-k]$$

$$x(t) * h(t) = \int_{-\infty}^{+\infty} x(\tau)h(t-\tau)d\tau$$

$$x[n] * (h_1[n] + h_2[n]) = x[n] * h_1[n] + x[n] * h_2[n]$$

$$x(t) * (h_1(t) + h_2(t)) = x(t) * h_1(t) + x(t) * h_2(t)$$



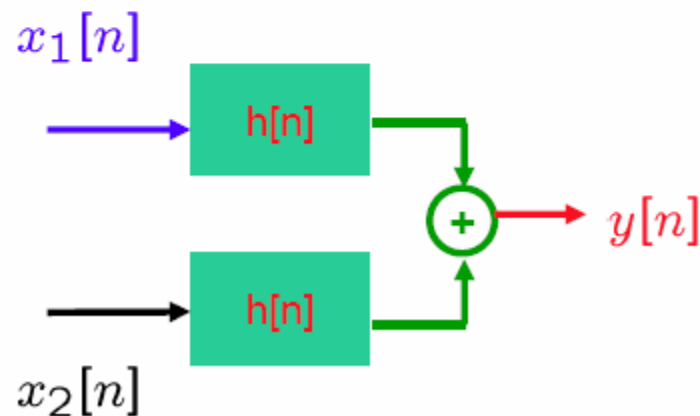
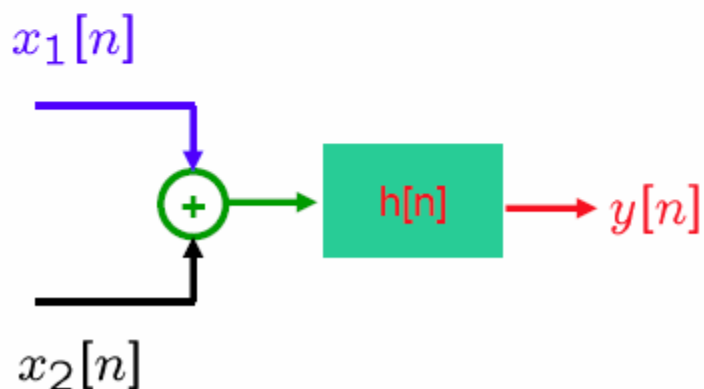
- Distributive Property:

$$x[n] * h[n] = \sum_{k=-\infty}^{+\infty} x[k]h[n-k]$$

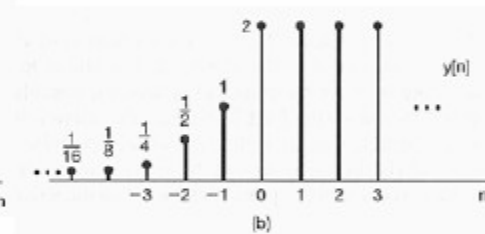
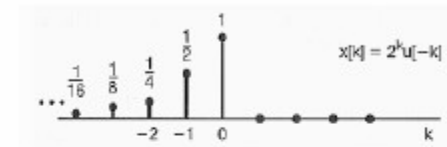
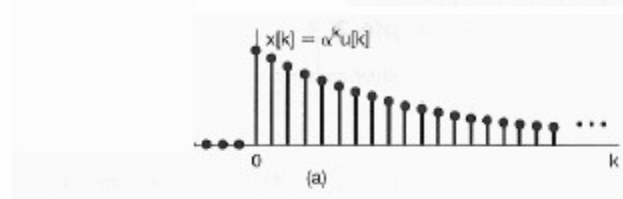
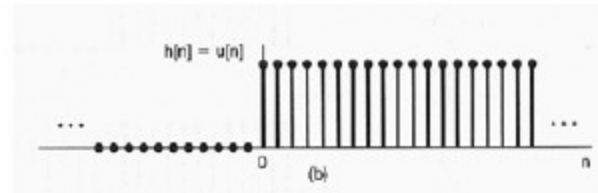
$$x(t) * h(t) = \int_{-\infty}^{+\infty} x(\tau)h(t-\tau)d\tau$$

$$(x_1[n] + x_2[n]) * h[n] = x_1[n] * h[n] + x_2[n] * h[n]$$

$$(x_1(t) + x_2(t)) * h(t) = x_1(t) * h(t) + x_2(t) * h(t)$$



■ Example 2.10

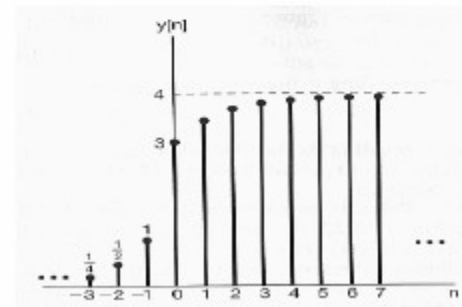


$$x[n] = \left(\frac{1}{2}\right)^n u[n] + 2^n u[-n]$$

$$y[n] = x[n] * h[n]$$

$$= \left(x_1[n] + x_2[n]\right) * h[n]$$

$$= x_1[n] * h[n] + x_2[n] * h[n]$$



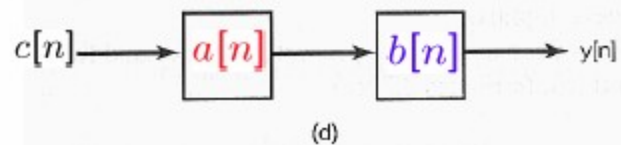
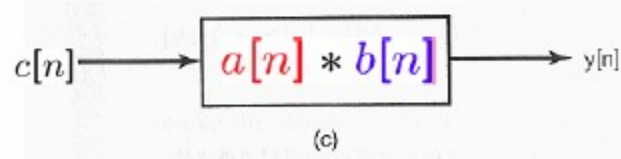
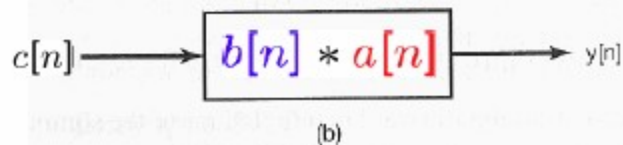
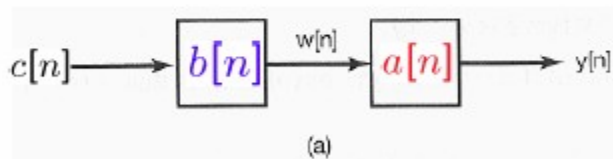
■ Associative Property:

$$a[n] * (b[n] * c[n]) = (a[n] * b[n]) * c[n]$$

$$a(t) * (b(t) * c(t)) = (a(t) * b(t)) * c(t)$$

$$x[n] * h[n] = \sum_{k=-\infty}^{+\infty} x[k]h[n-k]$$

$$x(t) * h(t) = \int_{-\infty}^{+\infty} x(\tau)h(t-\tau)d\tau$$



▪ Memoryless:

- A DT LTI system is memoryless if $h[n] = 0$ for $n \neq 0$

- The impulse response: $h[n] = K\delta[n]$, $K = h[0]$

- The convolution sum: $y[n] = x[n] * h[n] = Kx[n]$

- Similarly, for CT LTI system:

$$y(t) = x(t) * h(t) = Kx(t)$$

- Invertibility:

$$x(t) \rightarrow \boxed{h_1(t)} \rightarrow y(t) \rightarrow \boxed{h_2(t)} \rightarrow w(t)$$

$$y(t) = x(t) * h_1(t) \quad w(t) = y(t) * h_2(t)$$

$$\Rightarrow w(t) = x(t) * h_1(t) * h_2(t)$$

$$x(t) \rightarrow \boxed{\text{Identity System } \delta(t)} \rightarrow x(t)$$

$$x(t) = x(t) * \delta(t)$$

$$\Rightarrow h_2(t) * h_1(t) = \delta(t)$$

■ Example 2.11: Pure time shift



- $y(t) = x(t - t_0)$
- delay if $t_0 > 0$
- advance if $t_0 < 0$

$$\Rightarrow h_1(t) = \delta(t - t_0) \Rightarrow x(t - t_0) = x(t) * \delta(t - t_0)$$

$$\bullet w(t) = x(t) = y(t + t_0)$$

$$\Rightarrow h_2(t) = \delta(t + t_0) \Rightarrow y(t + t_0) = y(t) * \delta(t + t_0)$$

$$\Rightarrow h_1(t) * h_2(t) = \delta(t - t_0) * \delta(t + t_0) = \delta(t)$$

■ Example 2.12



$$h_1[n] = u[n]$$

$$\Rightarrow y[n] = \sum_{k=-\infty}^{\infty} x[k] u[n-k] = \sum_{k=-\infty}^n x[k]$$

$\Rightarrow$ a **running-sum** operation

- Its inverse is a **first difference** operation:

$$w[n] = y[n] - y[n-1] \Rightarrow h_2[n] = \delta[n] - \delta[n-1]$$

$$\Rightarrow h_1[n] * h_2[n] = u[n] - u[n-1] = \delta[n]$$

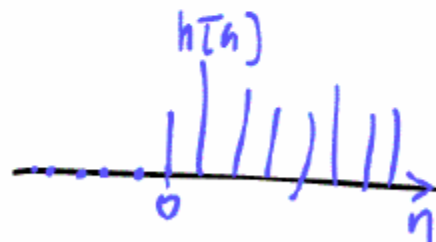
■ Causality:

- The **output** of a **causal** system depends only on the **present** and **past** values of the **input** to the system
- Specifically, $y[n]$ must **not** depend on $x[k]$, for $k > n$

$$h[n - k] = 0, \quad \text{for } k > n$$

$$h[n] = 0, \quad \text{for } n < 0$$

- It implies that the system is **initially rest**



- A **CT** LTI system is **causal** if $h(t) = 0$, for $t < 0$

- Convolution Sum & Integral

$$y[n] = \sum_{k=-\infty}^{+\infty} x[k] h[n-k]$$

$$= \sum_{k=-\infty}^n x[k] h[n-k]$$

$$= \sum_{k=0}^{\infty} h[k] x[n-k]$$

$$y(t) = \int_{-\infty}^{\infty} x(\tau) h(t-\tau) d\tau$$

$$= \int_{-\infty}^t x(\tau) h(t-\tau) d\tau$$

$$= \int_0^{\infty} h(\tau) x(t-\tau) d\tau$$

■ Stability:

- A system is **stable** if every **bounded input** produces a **bounded output**

$$x[n] \rightarrow \text{Stable LTI} \rightarrow y[n]$$

$$|x[n]| < B \quad \text{for all } n \quad \left| y[n] \right| = \left| \sum_{k=-\infty}^{+\infty} h[k] x[n-k] \right|$$

$$\Rightarrow |y[n]| \leq \sum_{k=-\infty}^{+\infty} |h[k]| |x[n-k]|$$

$$\Rightarrow |y[n]| \leq B \left(\sum_{k=-\infty}^{+\infty} |h[k]| \right)$$

$$\text{if } \sum_{k=-\infty}^{+\infty} |h[k]| < \infty \quad \text{absolutely summable} \quad \text{then, } y[n] \text{ is bounded}$$

■ Stability:

- For CT LTI stable system:

$$x(t) \rightarrow \text{Stable LTI} \rightarrow y(t)$$

$$|x(t)| < B \quad \text{for all } t \quad |y(t)| = \left| \int_{-\infty}^{+\infty} h(\tau) x(t - \tau) d\tau \right|$$

$$\Rightarrow |y(t)| \leq \int_{-\infty}^{+\infty} |h(\tau)| |x(t - \tau)| d\tau$$

$$\Rightarrow |y(t)| \leq B \left(\int_{-\infty}^{+\infty} |h(\tau)| d\tau \right)$$

if $\int_{-\infty}^{+\infty} |h(\tau)| d\tau < \infty$ then, $y(t)$ is bounded
absolutely integrable

■ Example 2.13: Pure time shift

• $y[n] = x[n - n_0]$ & $h[n] = \delta[n - n_0]$

• $y(t) = x(t - t_0)$ & $h(t) = \delta(t - t_0)$

$$\Rightarrow \sum_{n=-\infty}^{+\infty} |h[n]| = \sum_{n=-\infty}^{+\infty} |\delta[n - n_0]| = 1 \quad \text{absolutely summable}$$

$$\Rightarrow \int_{-\infty}^{+\infty} |h(\tau)| d\tau = \int_{-\infty}^{+\infty} |\delta(\tau - t_0)| d\tau = 1 \quad \text{absolutely integrable}$$

$\Rightarrow$ A (CT or DT) pure time shift is stable

■ Example 2.13: Accumulator

$$\bullet y[n] = \sum_{k=-\infty}^n x[k] \quad \& \quad h[n] = u[n]$$

$$\bullet y(t) = \int_{-\infty}^t x(\tau) d\tau \quad \& \quad h(t) = u(t)$$

$$\Rightarrow \sum_{n=-\infty}^{+\infty} |h[n]| = \sum_{n=0}^{+\infty} |u[n]| = \infty \quad \text{NOT absolutely summable}$$

$$\Rightarrow \int_{-\infty}^{+\infty} |h(\tau)| d\tau = \int_0^{\infty} |u(\tau)| d\tau = \infty \quad \text{NOT absolutely integrable}$$

$\Rightarrow$ A accumulator or integrator is NOT stable

■ Unit Step Response:

$$h[n] = \delta[n] * h[n]$$

- For an LTI system, its impulse response is:

$$\delta[n] \rightarrow \boxed{\text{DT LTI}} \rightarrow h[n] \quad \delta(t) \rightarrow \boxed{\text{CT LTI}} \rightarrow h(t)$$

- Its unit step response is:

$$u[n] \rightarrow \boxed{\text{DT LTI}} \rightarrow s[n] \quad u(t) \rightarrow \boxed{\text{CT LTI}} \rightarrow s(t)$$

$$\Rightarrow s[n] = u[n] * h[n]$$

$$\Rightarrow s(t) = u(t) * h(t)$$

$$= \sum_{k=-\infty}^n h[k]$$

$$= \int_{-\infty}^t h(\tau) d\tau$$

$$\Rightarrow h[n] = s[n] - s[n-1]$$

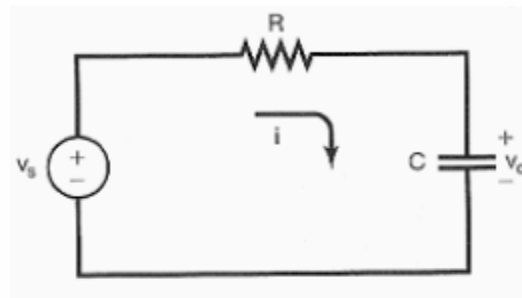
$$\Rightarrow h(t) = \frac{ds(t)}{dt}$$

- Discrete-Time Linear Time-Invariant Systems
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■ Linear Constant-Coefficient Differential Equations

- e.x., RC circuit

Input signal: $v_s(t)$



Output signal: $v_c(t)$

$$\Rightarrow \frac{dv_c(t)}{dt} + \frac{1}{RC}v_c(t) = \frac{1}{RC}v_s(t)$$

$$x(t) \rightarrow \text{RC Circuit} \rightarrow y(t) \quad \Rightarrow \quad \frac{d}{dt}y(t) + a y(t) = b x(t)$$

■ Linear Constant-Coefficient Differential Equations

- For a general CT LTI system,

$$x(t) \rightarrow \boxed{\text{CT LTI}} \rightarrow y(t)$$

$$a_N \frac{d^N}{dt^N} y(t) + a_{N-1} \frac{d^{N-1}}{dt^{N-1}} y(t) + \cdots + a_1 \frac{d}{dt} y(t) + a_0 y(t)$$

$$= b_M \frac{d^M}{dt^M} x(t) + b_{M-1} \frac{d^{M-1}}{dt^{M-1}} x(t) + \cdots + b_1 \frac{d}{dt} x(t) + b_0 x(t)$$

$$\Rightarrow \sum_{k=0}^N a_k \frac{d^k}{dt^k} y(t) = \sum_{k=0}^M b_k \frac{d^k}{dt^k} x(t)$$

$$\Rightarrow h(t) = ?$$

■ Linear Constant-Coefficient Difference Equations

- For a general DT LTI system,

$$x[n] \rightarrow \boxed{\text{DT LTI}} \rightarrow y[n]$$

$$a_0 y[n] + a_1 y[n-1] + \cdots + a_{N-1} y[n-N+1] + a_N y[n-N]$$

$$= b_0 x[n] + b_1 x[n-1] + \cdots + b_{M-1} x[n-M+1] + b_M x[n-M]$$

$$\Rightarrow \sum_{k=0}^N a_k y[n-k] = \sum_{k=0}^M b_k x[n-k]$$

$$\Rightarrow h[n] = ?$$

- Recursive Equation:

$$a_0 y[n] + a_1 y[n-1] + \cdots + a_{N-1} y[n-N+1] + a_N y[n-N]$$

$$= b_0 x[n] + b_1 x[n-1] + \cdots + b_{M-1} x[n-M+1] + b_M x[n-M]$$

$$\sum_{k=0}^N a_k y[n-k] = \sum_{k=0}^M b_k x[n-k]$$

$$\Rightarrow y[n] = \frac{1}{a_0} \left\{ \sum_{k=0}^M b_k x[n-k] - \sum_{k=1}^N a_k y[n-k] \right\}$$

■ Recursive Equation:

- For example, (Example 2.15)

$$y[n] - \frac{1}{2}y[n-1] = x[n]$$

$$\Rightarrow \begin{cases} y[0] = x[0] + \frac{1}{2}y[-1] \\ y[1] = x[1] + \frac{1}{2}y[0] \\ y[2] = x[2] + \frac{1}{2}y[1] \\ \vdots \\ y[n] = x[n] + \frac{1}{2}y[n-1] \end{cases}$$

$$\Rightarrow h[n] = \left(\frac{1}{2}\right)^n u[n]$$

$\Rightarrow$ an Infinite Impulse Response (IIR) system



$$y[n] = 0, \quad \text{for } n \leq -1$$

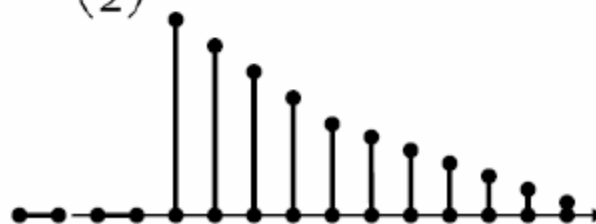
$$x[n] = K\delta[n]$$

$$= K$$

$$= \frac{1}{2}K$$

$$= \left(\frac{1}{2}\right)^2 K$$

$$= \left(\frac{1}{2}\right)^n K$$



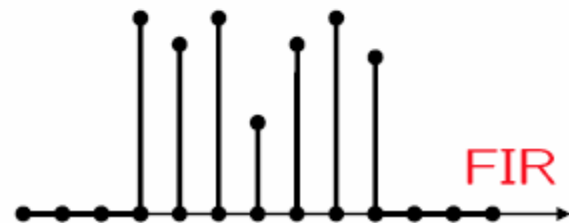
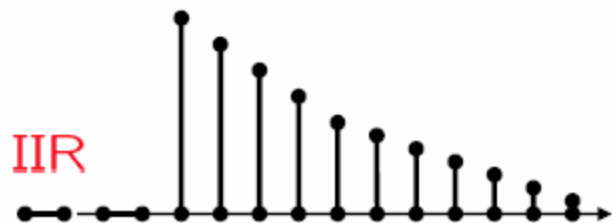
■ Nonrecursive Equation:

- When $N = 0$,

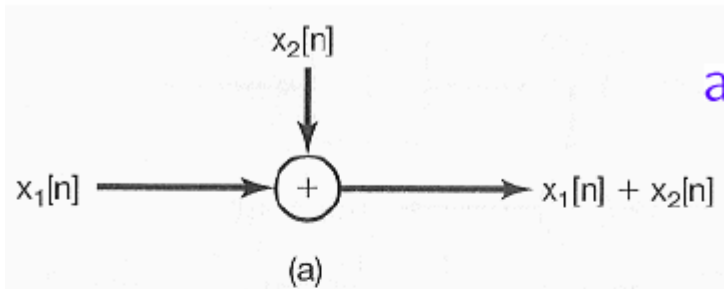
$$\Rightarrow y[n] = \sum_{k=0}^M \left(\frac{b_k}{a_0} \right) x[n-k]$$

$$\Rightarrow h[n] = \begin{cases} \frac{b_n}{a_0}, & 0 \leq n \leq M \\ 0, & \text{otherwise} \end{cases}$$

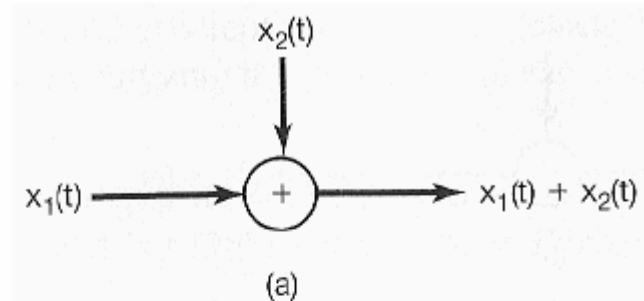
$\Rightarrow$ a **Finite Impulse Response (FIR)** system



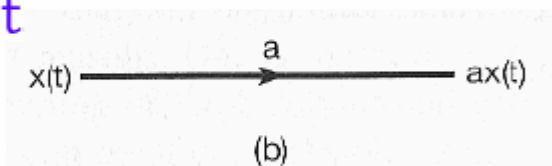
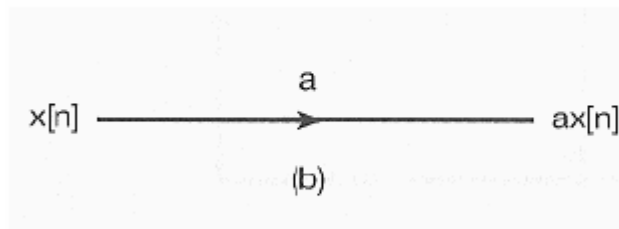
▪ Block Diagram Representations:



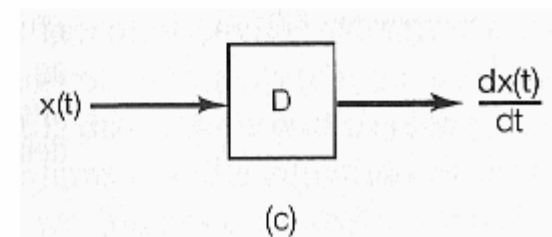
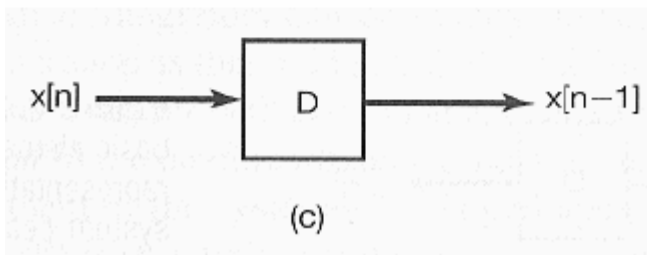
an adder



multiplication
by a coefficient



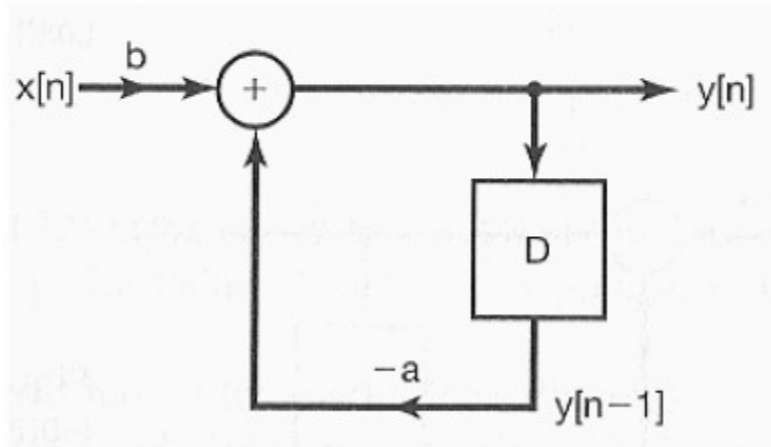
a unit delay/
differentiator



- Block Diagram Representations:

$$y[n] + ay[n-1] = bx[n]$$

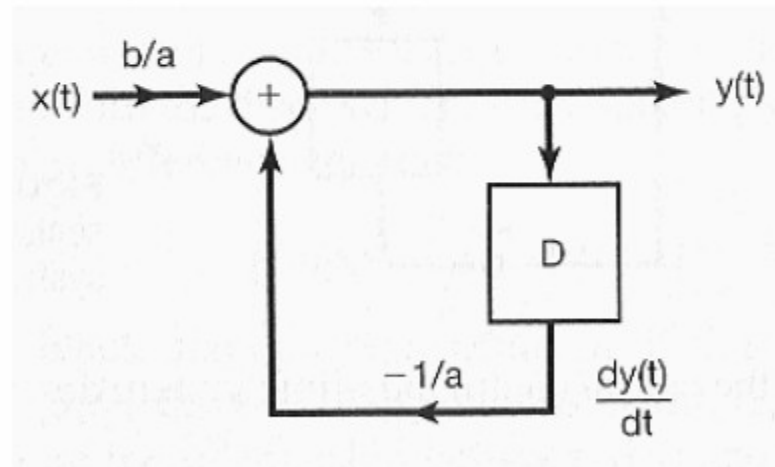
$$y[n] = -ay[n-1] + bx[n]$$



$$D \iff z^{-1}$$

$$\frac{d}{dt}y(t) + ay(t) = bx(t)$$

$$y(t) = -\frac{1}{a} \frac{d}{dt}y(t) + \frac{b}{a}x(t)$$



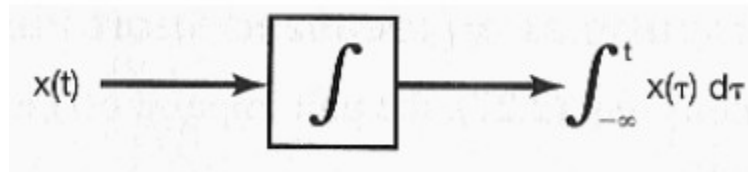
$$D \iff s$$

- Block Diagram Representations:

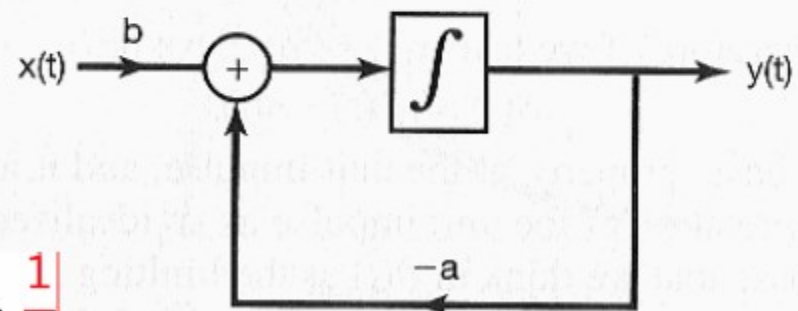
$$\frac{d}{dt}y(t) = bx(t) - ay(t)$$

$$\Rightarrow y(t) = \int_{-\infty}^t [bx(\tau) - ay(\tau)] d\tau$$

$$\Rightarrow y(t) = y(t_0) + \int_{t_0}^t [bx(\tau) - ay(\tau)] d\tau$$



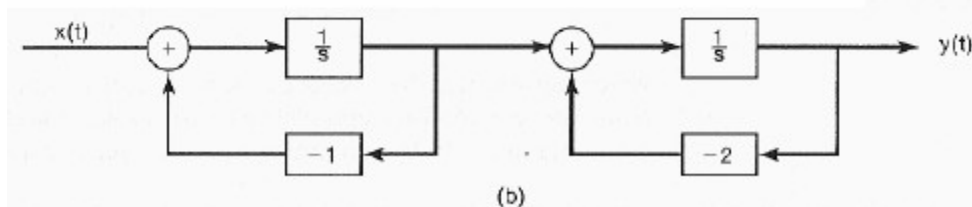
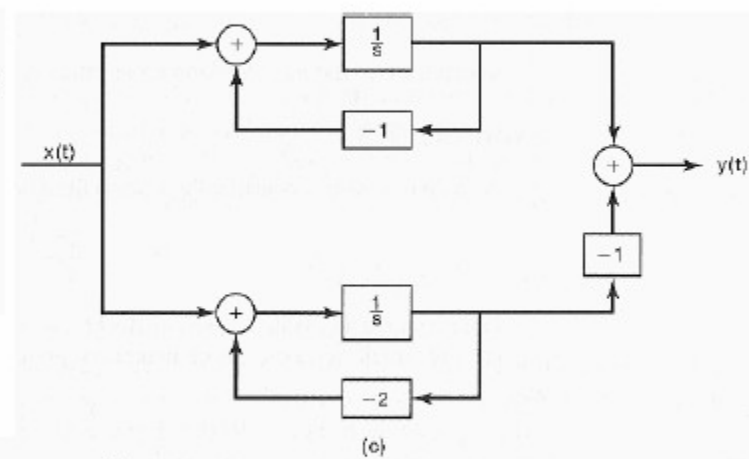
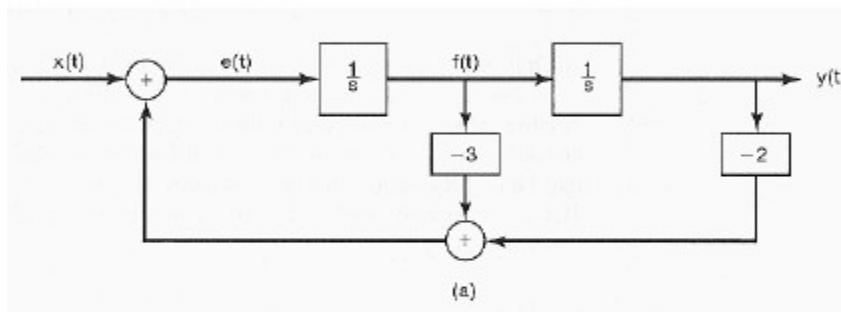
$$\int \Leftrightarrow \frac{1}{s}$$



■ Block Diagram Representations:

$$\int \Longleftrightarrow \frac{1}{s}$$

$$\frac{d^2}{dt^2}y(t) + 3\frac{d}{dt}y(t) + 2y(t) = x(t)$$

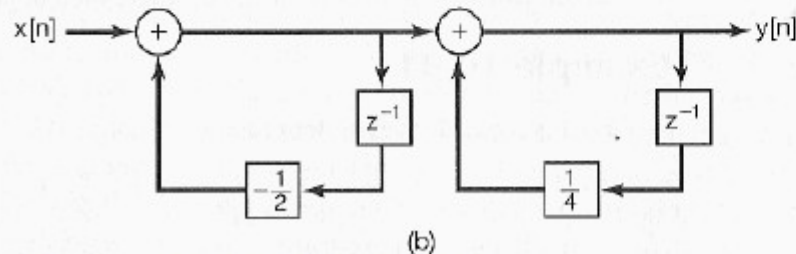
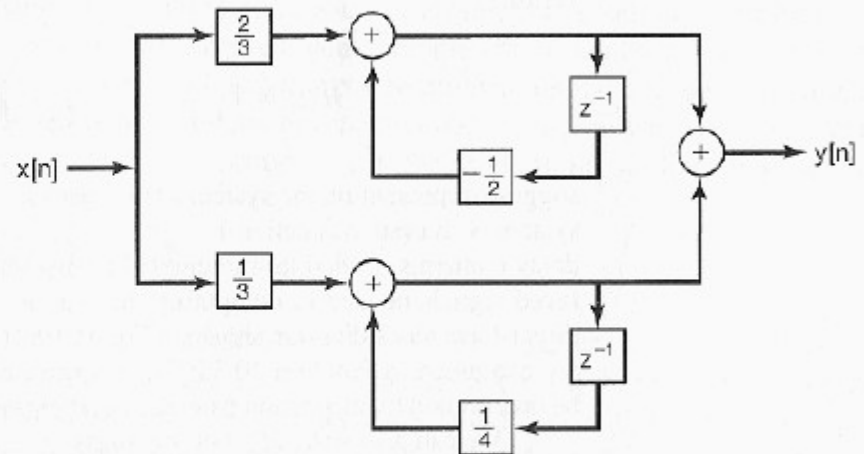
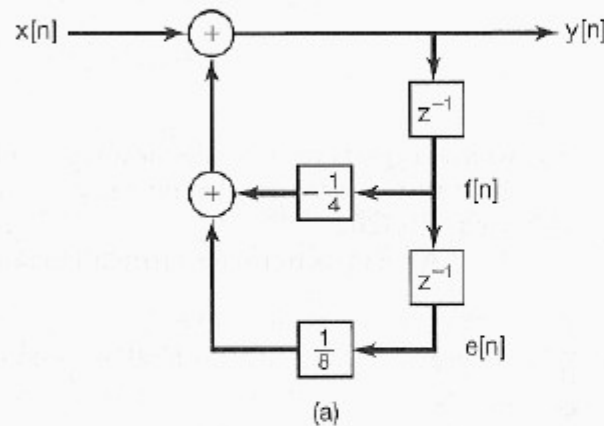


$$H(s) = \frac{1}{(s+1)(s+2)}$$

■ Example 9.30 (pp.711)

- Block Diagram Representations:

$$y[n] + \frac{1}{4}y[n-1] - \frac{1}{8}y[n-2] = x[n]$$



$$H(z) = \frac{1}{(1 + \frac{1}{2}z^{-1})(1 - \frac{1}{4}z^{-1})}$$

- Example 10.30 (pp.786)

- Discrete-Time Linear Time-Invariant Systems
 - The convolution sum
- Continuous-Time Linear Time-Invariant Systems
 - The convolution integral
- Properties of Linear Time-Invariant Systems
- Causal Linear Time-Invariant Systems
Described by Differential & Difference Equations
- Singularity Functions

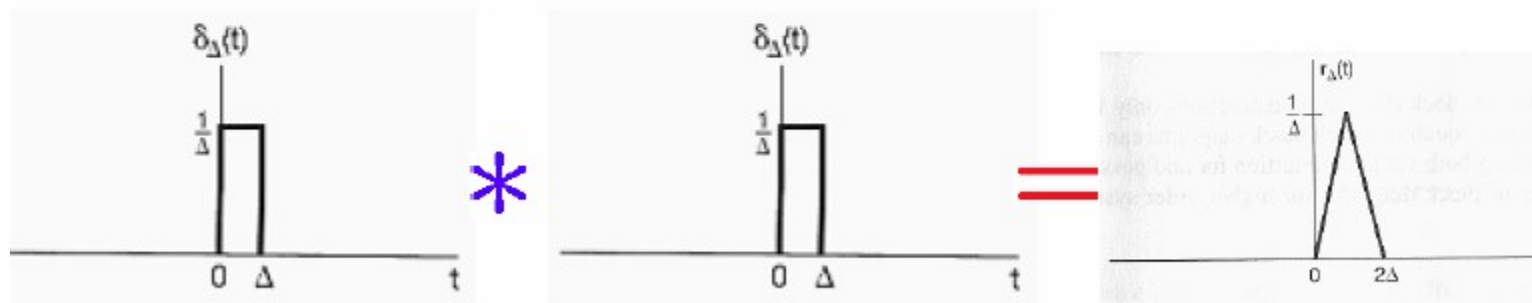
■ Singularity Functions

- CT unit impulse function is one of singularity functions

$$x(t) = x(t) * \delta(t)$$

$$\delta(t) = \delta(t) * \delta(t)$$

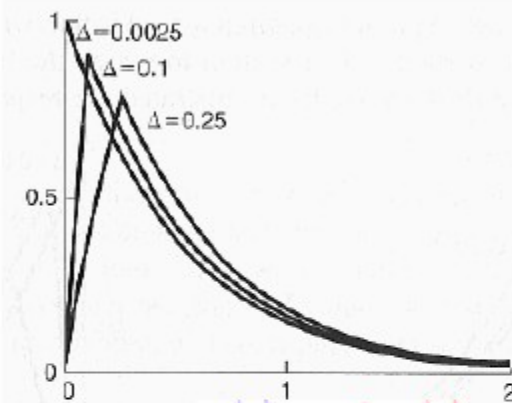
$$r_{\Delta}(t) = \delta_{\Delta}(t) * \delta_{\Delta}(t)$$



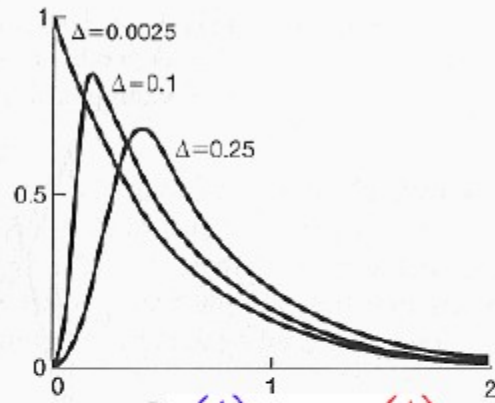
$$\lim_{\Delta \rightarrow 0} \delta_{\Delta}(t) = \delta(t)$$

$$\Rightarrow \lim_{\Delta \rightarrow 0} r_{\Delta}(t) = \delta(t)$$

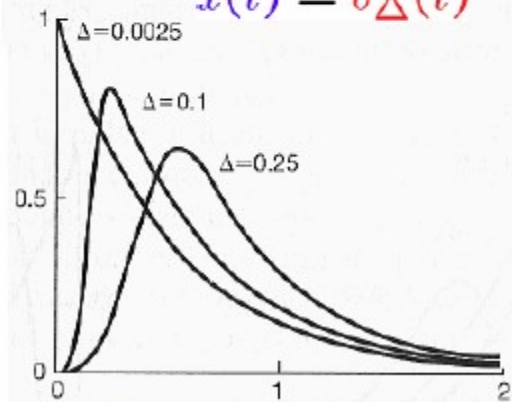
- Example 2.16 $\frac{d}{dt}y(t) + 2y(t) = x(t)$ with initial-rest condition



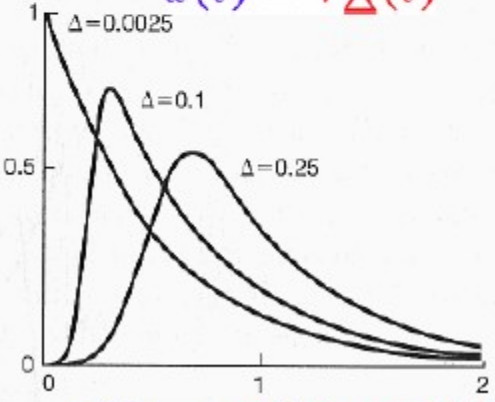
$$x(t) = \delta_{\Delta}(t)$$



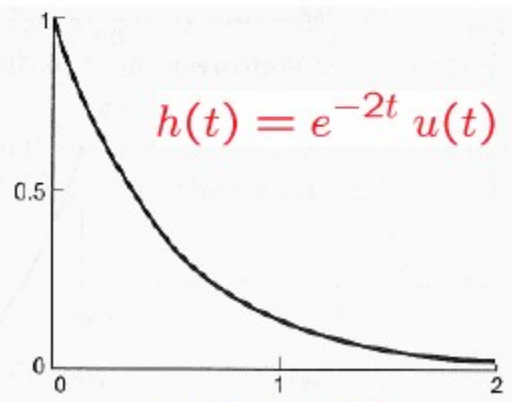
$$x(t) = r_{\Delta}(t)$$



$$x(t) = \delta_{\Delta}(t) * r_{\Delta}(t)$$



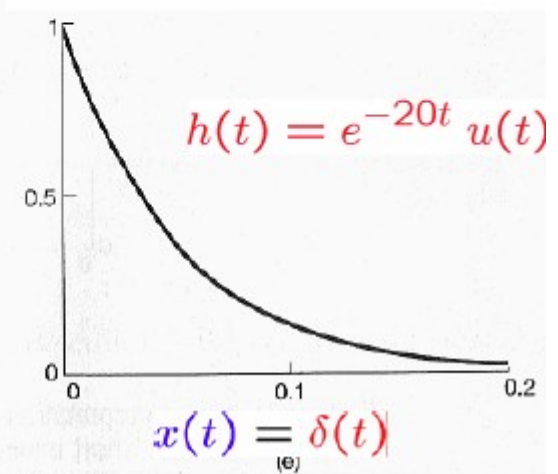
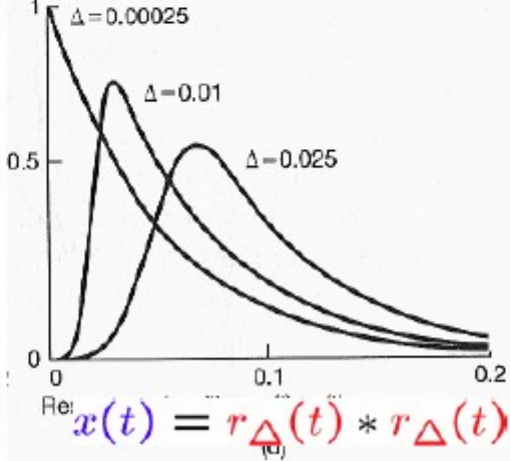
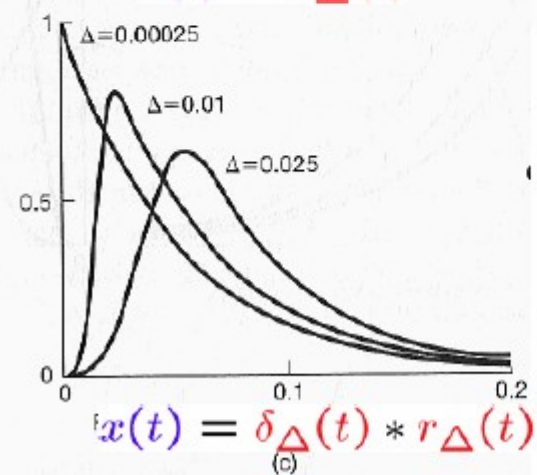
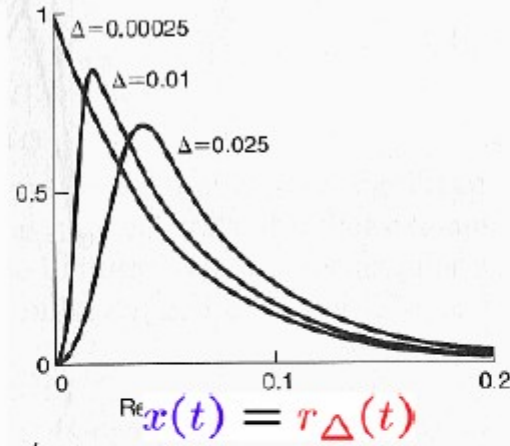
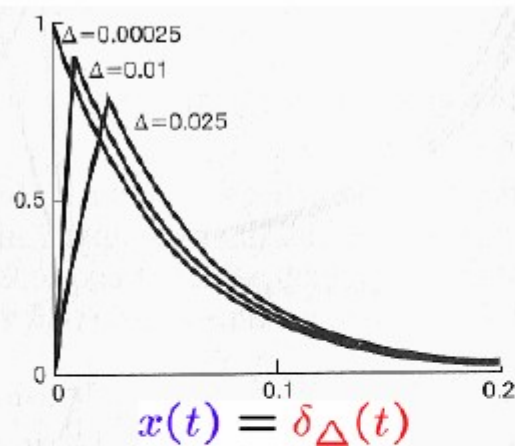
$$x(t) = r_{\Delta}(t) * r_{\Delta}(t)$$



$$h(t) = e^{-2t} u(t)$$

$$x(t) = \delta(t)$$

- Example 2.16** $\frac{d}{dt}y(t) + 20y(t) = x(t)$ with initial-rest condition



- Defining the Unit Impulse through Convolution:

$$x(t) = x(t) * \delta(t)$$

- Let $x(t) = 1$,

$$\begin{aligned} 1 &= x(t) = x(t) * \delta(t) = \delta(t) * x(t) \\ &= \int_{-\infty}^{\infty} \delta(\tau) x(t - \tau) d\tau = \int_{-\infty}^{\infty} \delta(\tau) d\tau \end{aligned}$$

- So that the unit impulse has unit area

- Defining Unit Impulse through Convolution:

- Alternatively, consider an arbitrary signal $g(t)$,

$$g(-t) = g(-t) * \delta(t) = \int_{-\infty}^{\infty} g(\tau - t) \delta(\tau) d\tau$$

$$g(0) = \int_{-\infty}^{\infty} g(\tau) \delta(\tau) d\tau$$

- Define $x(t - \tau) = g(\tau)$

$$x(t) = g(0) = \int_{-\infty}^{\infty} g(\tau) \delta(\tau) d\tau$$

$$= \int_{-\infty}^{\infty} x(t - \tau) \delta(\tau) d\tau = x(t) * \delta(t)$$

▪ Defining Unit Impulse through Convolution:

- Consider the signal $f(t)\delta(t)$

$$\int_{-\infty}^{\infty} g(\tau)f(\tau)\delta(\tau)d\tau = g(0)f(0)$$

- On the other hand, consider the signal $f(0)\delta(t)$

$$\int_{-\infty}^{\infty} g(\tau)f(0)\delta(\tau)d\tau = g(0)f(0)$$

- Therefore,

$$f(t)\delta(t) = f(0)\delta(t)$$

■ Unit Doublets of Derivative Operation:

- A system: Output is the derivative of input

$$y(t) = \frac{d}{dt}x(t)$$

⇒ The unit impulse response of the system is the derivative of the unit impulse, which is called the unit doublet $u_1(t)$

- That is, from $x(t) = x(t) * \delta(t)$, we have

$$\frac{d}{dt}x(t) = x(t) * u_1(t)$$

■ Unit Doublets of Derivative Operation:

- Similarly,

$$\frac{d^2}{dt^2}x(t) = x(t) * u_2(t)$$

- But,

$$\frac{d^2}{dt^2}x(t) = \frac{d}{dt} \left(\frac{d}{dt}x(t) \right) = \left(x(t) * u_1(t) \right) * u_1(t)$$

- Therefore,

$$u_2(t) = u_1(t) * u_1(t)$$

- In general,

$u_k(t)$, $k > 0$, the k th derivative of $\delta(t)$

$$u_k(t) = u_1(t) * \cdots * u_1(t)$$

■ Unit Doublets of Integration Operation:

- A system: Output is the **integral** of input

$$y(t) = \int_{-\infty}^{\infty} x(\tau) d\tau$$

- Therefore,

$$u(t) = \int_{-\infty}^{\infty} \delta(\tau) d\tau$$

- Hence, we have

$$x(t) * u(t) = \int_{-\infty}^t x(\tau) d\tau$$

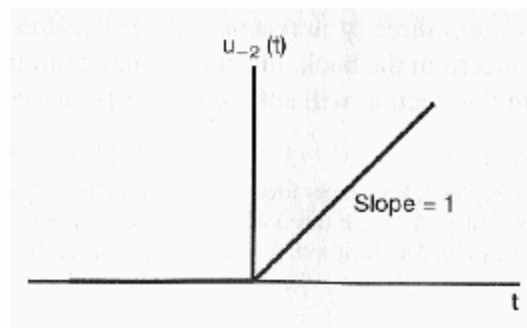
■ Unit Doublets of Integration Operation:

- Similarly,

$$u_{-2}(t) = u(t) * u(t) = \int_{-\infty}^t u(\tau) d\tau$$

- That is,

$$u_{-2}(t) = t u(t) \quad \text{the unit ramp function}$$



■ Unit Doublets of Integration Operation:

- Moreover,

$$\begin{aligned}x(t) * u_{-2}(t) &= x(t) * u(t) * u(t) \\&= \left(\int_{-\infty}^t x(\sigma) d\sigma \right) * u(t) \\&= \int_{-\infty}^t \left(\int_{-\infty}^{\tau} x(\sigma) d\sigma \right) d\tau\end{aligned}$$

- In general,

$$\begin{aligned}u_{-k}(t) &= u(t) * \cdots * u(t) = \int_{-\infty}^t u_{-(k-1)}(\tau) d\tau \\u_{-k}(t) &= \frac{t^{k-1}}{(k-1)!} u(t)\end{aligned}$$

■ In Summary

$$\delta(t) = u_0(t)$$

$$u(t) = u_{-1}(t)$$

$$u_k(t) \quad k > 0,$$

Impulse response of a cascade of k differentiators

$$k < 0,$$

Impulse response of a cascade of $|k|$ integrators

$$u(t) * u_1(t) = \delta(t) \quad \text{or,} \quad u_{-1}(t) * u_1(t) = u_0(t)$$

$$\Rightarrow u_k(t) * u_r(t) = u_{k+r}(t)$$

Signals & Systems [\(Chap 1\)](#)

LTI & Convolution [\(Chap 2\)](#)

Bounded/Convergent

Periodic

FS

[\(Chap 3\)](#)

– CT

– DT

Aperiodic

FT

– CT

[\(Chap 4\)](#)

– DT

[\(Chap 5\)](#)

Unbounded/Non-convergent

LT

– CT

[\(Chap 9\)](#)

zT

– DT

[\(Chap 10\)](#)

Time-Frequency [\(Chap 6\)](#)

Communication [\(Chap 8\)](#)

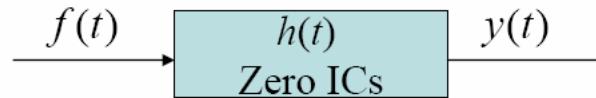
CT-DT [\(Chap 7\)](#)

Control [\(Chap 11\)](#)

C-T Convolution

Examples

Example 1:



Given : $f(t) = e^t u(-t)$ $h(t) = -\delta(t) + 2e^{-t}u(t)$

Find : *Zero - state response* : $y(t) = f(t) * h(t)$

Solution: $y(t) = f(t) * h(t) = f(t) * \left[-\delta(t) + 2e^{-t}u(t) \right]$

First we'll use properties of convolution to break this down into sub-problems.

By the distributive property we have:

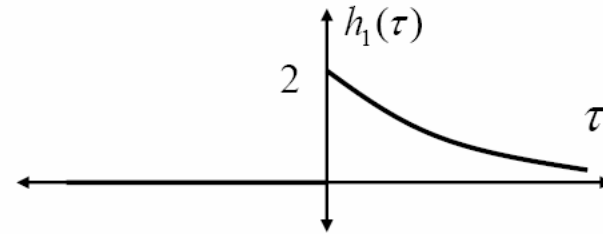
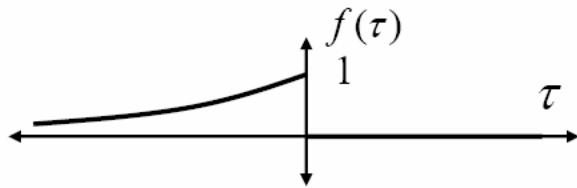
$$y(t) = \underbrace{-f(t) * \delta(t)}_{\text{Use Property of Convolution with impulses}} + \underbrace{f(t) * 2e^{-t}u(t)}_{\text{Call this convolution } y_1(t)}$$

call this $h_1(t)$

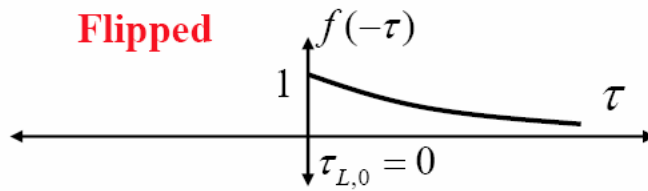
Use Property of Convolution with impulses Call this convolution $y_1(t)$

$$y(t) = -f(t) + y_1(t)$$

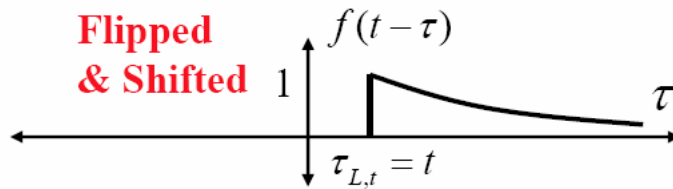
Write as functions of τ :



Flipped

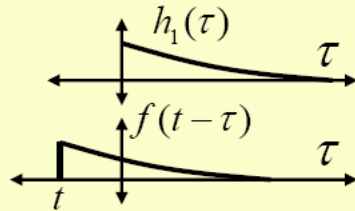


**Flipped
& Shifted**



Only two cases for product $h_1(\tau)f(t-\tau)$:

RI: $t < 0$



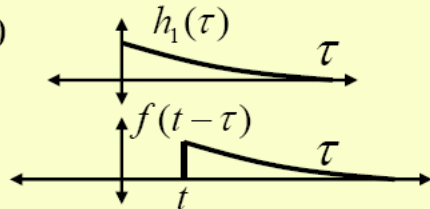
$$\Rightarrow y_1(t) = \int_0^{\infty} [e^{t-\tau}] 2e^{-\tau} d\tau$$

$$= 2e^t \int_0^{\infty} e^{-2\tau} d\tau$$

$$= 2e^t \left[\frac{-1}{2} e^{-2\tau} \right]_0^{\infty} = -e^t [0 - 1]$$

$$y_1(t) = e^t \text{ for } t < 0$$

RII: $t \geq 0$



$$y_1(t) = \int_t^{\infty} [e^{t-\tau}] 2e^{-\tau} d\tau$$

$$= -e^t [e^{-2\tau}]_t^{\infty} = -e^t [0 - e^{-2t}]$$

$$= e^{-t}$$

$$y_1(t) = e^{-t} \text{ for } t \geq 0$$

Now... assemble these results together:

$$y(t) = -f(t) + y_1(t)$$

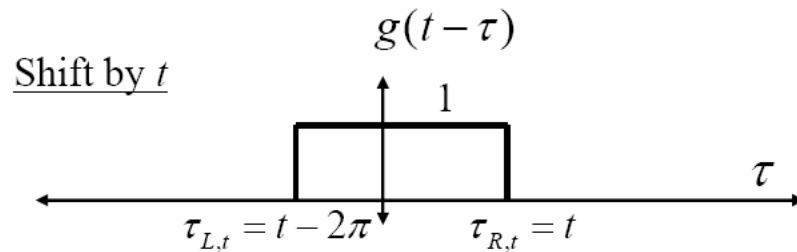
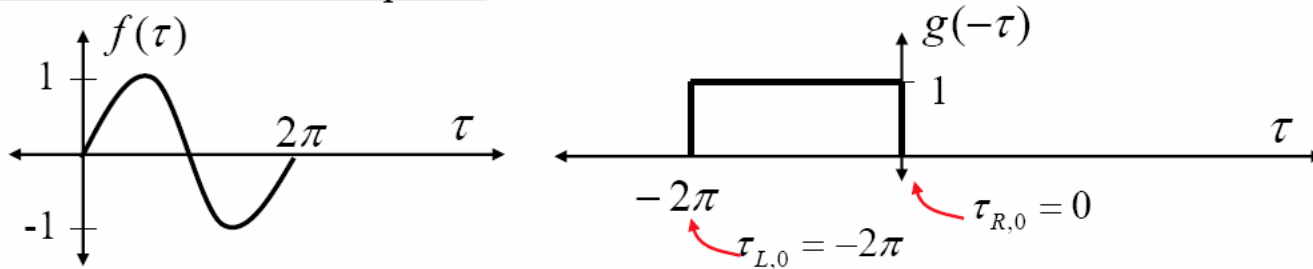
So:

$$y(t) = \begin{cases} \underbrace{-e^t}_{-f(t)} + \underbrace{e^t}_{y_1(t)} = 0, & \text{for } t < 0 \\ \underbrace{0}_{-f(t)} + \underbrace{e^{-t}}_{y_1(t)} = e^{-t}, & \text{for } t \geq 0 \end{cases}$$

$$y(t) = \begin{cases} 0, & \text{for } t < 0 \\ e^{-t}, & \text{for } t \geq 0 \end{cases}$$

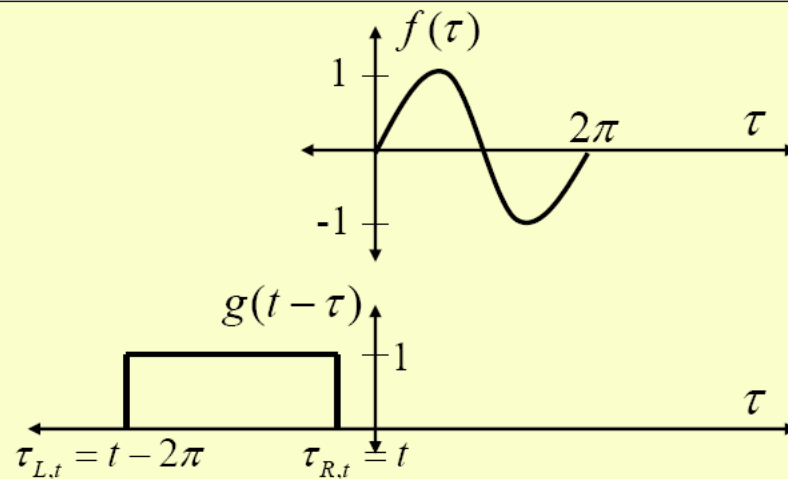
Example 2: Given : $f(t) = \sin(t)[u(t) - u(t - 2\pi)]$
 $g(t) = [u(t) - u(t - 2\pi)]$
 Find : $c(t) = f(t) * g(t)$

First write “as τ ” and flip one:



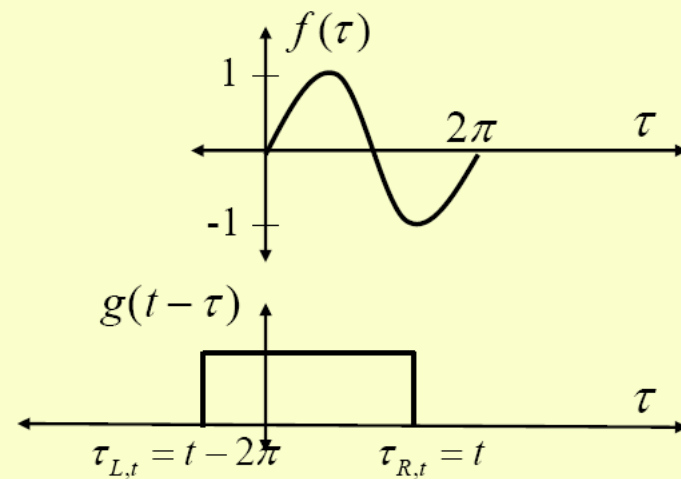
Now find the Regions of overlap... there are Four Regions

$RI: \quad t \leq 0$

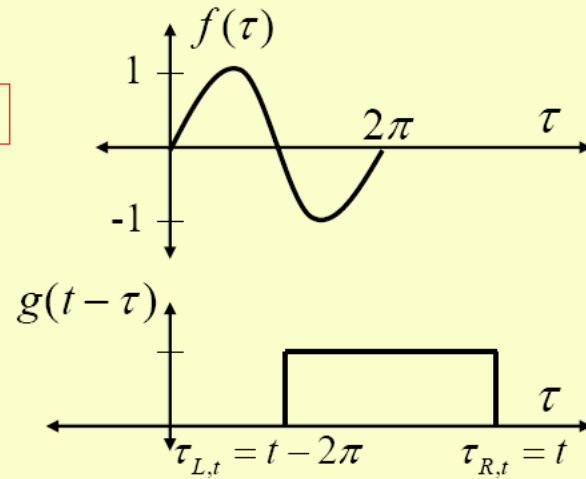


$RII: \quad \left. \begin{array}{l} t > 0 \\ t - 2\pi \leq 0 \end{array} \right\} 0 < t \leq 2\pi$

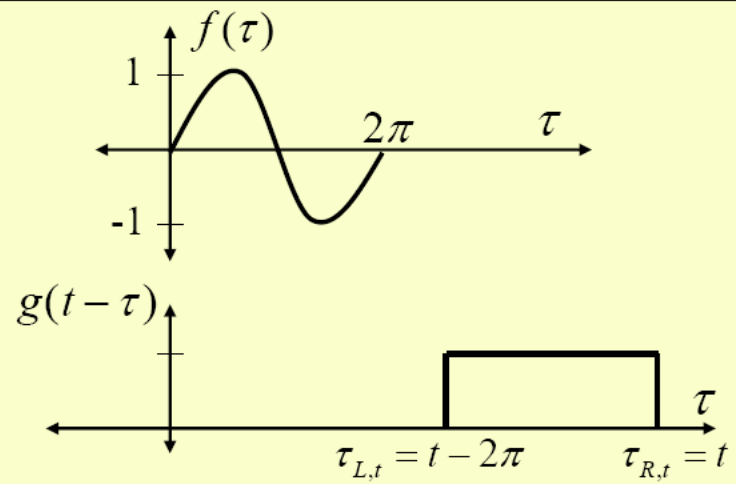
Note: this region includes the single t value at which you get complete overlap



$$RIII : \left. \begin{array}{l} t - 2\pi \leq 2\pi \\ t > 2\pi \end{array} \right\} \boxed{2\pi < t \leq 4\pi}$$



$$RIV : \quad t - 2\pi > 2\pi \quad \boxed{t > 4\pi}$$

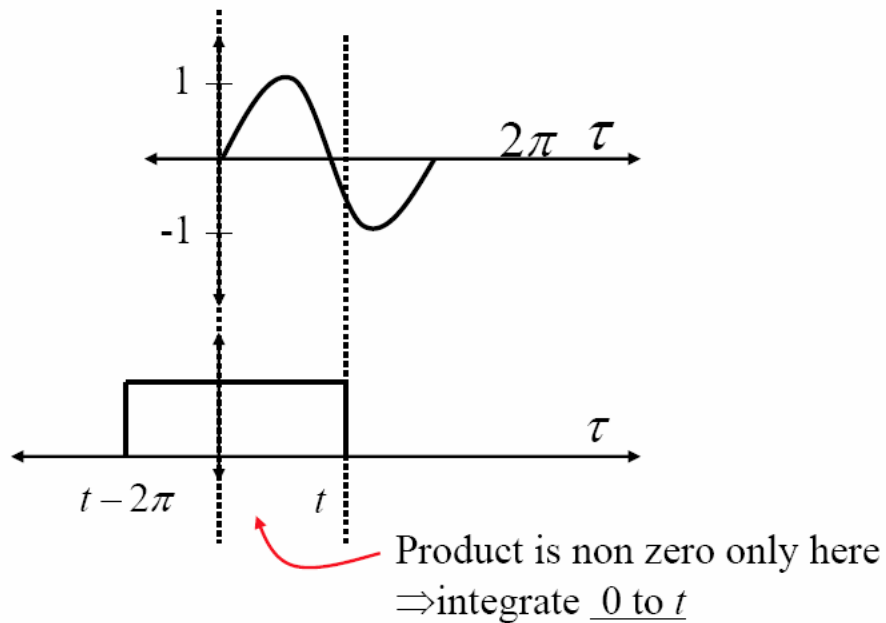


RI and RIV Integrals

No Overlap $\Rightarrow$ Integrand = 0 $\Rightarrow$

$$c(t) = 0 \text{ for } t \leq 0 \\ \text{and } t > 4\pi$$

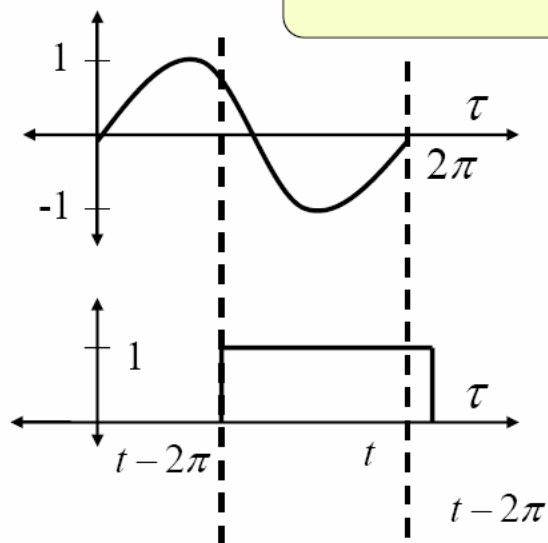
RII Integral



$$\int_0^t 1 \sin(\tau) d\tau = [-\cos(\tau)]_0^t = -[\cos(t) - 1] \\ = 1 - \cos(t) \quad \text{for } 0 < t \leq 2\pi$$

RIII Integral

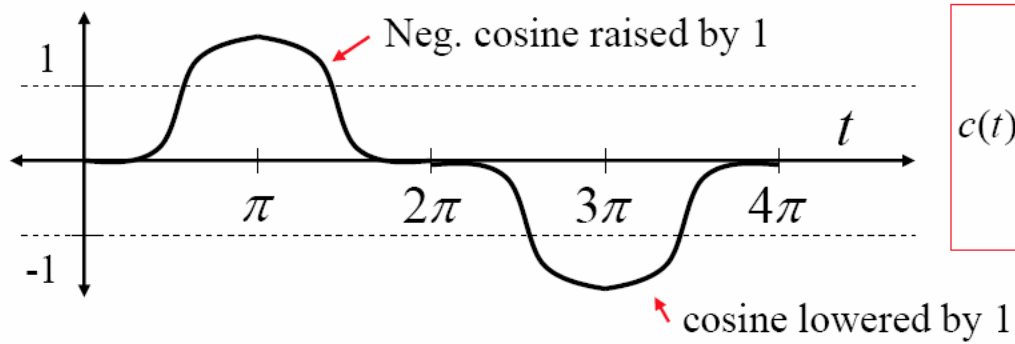
Comment: note that you do not get a sinusoid out, because you only put part of a sinusoid in!



$2\pi \Rightarrow$ Integrate $(t - 2\pi)$ to 2π

$$\begin{aligned} \int_{(t-2\pi)}^{2\pi} 1 \sin(\tau) d\tau &= \left[-\cos(\tau) \right]_{t-2\pi}^{2\pi} = -[\cos(2\pi) - \underbrace{\cos(t-2\pi)}_{= \cos(t) \text{ by periodicity}}] \\ &= \underbrace{\cos(t) - 1}_{\text{cosine lowered by 1}} \quad \text{for } 2\pi < t \leq 4\pi \end{aligned}$$

Now... assemble the parts that make up the output:



$$c(t) = \begin{cases} 0, & t \leq 0 \\ 1 - \cos t, & 0 < t \leq 2\pi \\ \cos t - 1, & 2\pi < t \leq 4\pi \\ 0, & t > 4\pi \end{cases}$$

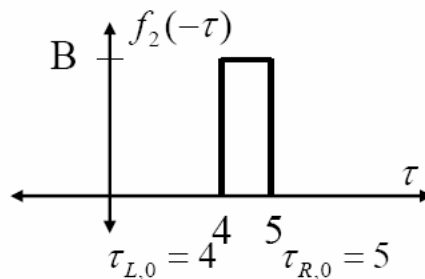
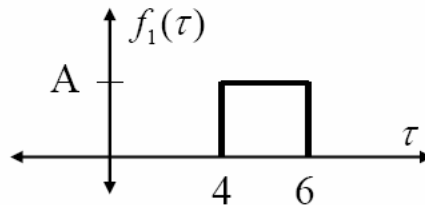
Example 3:

Given: $f_1(t) = A[u(t-4) - u(t-6)]$

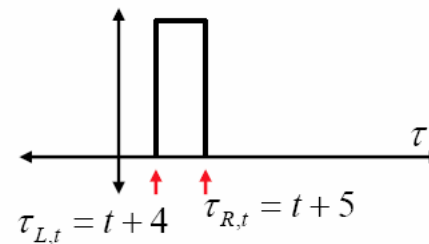
$$f_2(t) = [u(t+5) - u(t+4)]$$

Find: $c(t) = f_1(t) * f_2(t)$

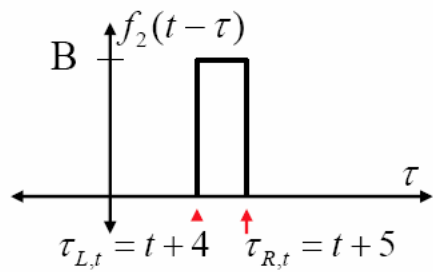
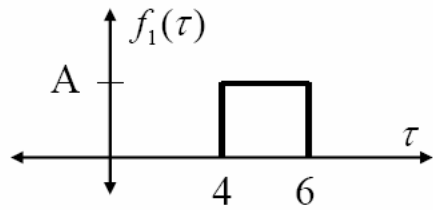
Write “as τ ” and flip one:



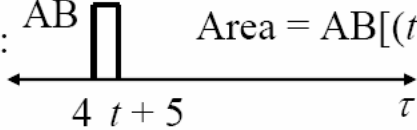
“Shift by t ”



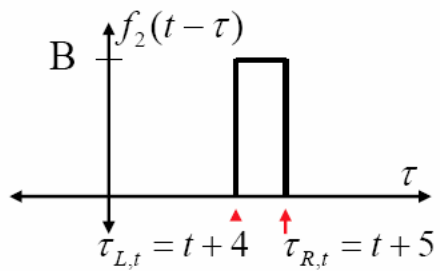
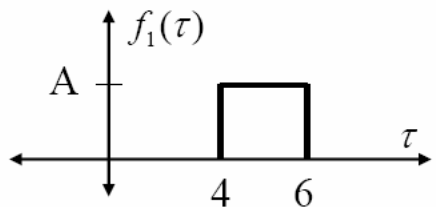
RI: $t+5 < 4 \Rightarrow t < -1$ prod. = 0 $\Rightarrow c(t) = 0 \quad \forall t < -1$



$$\text{RII: } \left. \begin{array}{l} t+5 \geq 4 \\ t+4 < 4 \end{array} \right\} -1 \leq t < 0$$

product:  Area = $AB[(t+5)-4] = ABt + AB$

$$c(t) = \int_4^{t+5} AB d\tau = \boxed{ABt + AB \quad \text{for} \quad -1 \leq t < 0}$$

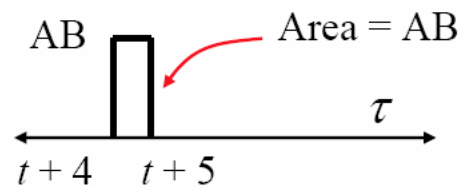


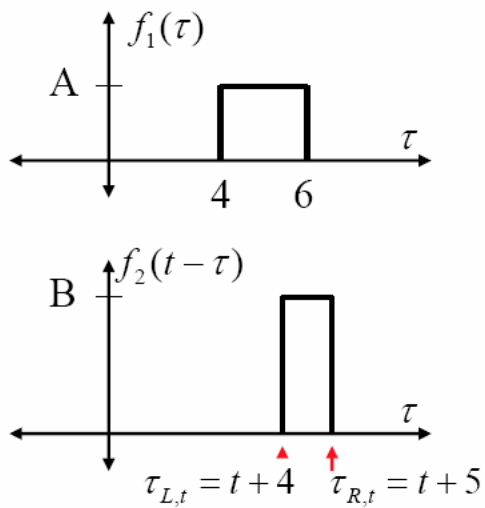
RIII: $\left. \begin{array}{l} t+4 \geq 4 \\ t+5 < 6 \end{array} \right\} 0 \leq t < 1$

$$c(t) = \int_{t+4}^{t+5} AB d\tau$$

$$c(t) = AB \text{ for } 0 \leq t < 1$$

product:

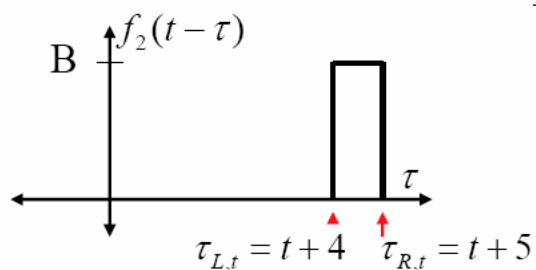
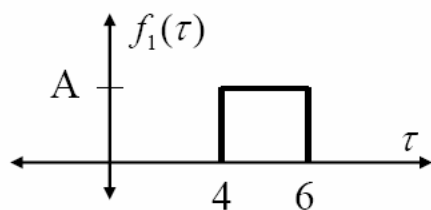




RIV: $\left. \begin{array}{l} t+5 \geq 6 \\ t+4 < 6 \end{array} \right\} 1 \leq t < 2$ product:

Graph 3: The product function AB vs τ . The function is zero for $\tau < t+4$ and $\tau > 6$. It has a constant value of AB for $t+4 \leq \tau \leq 6$. A red arrow points to the area of the rectangle, which is labeled as $\text{Area} = AB(6 - (t+4))$.

$$c(t) = \int_{t+4}^6 AB d\tau = 2AB - ABt \quad \text{for } 1 \leq t < 2$$

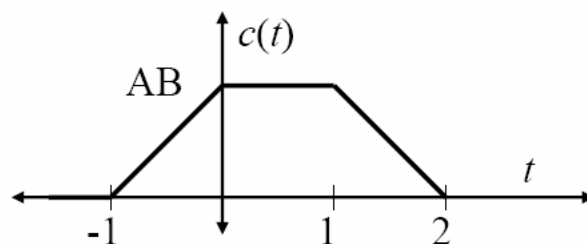


RV: $t + 4 \geq 6 \Rightarrow t \geq 2$ product = 0

$c(t) = 0$ for $t \geq 2$

Now... assemble the pieces:

$$c(t) = \begin{cases} 0, & t < -1 \\ ABt + AB, & -1 \leq t < 0 \\ AB, & 0 \leq t < 1 \\ 2AB - ABt, & 1 \leq t < 2 \\ 0, & t \geq 2 \end{cases}$$

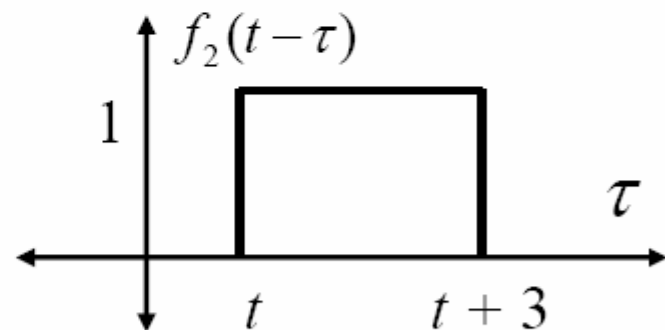
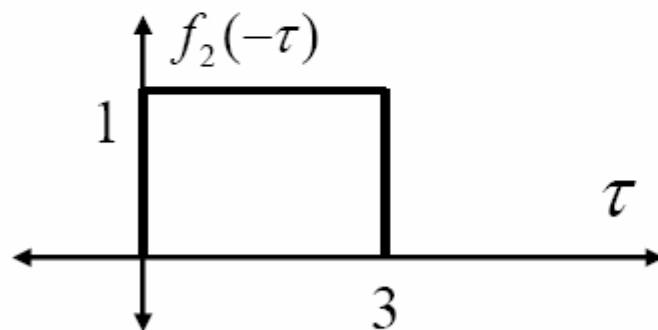
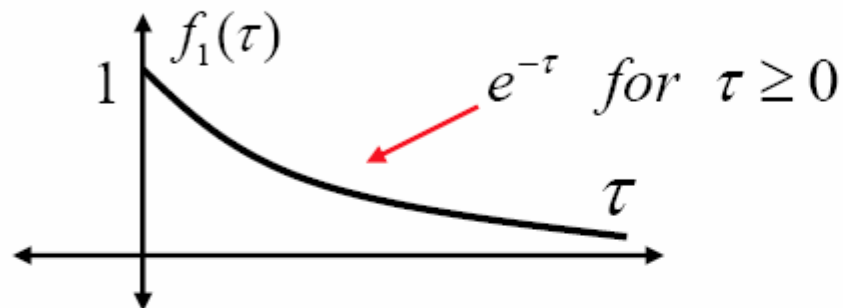


Example 4:

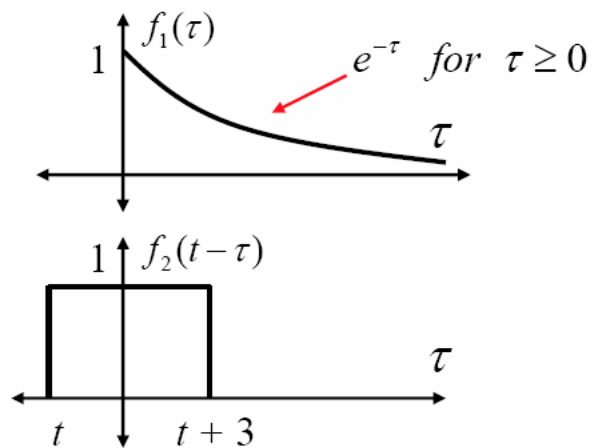
Given: $f_1(t) = e^{-t} * u(t)$

$$f_2(t) = u(t+3) - u(t)$$

Find: $c(t) = f_1(t) * f_2(t)$



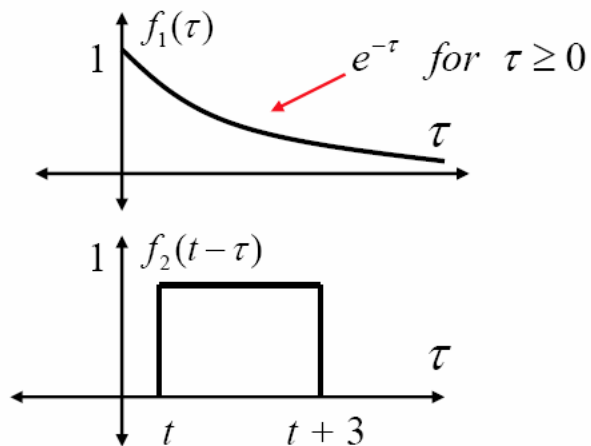
RI: $t+3 < 0 \Rightarrow t < -3$ prod = 0 $\Rightarrow c(t) = 0 \quad t < -3$



RII: $t+3 \geq 0$ $\left\{ \begin{array}{l} -3 \leq t < 0 \\ t < 0 \end{array} \right.$ product:

$\Rightarrow c(t) = \int_0^{t+3} e^{-\tau} d\tau = -[e^{-\tau}]_0^{t+3}$

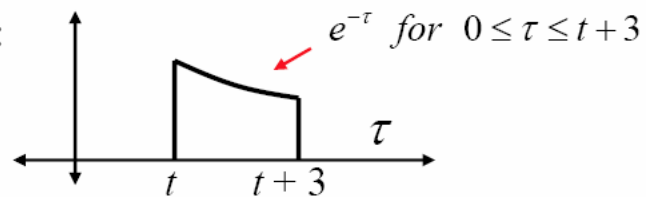
$= -[e^{-(t+3)} - 1] = 1 - e^{-(t+3)}$ for $-3 \leq t < 0$



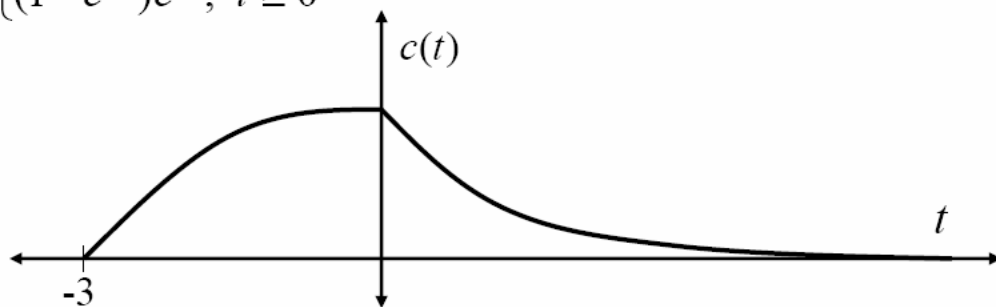
RIII: $t \geq 0$

product:

$$\begin{aligned} \Rightarrow c(t) &= \int_t^{t+3} e^{-\tau} d\tau = -\left[e^{-\tau}\right]_t^{t+3} = e^{-t} - e^{-(t+3)} \\ &= e^{-t} - e^{-3}e^{-t} = \boxed{(1 - e^{-3})e^{-t} \text{ for } t \geq 0} \end{aligned}$$



$$c(t) = \begin{cases} 0, & t < -3 \\ 1 - e^{-(t+3)}, & -3 \leq t < 0 \\ (1 - e^{-3})e^{-t}, & t \geq 0 \end{cases}$$



Graphical Convolution Steps

Can do convolution this way when signals are known numerically or by equation

- Convolution involves the sum of a product of two signals: $x[i]h[n - i]$

- At each output index n , the product changes

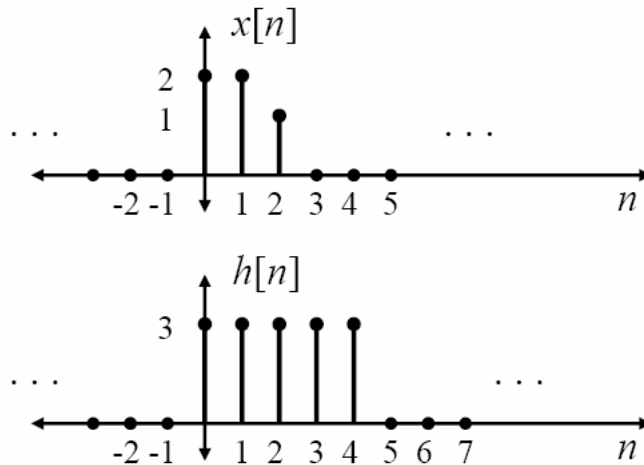
“Commutativity” says we can flip either $x[i]$ or $h[i]$ and get the same answer

Step 1: Write both as functions of i : $x[i]$ & $h[i]$

Step 2: Flip $h[i]$ to get $h[-i]$ (The book calls this “fold”)

Repeat for each n { **Step 3**: For each output index n value of interest, shift by n to get $h[n - i]$
(Note: positive n gives right shift!!!!)
Step 4: Form product $x[i]h[n - i]$ and sum its elements to get the number $y[n]$

Example of Graphical Convolution



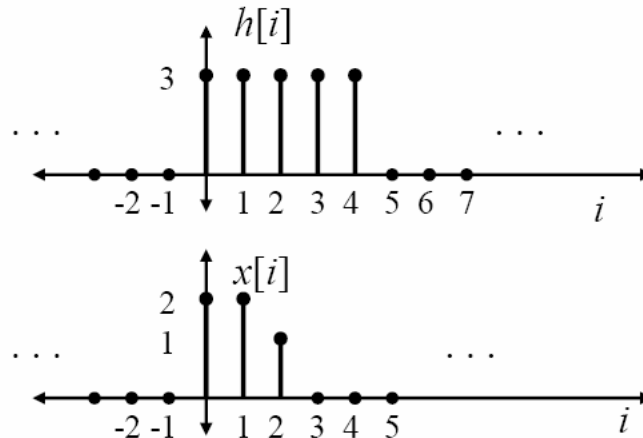
Find $y[n]=x[n]*h[n]$
for all
integer values of n

Solution

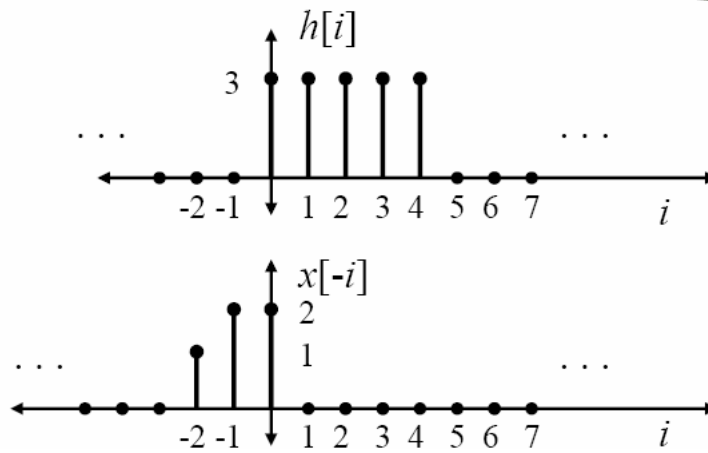
For this problem I choose to flip $x[n]$

My personal preference is to flip the shorter signal although I sometimes don't follow that "rule"... only through lots of practice can you learn how to best choose which one to flip.

Step 1: Write both as functions of i : $x[i]$ & $h[i]$



Step 2: Flip $x[i]$ to get $x[-i]$



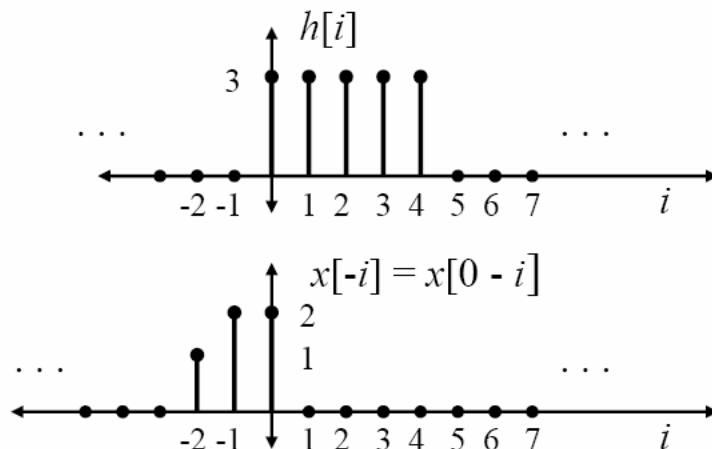
“Commutativity” says we can flip either $x[i]$ or $h[i]$ and get the same answer...
Here I flipped $x[i]$

We want a solution for $n = \dots -2, -1, 0, 1, 2, \dots$ so must do Steps 3&4 for all n .

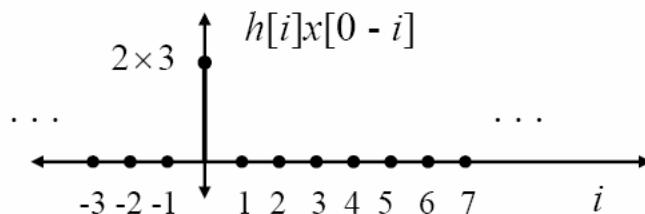
But... let's first do: **Steps 3&4 for $n = 0$** and then proceed from there.

Step 3: For $n = 0$, shift by n to get $x[n - i]$

For $n = 0$ case there is no shift!
 $x[0 - i] = x[-i]$



Step 4: For $n = 0$, Form the product $x[i]h[n - i]$ and sum its elements to give $y[n]$



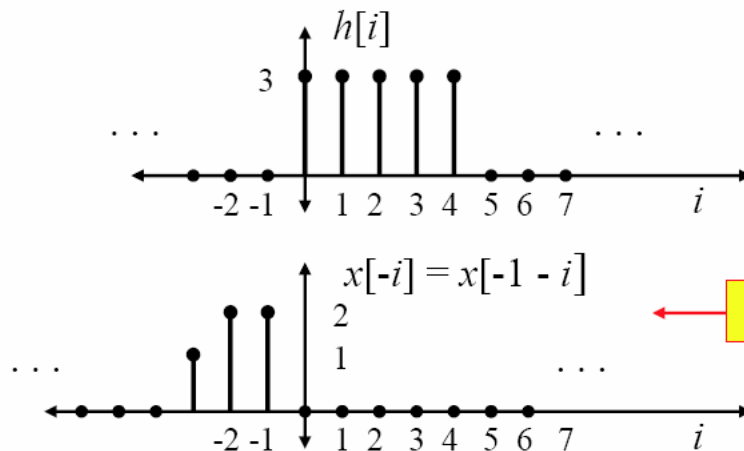
Sum over $i \Rightarrow$

$$y[0] = 6$$

Steps 3&4 for all $n < 0$

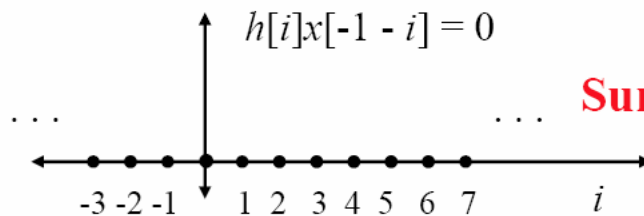
Step 3: For $n < 0$, shift by n to get $x[n - i]$

Negative n gives a left-shift



Shown here for $n = -1$

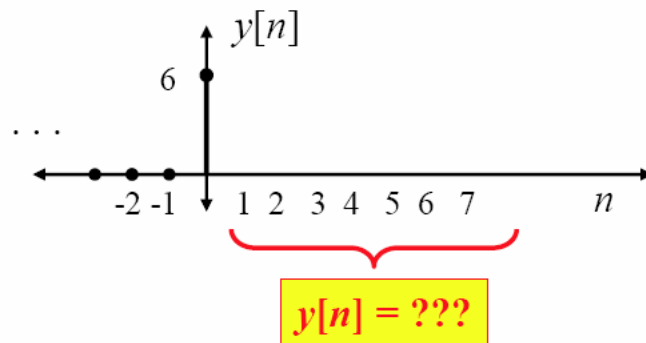
Step 4: For $n < 0$, Form the product $x[i]h[n - i]$ and sum its elements to give $y[n]$



Sum over $i \Rightarrow y[n] = 0 \quad \forall n < 0$

So... what we know so far is that:

$$y[n] = \begin{cases} 0, & \forall n < 0 \\ 6, & n = 0 \end{cases}$$

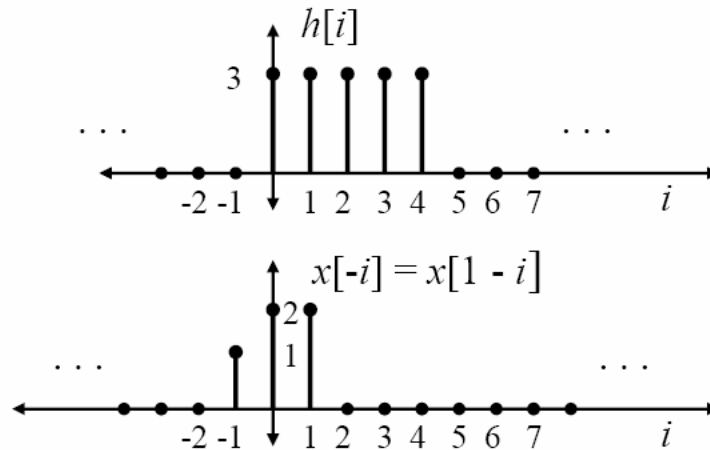


So now we have to do Steps 3&4 for $n > 0$...

Steps 3&4 for $n = 1$

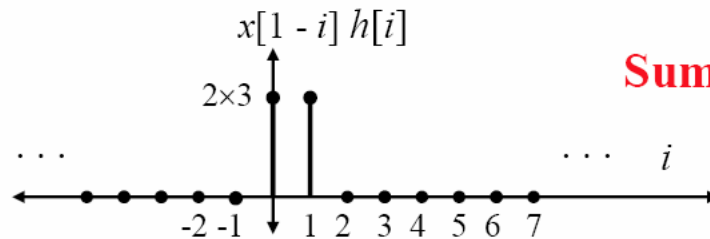
Step 3: For $n = 1$, shift by n to get $x[n - i]$

Positive n gives a Right-shift



shifted to the
right by one

Step 4: For $n = 1$, Form the product $x[i]h[n - i]$ and sum its elements to give $y[n]$

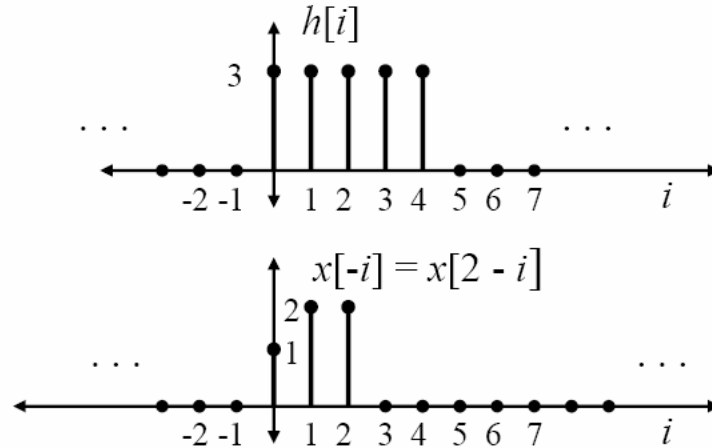


Sum over $i \Rightarrow y[1] = 6 + 6 = 12$

Steps 3&4 for $n = 2$

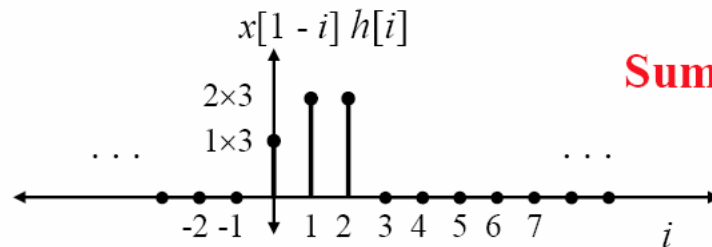
Step 3: For $n = 2$, shift by n to get $x[n - i]$

Positive n gives a Right-shift



shifted to the right by two

Step 4: For $n = 2$, Form the product $x[i]h[n - i]$ and sum its elements to give $y[n]$



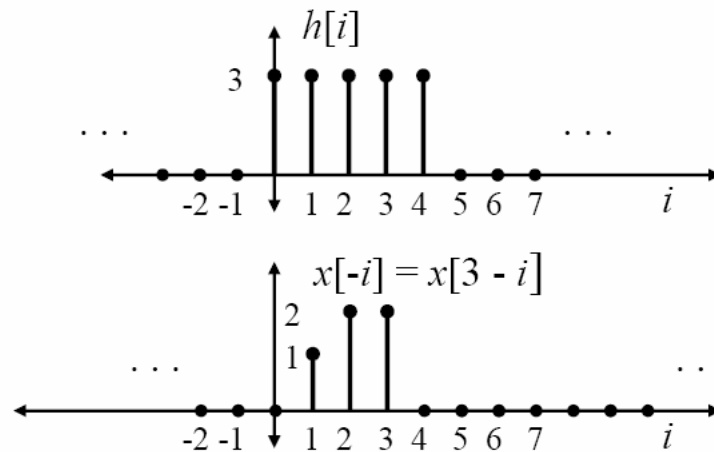
Sum over $i \Rightarrow$

$$y[2] = 3 + 6 + 6 = 15$$

Steps 3&4 for $n = 3$

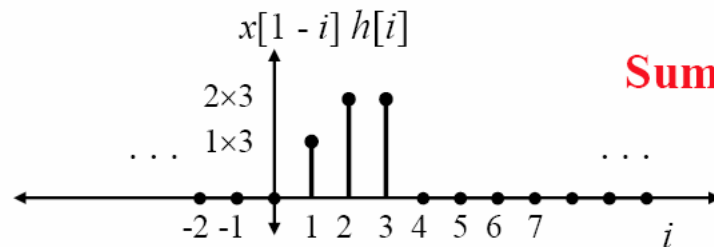
Step 3: For $n = 3$, shift by n to get $x[n - i]$

Positive n gives a Right-shift



shifted to the
right by three

Step 4: For $n = 3$, Form the product $x[i]h[n - i]$ and sum its elements to give $y[n]$

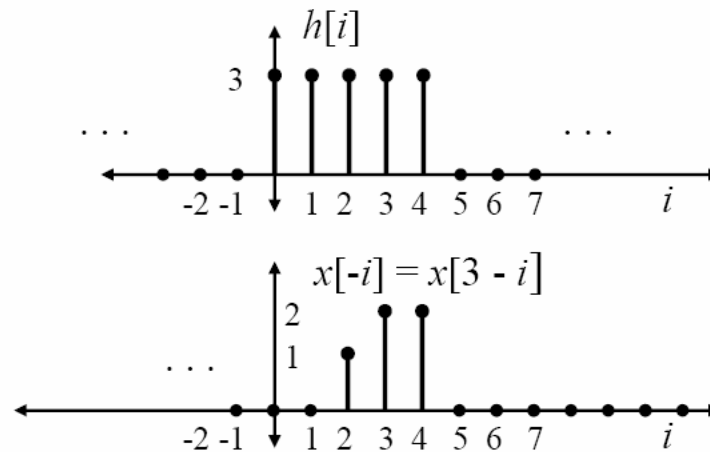


Sum over $i \Rightarrow y[3] = 3 + 6 + 6 = 15$

Steps 3&4 for $n = 4$

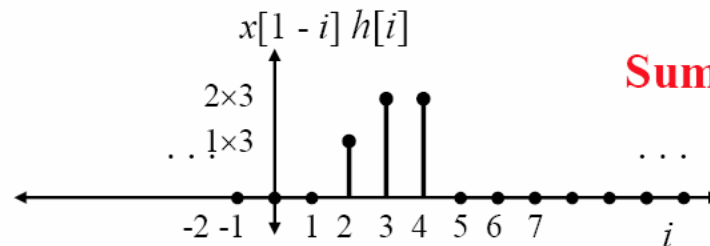
Step 3: For $n = 4$, shift by n to get $x[n - i]$

Positive n gives a Right-shift



shifted to the right by four

Step 4: For $n = 4$, Form the product $x[i]h[n - i]$ and sum its elements to give $y[n]$



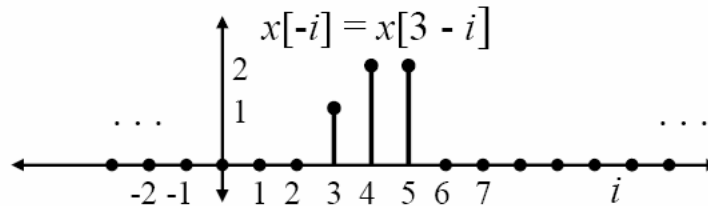
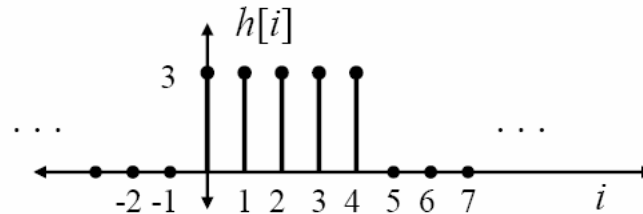
Sum over $i \Rightarrow$

$$y[4] = 3 + 6 + 6 = 15$$

Steps 3&4 for $n = 5$

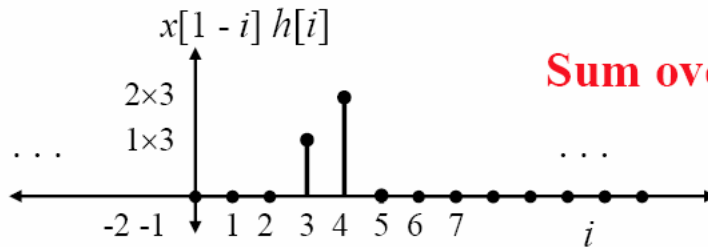
Step 3: For $n = 5$, shift by n to get $x[n - i]$

Positive n gives a Right-shift



shifted to the
right by five

Step 4: For $n = 5$, Form the product $x[i]h[n - i]$ and sum its elements to give $y[n]$



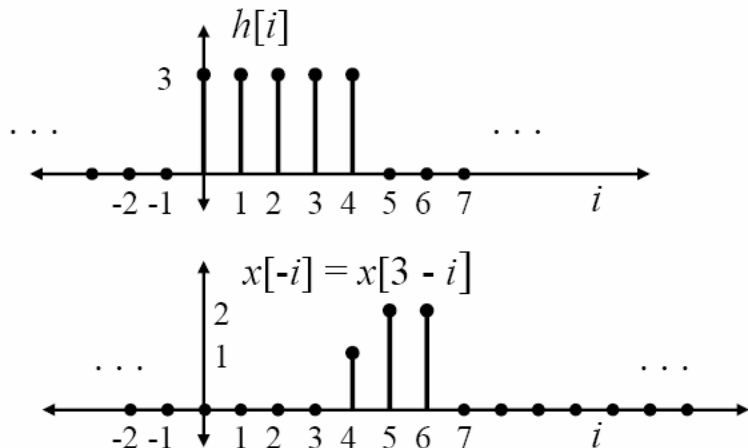
Sum over $i \Rightarrow$

$$y[5] = 3 + 6 = 9$$

Steps 3&4 for $n = 6$

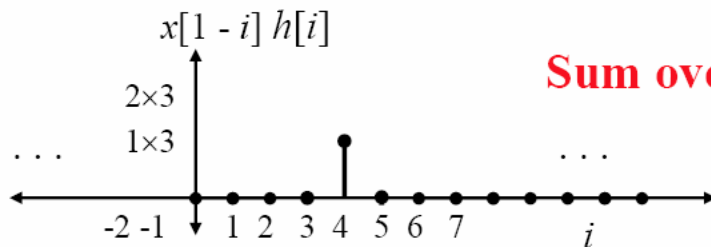
Step 3: For $n = 6$, shift by n to get $x[n - i]$

Positive n gives a Right-shift



shifted to the
right by six

Step 4: For $n = 6$, Form the product $x[i]h[n - i]$ and sum its elements to give $y[n]$

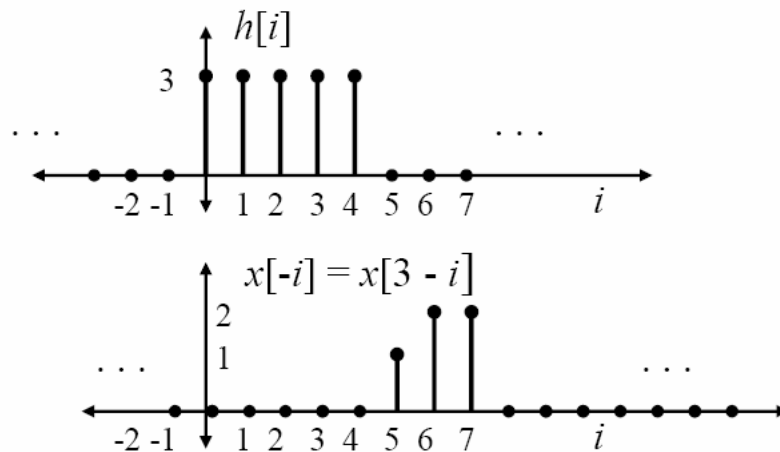


Sum over $i \Rightarrow y[6] = 3$

Steps 3&4 for all $n > 6$

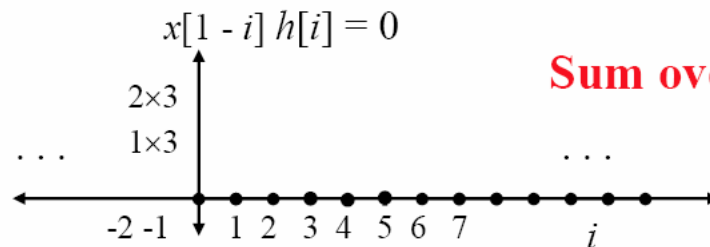
Step 3: For $n > 6$, shift by n to get $x[n - i]$

Positive n gives a Right-shift



shifted to the
right by seven

Step 4: For $n > 6$, Form the product $x[i]h[n - i]$ and sum its elements to give $y[n]$



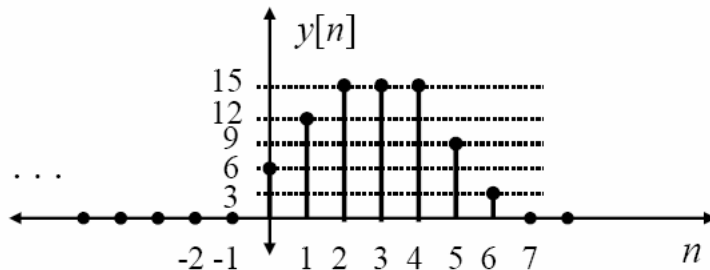
Sum over $i \Rightarrow$

$$y[n] = 0 \quad \forall n > 6$$

So... now we know the values of $y[n]$ for all values of n

We just need to put it all together as a function...

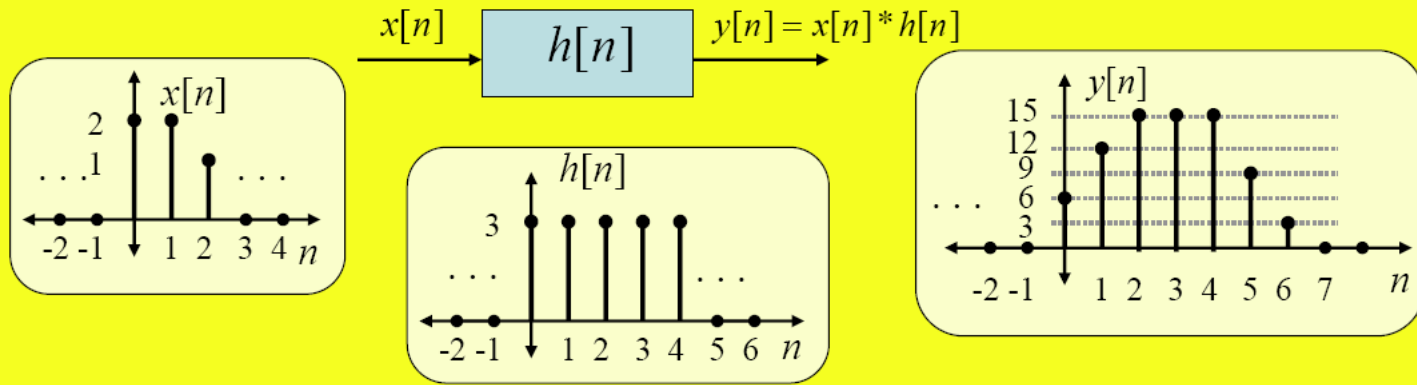
Here it is easiest to just plot it... you could also list it as a table.



Note that convolving these kinds of signals gives a “ramp-up” at the beginning and a “ramp-down” at the end.

Various kinds of “transients” at the beginning and end of a convolution are common.

BIG PICTURE: So... what we have just done is found the zero-state output of a system having an impulse response given by this $h[n]$ when the input is given by this $x[n]$:



[Link:](#) Web Demos of Graphical D-T Convolution

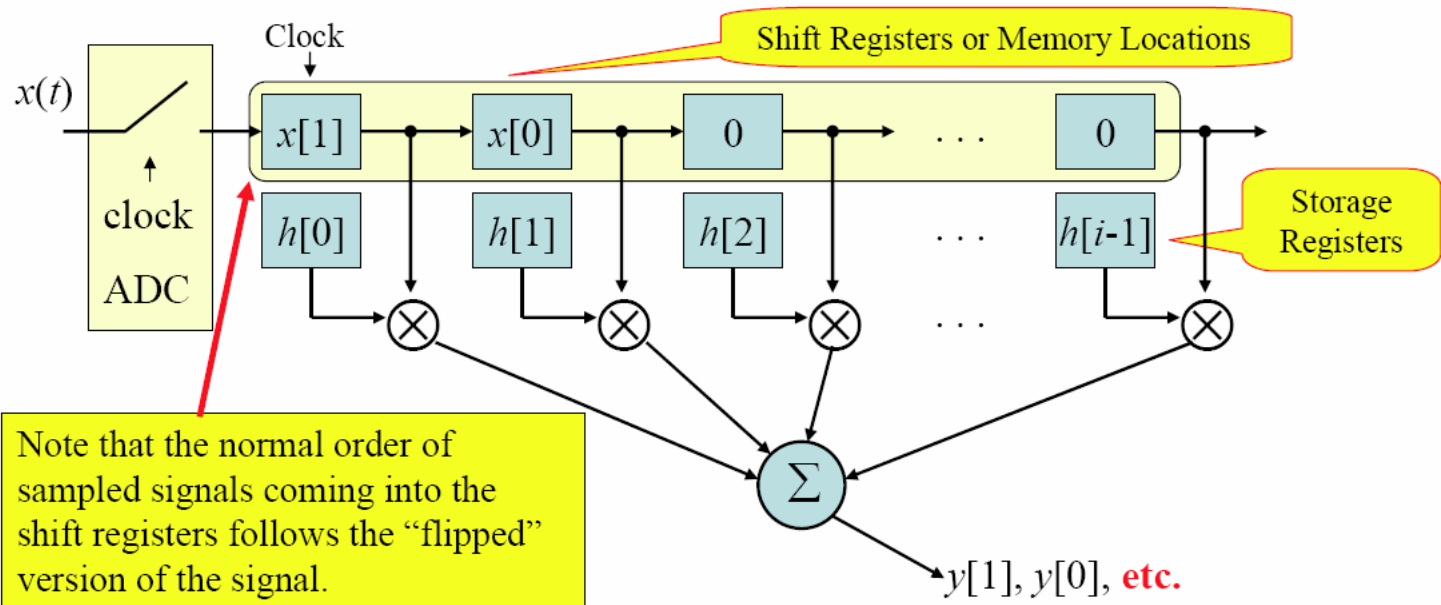
This is a good site that provides insight into how to visualize D-T convolution...

However, be sure you can do graphical convolution by hand without the aid of this site!!

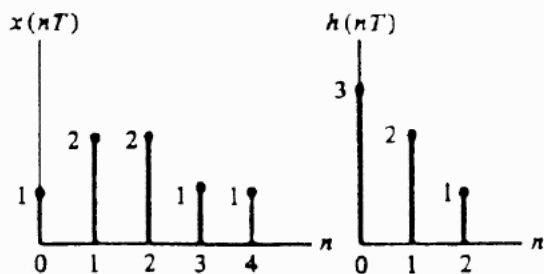
Implementation Issues

Consider a D-T system with impulse response $h[n]$ that has finite duration...

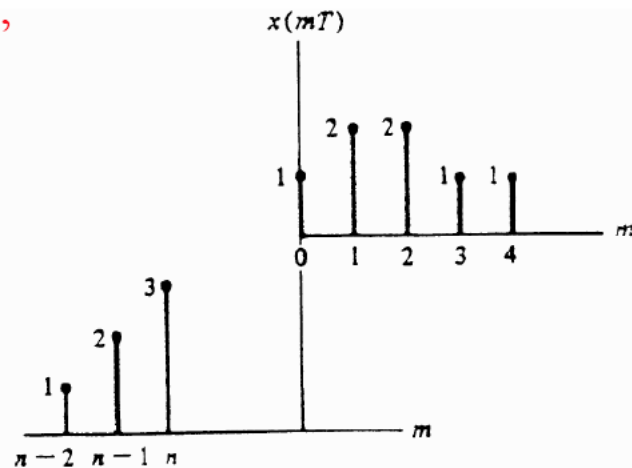
Could Build a digital hardware system or a software program for D-T convolution like this:



Convolution Example “Table view”



(a) Input and unit pulse response



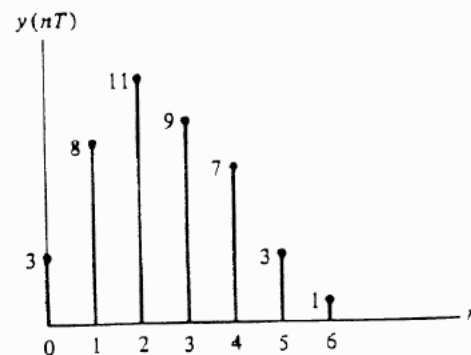
(b) Functions for computing convolution sum

Samples $x(mT)$

	0	0	1	2	2	1	1	0
$n=0$	1	2	3	0	0	0	0	0
$n=1$	0	1	2	3	0	0	0	0
$n=2$	0	0	1	2	3	0	0	0
$n=3$	0	0	0	1	2	3	0	0
$n=4$	0	0	0	0	1	2	3	0
$n=5$	0	0	0	0	0	1	2	3
$n=6$	0	0	0	0	0	0	1	2

Red arrows point to the columns labeled $h(-m)$ and $h(1-m)$.

(c) Table for evaluating summation



(d) Output

Discrete-Time Convolution Example: “Sliding Tape View”

	k=-6	-5	-4	-3	-2	-1	0	1	2	3	4	5	6	7	8	9	
$h(k)$	0	0	0	0	0	0	3	2	1	0	0	0	0	0	0	0	Rewrite signals in k
$x(k)$	0	0	0	0	0	0	1	2	2	1	1	0	0	0	0	0	
$x(-k)$	0	0	1	1	2	2	1	0	0	0	0	0	0	0	0	0	Flip $x(k)$

n=-1	$x(-1-k)$	0	1	1	2	2	1	0	0	0	0	0	0	0	0	0	0	← Shifted $x(-k)$ by -1
	$h(k)$	0	0	0	0	0	0	3	2	1	0	0	0	0	0	0	0	
	$h(k)x(-1-k)$	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	Sum → $y(-1) = 0$

n=0	$x(-k)$	0	0	1	1	2	2	1	0	0	0	0	0	0	0	0	0	← Shifted $x(-k)$ by 0
	$h(k)$	0	0	0	0	0	0	3	2	1	0	0	0	0	0	0	0	
	$h(k)x(-k)$	0	0	0	0	0	0	3	0	0	0	0	0	0	0	0	0	Sum → $y(0) = 3$

n=1	$x(1-k)$	0	0	0	1	1	2	2	1	0	0	0	0	0	0	0	0	← Shifted $x(-k)$ by 1
	$h(k)$	0	0	0	0	0	0	3	2	1	0	0	0	0	0	0	0	
	$h(k)x(1-k)$	0	0	0	0	0	0	6	2	0	0	0	0	0	0	0	0	Sum → $y(1) = 8$

n=2	$x(2-k)$	0	0	0	0	1	1	2	2	1	0	0	0	0	0	0	0	← Shifted $x(-k)$ by 2
	$h(k)$	0	0	0	0	0	0	3	2	1	0	0	0	0	0	0	0	
	$h(k)x(2-k)$	0	0	0	0	0	0	6	4	1	0	0	0	0	0	0	0	Sum → $y(2) = 11$

n=3	$x(3-k)$	0	0	0	0	0	1	1	2	2	1	0	0	0	0	0	0	← Shifted $x(-k)$ by 3
	$h(k)$	0	0	0	0	0	0	3	2	1	0	0	0	0	0	0	0	
	$h(k)x(3-k)$	0	0	0	0	0	0	3	4	2	0	0	0	0	0	0	0	Sum → $y(3) = 9$

n=4	$x(4-k)$	0	0	0	0	0	0	1	1	2	2	1	0	0	0	0	0	← Shifted $x(-k)$ by 4
	$h(k)$	0	0	0	0	0	0	3	2	1	0	0	0	0	0	0	0	
	$h(k)x(4-k)$	0	0	0	0	0	0	3	2	2	0	0	0	0	0	0	0	Sum → $y(4) = 7$

n=5	$x(5-k)$	0	0	0	0	0	0	0	1	1	2	2	1	0	0	0	0	← Shifted $x(-k)$ by 5
	$h(k)$	0	0	0	0	0	0	0	3	2	1	0	0	0	0	0	0	
	$h(k)x(5-k)$	0	0	0	0	0	0	0	2	1	0	0	0	0	0	0	0	Sum → $y(5) = 3$

n=6	$x(6-k)$	0	0	0	0	0	0	0	0	1	1	2	2	1	0	0	0	← Shifted $x(-k)$ by 6
	$h(k)$	0	0	0	0	0	0	0	3	2	1	0	0	0	0	0	0	
	$h(k)x(6-k)$	0	0	0	0	0	0	0	0	1	0	0	0	0	0	0	0	Sum → $y(6) = 1$

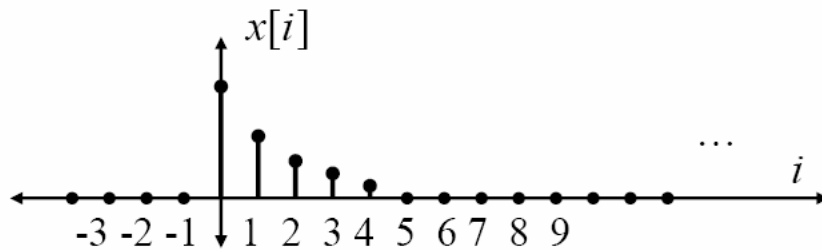
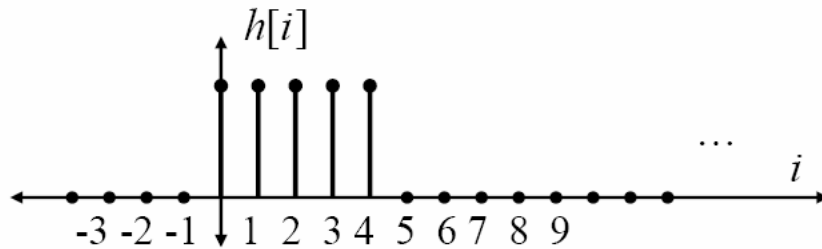
n=7	$x(7-k)$	0	0	0	0	0	0	0	0	0	1	1	2	2	1	0	0	← Shifted $x(-k)$ by 7
	$h(k)$	0	0	0	0	0	0	0	3	2	1	0	0	0	0	0	0	
	$h(k)x(7-k)$	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	Sum → $y(7) = 0$

n=8	$x(8-k)$	0	0	0	0	0	0	0	0	0	0	1	1	2	2	1	0	← Shifted $x(-k)$ by 8
	$h(k)$	0	0	0	0	0	0	0	3	2	1	0	0	0	0	0	0	
	$h(k)x(8-k)$	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	Sum → $y(8) = 0$

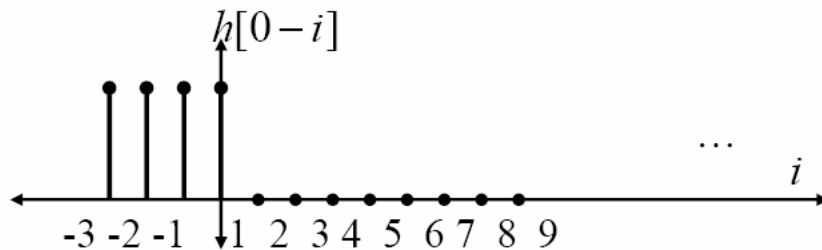
D-T Convolution Examples

$$x[n] = \left(\frac{1}{2}\right)^n u[n]$$

$$h[n] = u[n] - u[n-4]$$



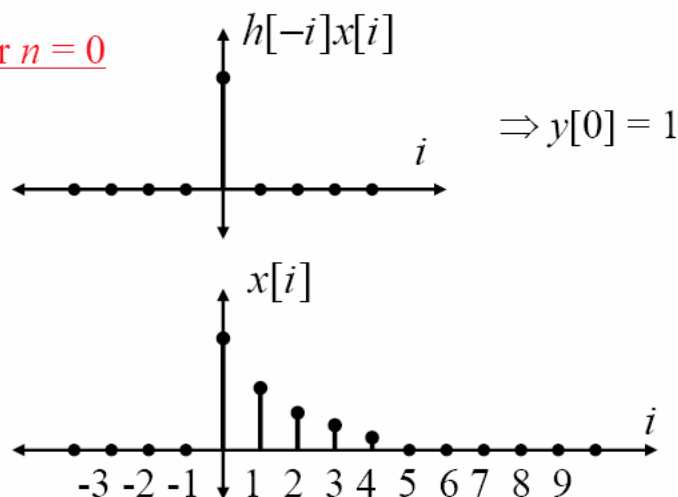
Choose to flip and slide $h[n]$ This shows $h[n-i]$ for $n = 0$



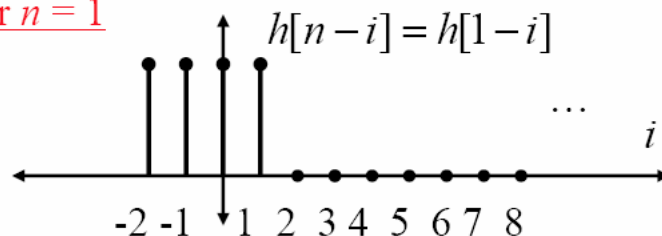
For $n < 0$ $h[n-i]x(i) = 0 \quad \forall i$

$$\Rightarrow y[n] = 0 \quad \text{for } n < 0$$

For $n = 0$



For $n = 1$



$$\Rightarrow y(1) = 1 + 1/2 = 3/2$$

Notice that for $n = 0, n = 1, \dots, n = 3$

The general result is:

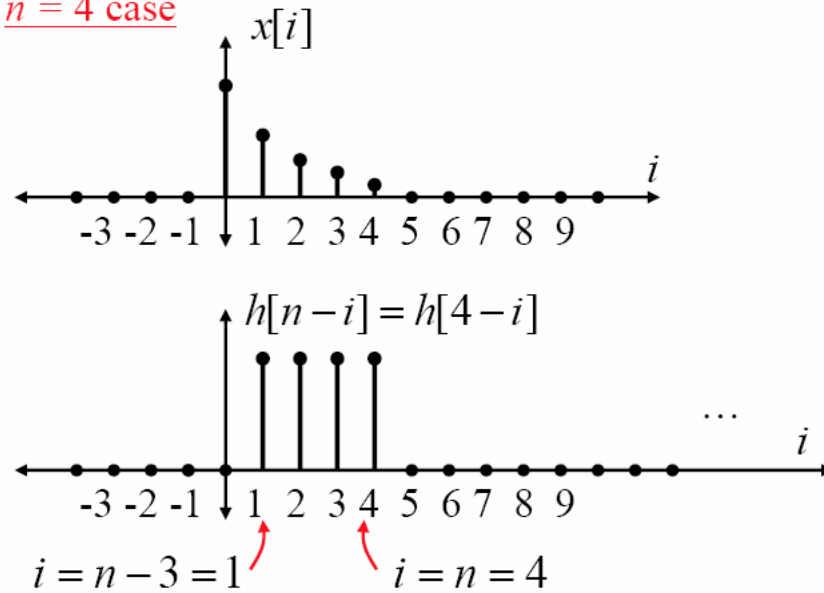
$$y[n] = \sum_{i=0}^n \left(\frac{1}{2}\right)^i \quad \text{for } n = 0, 1, 2, 3$$

$$= \frac{1 - \left(\frac{1}{2}\right)^{n+1}}{1 - \frac{1}{2}} \quad (\text{Geometric Sum})$$

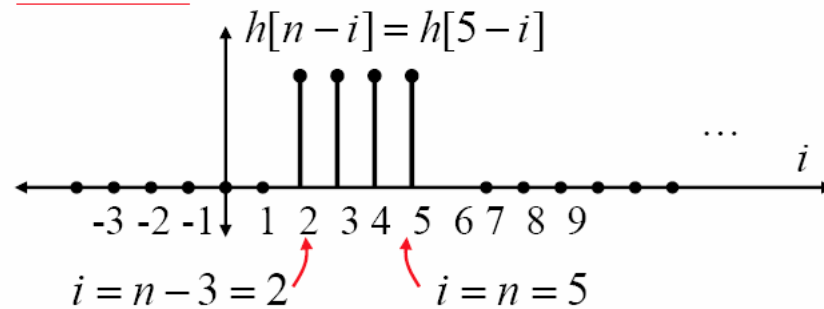
$$y[n] = 2 \left[1 - \left(\frac{1}{2}\right)^{n+1} \right] \quad \text{for } n = 0, 1, 2, 3$$

Now for $n = 4, n = 5, \dots$

$n = 4$ case



$n = 5$ case

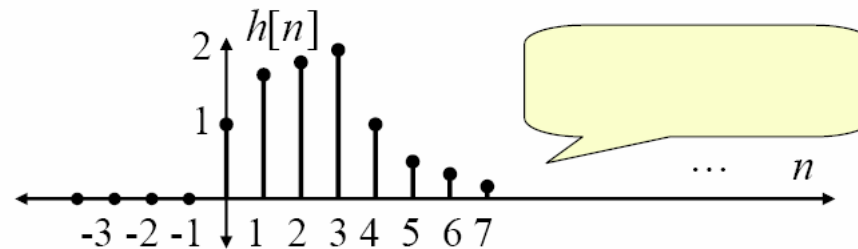


Notice that: for $n = 4, 5, 6, \dots$

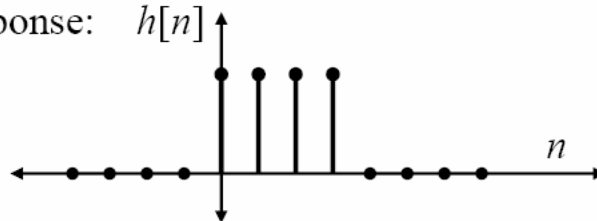
$$y[n] = \sum_{i=n-3}^n \left(\frac{1}{2}\right)^i \quad \text{for } n = 4, 5, 6, \dots$$
$$= \frac{\left(\frac{1}{2}\right)^{n-3} - \left(\frac{1}{2}\right)^{n+1}}{1 - \frac{1}{2}} \quad \text{then simplify!}$$

Then we can write out the solution as:

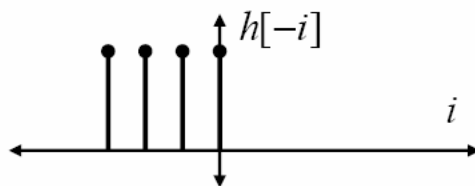
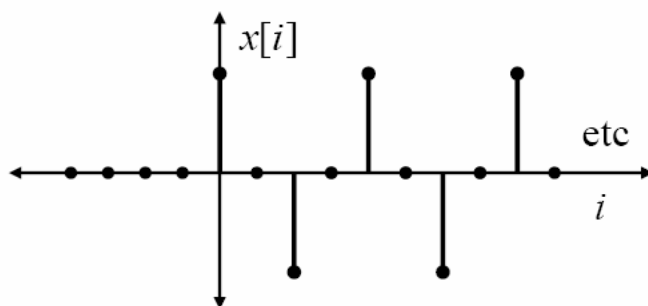
$$y[n] = \begin{cases} 0, & n < 0 \\ 2\left[1 - \left(\frac{1}{2}\right)^{n+1}\right], & n = 0, 1, 2, 3 \\ 2\left[\left(\frac{1}{2}\right)^{n-3} - \left(\frac{1}{2}\right)^{n+1}\right], & n = 4, 5, 6, \dots \end{cases}$$



2. Same Impulse Response: $h[n]$



$$x[n] = \cos\left(\frac{\pi}{2}n\right)\mu[n]$$



Again $y[n] = 0$ for $n < 0$:

$$y[0] = 1$$

$$y[1] = 1 + 0 = 1$$

$$y[2] = 1 + 0 - 1 = 0$$

$$y[3] = 1 + 0 - 1 + 0 = 0$$

$$y[4] = 0 - 1 + 0 + 1 = 0$$

$$y[5] = -1 + 0 + 1 + 0 = 0$$

Notice: $y[n] = 0 \quad \forall n = 2, 3, 4, 5, \dots!$

