

Chapter V

Discrete-Time Frequency Domain Methods

Fourier Series Representation of DT Periodic Signals

Let $x[n] = z^n$ be the input to DT system with $h[n]$ (z is a complex number)

$$\rightarrow y[n] = \sum_k h[k] z^{n-k} = z^n \sum_k h[k] z^{-k} = z^n H(z),$$

$$\text{where } H(z) = \sum_k h[k] z^{-k}$$

→ Response of LTI, DT system to z^n is of the same type

→ z^n : *eigenfunction* of LTI DT system

→ $H(z)$: *eigenvalue* of LTI DT system

If the input can be represented as $x[n] = \sum_k a_k z_k^n$

then the output is

$$y[n] = \sum_k a_k H(z_k) z_k^n$$

Fourier Series Representation of DT Periodic Signals

Recall:

$$\phi_k[n] = e^{jk\Omega_o n} = e^{jk(2\pi/N)n}, \quad k = 0, \pm 1, \pm 2, \dots$$

are periodic with N and have fundamental frequencies that are multiples of $2\pi/N$ (*harmonically related*).

→ *There are only N distinct signals* in $\phi_k[n]$

$$\phi_k[n] = \phi_{k+rN}[n]$$

as DT comp. exp. which differ in freq. by a multiple of $2\pi/N$ are identical.

Fourier Series Representation of DT Periodic Signals

$$\text{For } z_k^n = \phi_k[n] \Rightarrow x[n] = \sum_k a_k \phi_k[n] = \sum_k a_k e^{jk \left(\frac{2\pi}{N} \right) n}$$

As $\phi_k[n]$ is distinct only over N successive values of k:

$$x[n] = \sum_{k \in N} a_k \phi_k[n] = \sum_{k \in N} a_k e^{jk \left(\frac{2\pi}{N} \right) n} \quad \text{:DT Fourier Series}$$

k could take values $k=0, 1, \dots, N-1$ or $k=3, 4, \dots, N+2$

Fourier Series Representation of DT Periodic Signals

Assume $x[n]$ is periodic with N , how to find a_k ?

Similar to CT case, we use the relation:

$$\sum_{n=\langle N \rangle} e^{jk\left(\frac{2\pi}{N}\right)n} = \begin{cases} N, & k = 0, \pm N, \pm 2N, \dots \\ 0, & \text{o/w} \end{cases}$$

(Sum over one period of the values of a periodic complex exponential is zero unless the comp. exp. is a constant)

Fourier Series Representation of DT Periodic Signals

Multiply both sides of DTFS by $e^{-jr\left(\frac{2\pi}{N}\right)n}$ and sum over N terms:

$$\sum_{n=\langle N \rangle} x[n] e^{-jr\left(\frac{2\pi}{N}\right)n} = \sum_{n=\langle N \rangle} \sum_{k=\langle N \rangle} a_k e^{j(k-r)\left(\frac{2\pi}{N}\right)n} = \sum_{k=\langle N \rangle} a_k \underbrace{\sum_{n=\langle N \rangle} e^{j(k-r)\left(\frac{2\pi}{N}\right)n}}$$

$$\begin{cases} N, & k=r \text{ (or } (k-r)=mN \text{)} \\ 0, & k \neq r \end{cases}$$

$$\Rightarrow \sum_{n=\langle N \rangle} x[n] e^{-jr\left(\frac{2\pi}{N}\right)n} = a_r N$$

$$\Rightarrow a_r = \frac{1}{N} \sum_{n=\langle N \rangle} x[n] e^{-jr\left(\frac{2\pi}{N}\right)n}$$

m is an integer, and here we choose values for r over the same range as that over which k varies ($k=\langle N \rangle$ in the outer summation), Therefore, for $k=r$, the innermost sum is N , and for $k \neq r$ it is 0 .

Fourier Series Representation of DT Periodic Signals

$\Rightarrow$

$$x[n] = \sum_{k \in \langle N \rangle} a_k e^{jk \left(\frac{2\pi}{N} \right) n}$$
$$a_k = \frac{1}{N} \sum_{n \in \langle N \rangle} x[n] e^{-jk \left(\frac{2\pi}{N} \right) n}$$

Synthesis Equation

DTFS Pairs

Analysis Equation

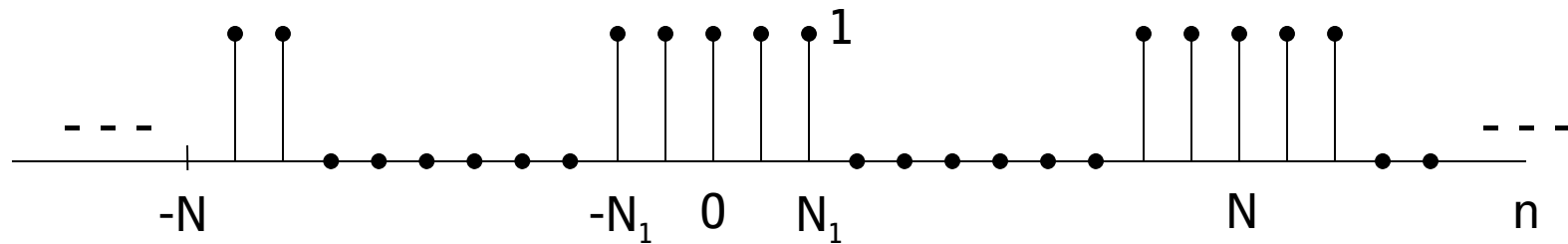
Spectral Coefficients

- *No convergence issue!*
- Note that $a_k = a_{k+N}$: *DTFS coefficients are PERIODIC !*

Fourier Series Representation of DT Periodic Signals

Example:

Consider DT periodic square-wave with period N .



$$a_k = \frac{1}{N} \sum_{-N_1}^{N_1} e^{-jk\left(\frac{2\pi}{N}\right)n}$$

For $m=n+N_1 \rightarrow$

$$a_k = \frac{1}{N} \sum_{m=0}^{2N_1} e^{-jk\left(\frac{2\pi}{N}\right)(m-N_1)}$$

Fourier Series Representation of DT Periodic Signals

$$= \frac{1}{N} e^{jk(\frac{2\pi}{N})N_1} \sum_{m=0}^{2N_1} e^{-jk(\frac{2\pi}{N})m}$$

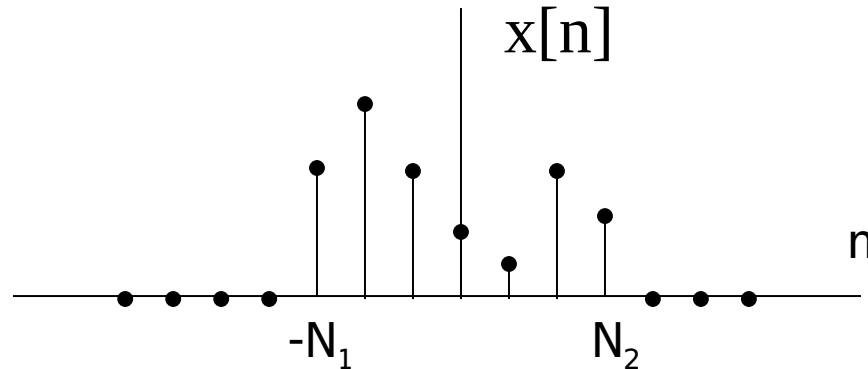
$$= \frac{1}{N} e^{jk(\frac{2\pi}{N})N_1} \left(\frac{1 - e^{-jk2\pi(2N_1+1)/N}}{1 - e^{-jk(2\pi/N)}} \right)$$

$$a_k = \frac{1}{N} \frac{\sin[2\pi k(N_1 + \frac{1}{2})/N]}{\sin(\pi k/N)} \quad \text{for } k \neq 0, \mp N, \mp 2N, \dots$$

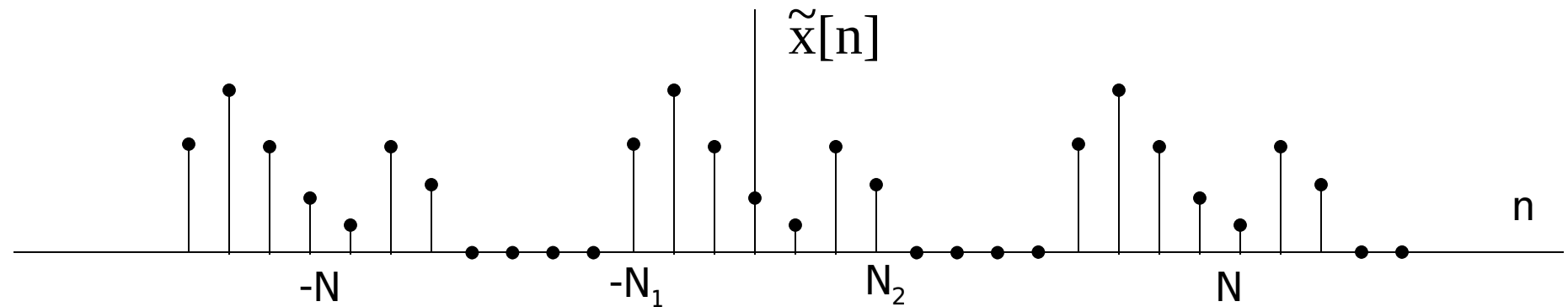
$$\text{and } a_k = \frac{2N_1 + 1}{N} \quad \text{for } k = 0, \mp N, \mp 2N, \dots$$

Discrete Time Fourier Transform

Consider an aperiodic finite duration sequence $x[n]$



Construct a periodic sequence $\tilde{x}[n]$ for which $x[n]$ is one period

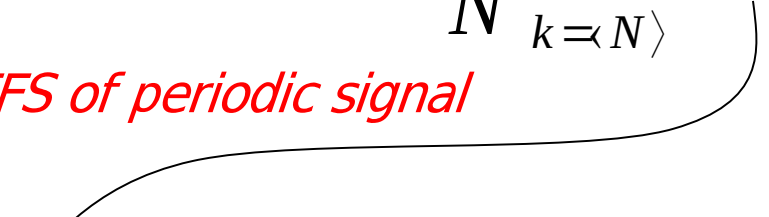


such that as $N \rightarrow \infty$, $\tilde{x}[n] = x[n]$ for any finite n .

Discrete Time Fourier Transform

$$\tilde{x}[n] = \sum_{k=-N}^N a_k e^{jk(\frac{2\pi}{N})n} \quad ; \quad a_k = \frac{1}{N} \sum_{n=-N}^N \tilde{x}[n] e^{-jk(\frac{2\pi}{N})n}$$

DTFS of periodic signal



$$\Rightarrow a_k = \frac{1}{N} \sum_{n=-N_1}^{N_2} x[n] e^{-jk(\frac{2\pi}{N})n} = \frac{1}{N} \sum_{n=-\infty}^{\infty} x[n] e^{-jk(\frac{2\pi}{N})n}$$

Define $X(\Omega) = \sum_{n=-\infty}^{\infty} x[n] e^{-j\Omega n}$

$$\Rightarrow a_k = \frac{1}{N} X(k\Omega_0) \text{ for } \Omega_0 = \frac{2\pi}{N}$$

Discrete Time Fourier Transform

$$\text{as } \frac{1}{N} = \frac{\Omega_0}{2\pi} \rightarrow \tilde{x}[n] = \sum_{k=-N}^N \frac{1}{N} X(k\Omega_0) e^{jk\Omega_0 n}$$

$$= \frac{1}{2\pi} \sum_{k=-N}^N X(k\Omega_0) e^{jk\Omega_0 n} \Omega_0$$

$$\text{As } N \rightarrow \infty, \Omega_0 \rightarrow 0, \sum \rightarrow \int, \tilde{x}[n] \rightarrow x[n]$$

Discrete Time Fourier Transform

$$x[n] = \frac{1}{2\pi} \int_{2\pi} X(\Omega) e^{j\Omega n} d\Omega$$

$$X(\Omega) = \sum_{n=-\infty}^{\infty} x[n] e^{-j\Omega n}$$

DTFT pairs

$X(\Omega)$ is periodic with 2π

how an aperiodic sequence can be thought of as a linear combination of complex exponentials!

$X(\Omega)$ at $\Omega=0$ and $\Omega=2\pi$ are same!

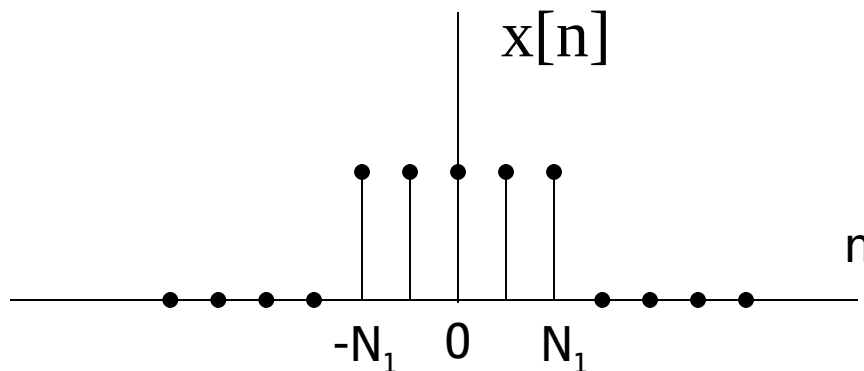
around $\Omega=0, 2\pi$ low frequency

around $\Omega=\pi$ high frequency

Discrete Time Fourier Transform

Example:

Consider the rectangular pulse $x[n] = \begin{cases} 1, & |n| \leq N_1 \\ 0, & |n| > N_1 \end{cases}$

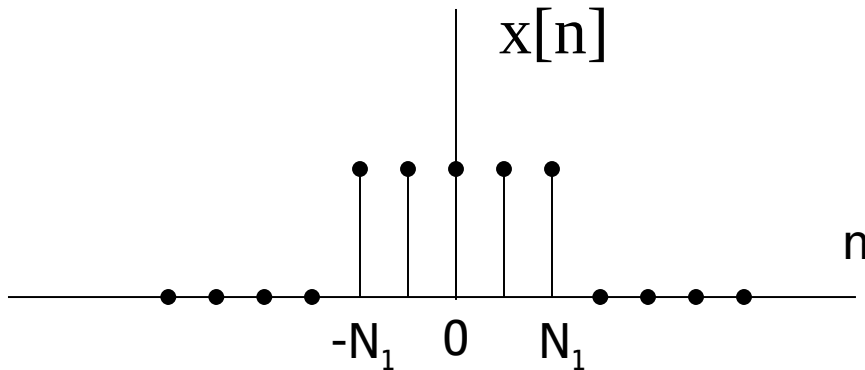


$$X(\Omega) = \sum_{n=-N_1}^{N_1} e^{-j\Omega n} = \frac{\sin \Omega(N_1 + \frac{1}{2})}{\sin(\Omega/2)}$$

Discrete Time Fourier Transform

Example:

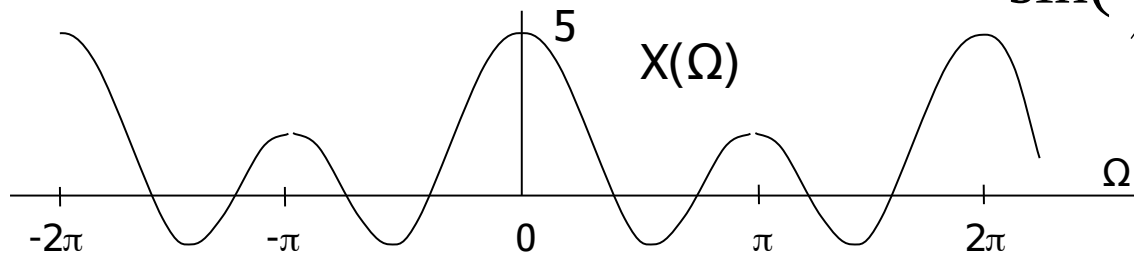
Consider the rectangular pulse $x[n] = \begin{cases} 1, & |n| \leq N_1 \\ 0, & |n| > N_1 \end{cases}$



$$X(\Omega) = \sum_{n=-N_1}^{N_1} e^{-j\Omega n}$$

$$= \frac{\sin \Omega(N_1 + \frac{1}{2})}{\sin(\Omega/2)}$$

For $N_1=2$



$X(\Omega)$ is periodic with 2π !

Convergence of DT Fourier Transform

DTFT exists if $\sum_{n=-\infty}^{\infty} |x[n]| < \infty$ or $\sum_{n=-\infty}^{\infty} |x[n]|^2 < \infty$

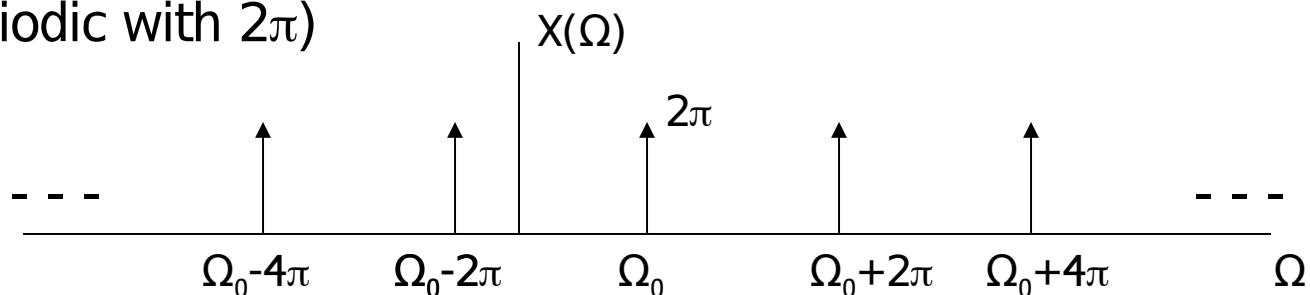
No convergence issue as the inverse integral is evaluated over finite interval.

DTFT for Periodic Signals :

Consider the signal $x[n] = e^{j\Omega_0 n} \rightarrow X(\Omega) = \sum_{l=-\infty}^{\infty} 2\pi\delta(\Omega - \Omega_0 - 2\pi l)$

Check this by evaluating the inverse DTFT

(DTFT is periodic with 2π)



Convergence of DT Fourier Transform

Consider $x[n] = \sum_{k=-N}^N a_k e^{jk\left(\frac{2\pi}{N}\right)n}$

$$\rightarrow X(\Omega) = \sum_{k=-\infty}^{\infty} 2\pi a_k \delta\left(\Omega - \frac{2\pi k}{N}\right) \quad : \text{DTFT of periodic signal}$$

(verify as an exercise!)

Example :

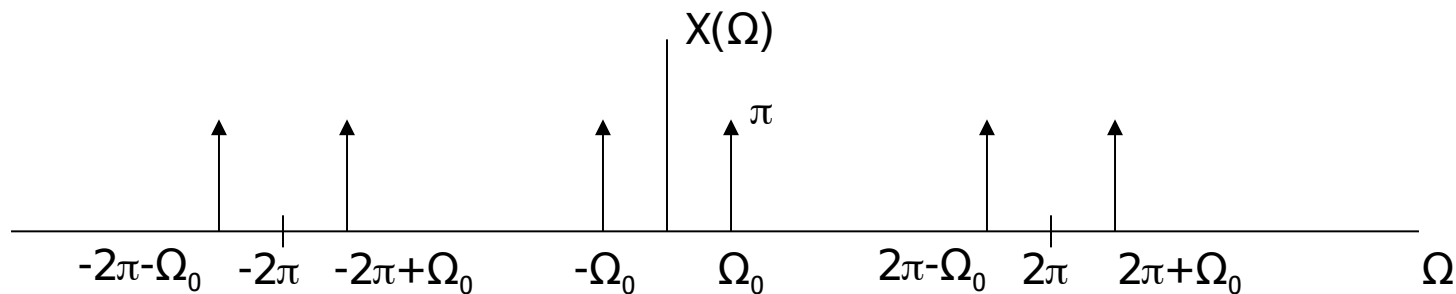
Consider $x[n] = \cos \Omega_0 n$ with $\Omega_0 = \frac{2\pi}{5}$

$$x[n] = \frac{1}{2} e^{j\Omega_0 n} + \frac{1}{2} e^{-j\Omega_0 n}$$

Convergence of DT Fourier Transform

$$\Rightarrow X(\Omega) = \sum_{k=-\infty}^{\infty} \pi \delta\left(\Omega - \frac{2\pi}{5} - 2\pi k\right) + \sum_{k=-\infty}^{\infty} \pi \delta\left(\Omega + \frac{2\pi}{5} - 2\pi k\right)$$

$$= \pi \delta\left(\Omega - \frac{2\pi}{5}\right) + \pi \delta\left(\Omega + \frac{2\pi}{5}\right) \quad \text{for } -\pi \leq \Omega < \pi$$



$X(\Omega)$ is periodic with 2π .

Properties of DT Fourier Series

Notation: $x[n] \xleftarrow{FS} a_k$

1. Multiplication: if $x[n] \xleftarrow{FS} a_k$ and $y[n] \xleftarrow{FS} b_k$, then

$$x[n]y[n] \xleftarrow{FS} d_k = \sum_{l \in N} a_l b_{k-l} \quad (\text{periodic convolution!})$$

Solve problem 3.46 from Oppenheim.

2. First Difference: if $x[n] \xleftarrow{FS} a_k$, then

$$x[n] - x[n-1] \xleftarrow{FS} \left(1 - e^{-jk(2\pi/N)}\right) a_k$$

Properties of DT Fourier Series

3. Parseval's Relation:

$$\underbrace{\frac{1}{N} \sum_{n=-N}^{N-1} |x[n]|^2}_{\substack{\text{average power in} \\ \text{one period for} \\ \text{periodic signal } x[n]}} = \sum_{k=-N}^{N-1} \underbrace{|a_k|^2}_{\substack{\text{average power in} \\ k^{\text{th}} \text{ harmonic}}}$$

Study and show the basic properties of DTFS on Table 3.2 @ Oppenheim

Properties of DT Fourier Transform

Notation: $x[n] \xleftrightarrow{F} X(\Omega)$ $X(\Omega) = F\{x[n]\}$

1. Periodicity: $X(\Omega + 2\pi) = X(\Omega)$: DTFT is always periodic with period 2π

2. Time Shifting and Frequency Shifting: if $x[n] \xleftrightarrow{F} X(\Omega)$, then

$$x[n - n_0] \xleftrightarrow{F} e^{-j\Omega n_0} X(\Omega) \quad (\text{time shifting})$$

$$e^{j\Omega_0 n} x[n] \xleftrightarrow{F} X(\Omega - \Omega_0) \quad (\text{frequency shifting})$$

Properties of DT Fourier Transform

3. Conjugation / Conjugate Symmetry: if $x[n] \xleftrightarrow{F} X(\Omega)$, then

$$x^*[n] \xleftrightarrow{F} X^*(-\Omega)$$

if $x[n]$ is real, $X(\Omega) = X^*(-\Omega)$

4. Differencing / Accumulation: if $x[n] \xleftrightarrow{F} X(\Omega)$, then

$$x[n] - x[n-1] \xleftrightarrow{F} (1 - e^{-j\Omega})X(\Omega) \quad (\text{show by shifting})$$

$$\sum_{m=-\infty}^n x[m] \xleftrightarrow{F} \frac{1}{1 - e^{-j\Omega}} X(\Omega) + \pi X(0) \sum_{k=-\infty}^{\infty} \delta(\Omega - 2\pi k)$$

Properties of DT Fourier Transform

5. Time Reversal: if $x[n] \xleftrightarrow{F} X(\Omega)$; $x[-n] \xleftrightarrow{\quad} X(-\Omega)$

6. Time Expansion:

Remember
$$x(at) \xleftrightarrow{F} \frac{1}{|a|} X\left(\frac{\omega}{a}\right)$$

if $x[an] \longrightarrow$ a should be integer.

to slow down the signal $a < 1 \implies$ Not possible !

to speed up $a \geq \mp 1$ e.g. $x[2n] \longrightarrow$ Not really speeding up!

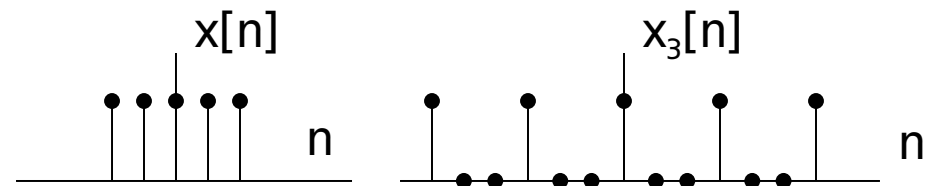
$x[2n]$: consists of even samples of $x[n]$ only.

Properties of DT Fourier Transform

⇒ Let k be a positive integer and define the signal

$$x_{(k)}[n] = \begin{cases} x[\frac{n}{k}], & \text{if } n \text{ is a multiple of } k \\ 0 & , \text{ if } n \text{ is not a multiple of } k \end{cases}$$

(insert $k-1$ zeros between successive values of $x[n]$, hence $x_{(k)}[n]$ is slowed-down version of $x[n]$)



$$x_k[n]=0 \quad \text{unless } n=rk$$

$$\Rightarrow X_{(k)}(\Omega) = \sum_{n=-\infty}^{\infty} x_{(k)}[n]e^{-j\Omega n} = \sum_{r=-\infty}^{\infty} x_{(k)}[rk]e^{-j\Omega rk}$$

Properties of DT Fourier Transform

$$= \sum_{r=-\infty}^{\infty} x[r] e^{-j(k\Omega)r} = X(k\Omega)$$

$$\Rightarrow x_k[n] \xleftrightarrow{F} X(k\Omega)$$

7. Differentiation in Frequency: if $x[n] \longleftrightarrow X(\Omega)$, then

$$\frac{dX(\Omega)}{d\Omega} = \sum_{n=-\infty}^{\infty} -jnx[n]e^{-j\Omega n} \quad (\text{multiply both sides with } j)$$

$$\Rightarrow nx[n] \longleftrightarrow j \frac{dX(\Omega)}{d\Omega}$$

Properties of DT Fourier Transform

8. Parseval's Relation: if $x[n] \longleftrightarrow X(\Omega)$, then

$$\underbrace{\sum_{n=-\infty}^{\infty} |x[n]|^2}_{\text{total energy in signal } x[n]} = \frac{1}{2\pi} \int_{2\pi} \underbrace{|X(\Omega)|^2}_{\substack{\text{energy-density} \\ \text{spectrum}}} d\Omega$$

9. Convolution Property:

$$y[n] = x[n] \star h[n] \xrightarrow{F} Y(\Omega) = X(\Omega)H(\Omega)$$

Properties of DT Fourier Transform

10. Multiplication Property:

Consider $y[n] = x_1[n]x_2[n]$

$$Y(\Omega) = \sum_{n=-\infty}^{\infty} y[n]e^{-j\Omega n} = \sum_{n=-\infty}^{\infty} x_1[n]x_2[n]e^{-j\Omega n}$$

Since $x_1[n] = \frac{1}{2\pi} \int_{-\pi}^{\pi} X_1(\theta)e^{j\theta n} d\theta$

$$Y(\Omega) = \sum_{n=-\infty}^{\infty} x_2[n] \left\{ \frac{1}{2\pi} \int_{-\pi}^{\pi} X_1(\theta)e^{j\theta n} d\theta \right\} e^{-j\Omega n}$$

Properties of DT Fourier Transform

$$= \frac{1}{2\pi} \int_{2\pi} X_1(\theta) \left\{ \sum_{n=-\infty}^{\infty} x_2[n] e^{-j(\Omega - \theta)n} \right\} d\theta$$

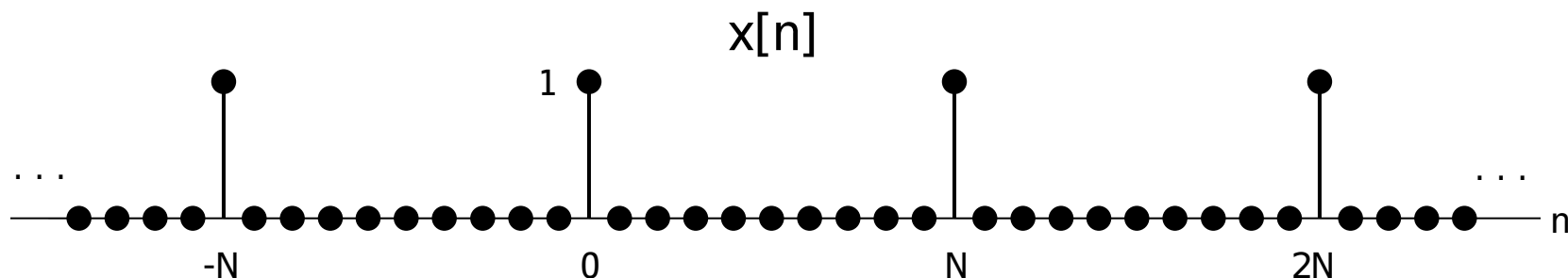
$$Y(\Omega) = \frac{1}{2\pi} \int_{2\pi} X_1(\theta) X_2(\Omega - \theta) d\theta$$

: periodic
convolution

Example 5.6

Find DTFT of the discrete-time periodic impulse train.

$$x[n] = \sum_{k=-\infty}^{\infty} \delta[n - kN]$$



The Fourier series coefficients for this signal

$$a_k = \frac{1}{N} \sum_{n=\langle N \rangle} x[n] e^{-jk(2\pi/N)n}$$

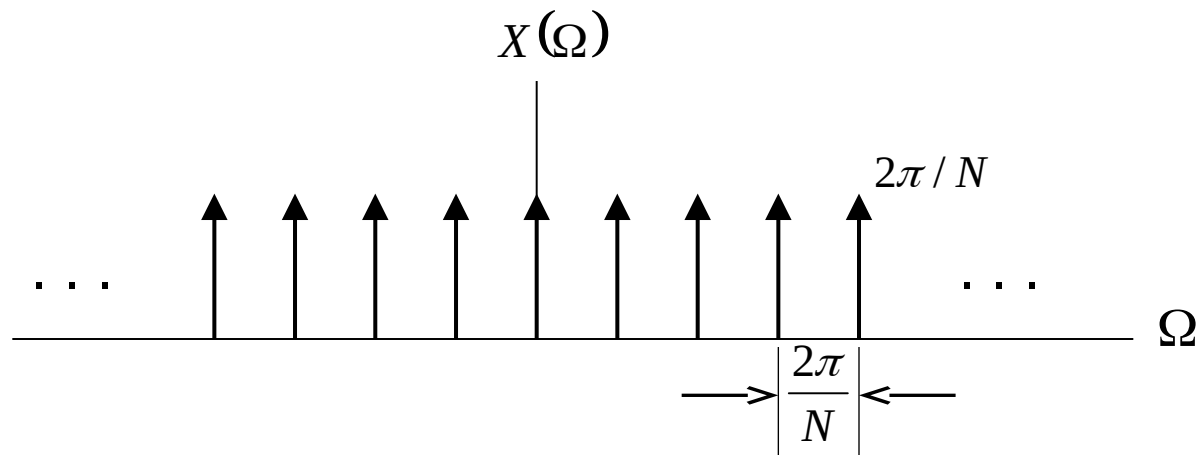
Example 5.6

Choosing the interval of summation as $0 \leq n \leq (N-1)$:

$$a_k = \frac{1}{N}$$

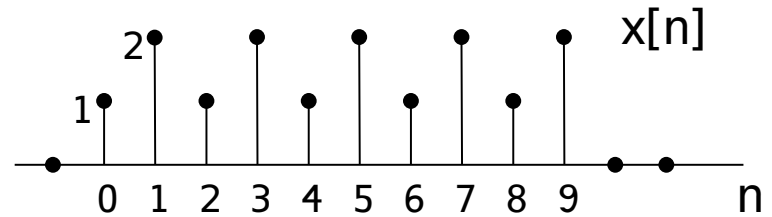
The Fourier transform of the signal is represented as

$$X(\Omega) = \frac{2\pi}{N} \sum_{k=-\infty}^{\infty} \delta\left(\Omega - \frac{2\pi k}{N}\right)$$

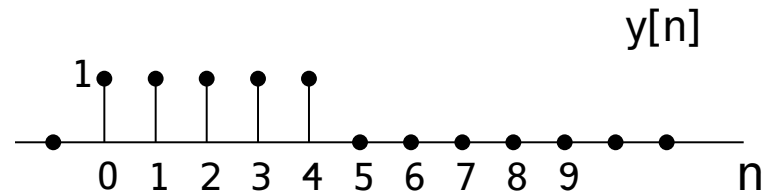


Example 5.9

Consider the following sequence $x[n]$, find its DTFT.

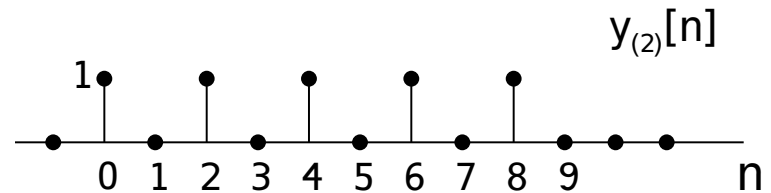


This sequence can be related to the simpler sequence $y[n]$ as following:

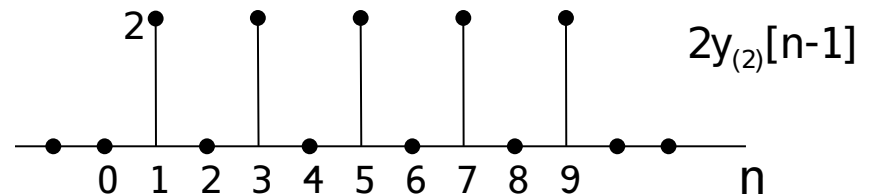


$$x[n] = y_{(2)}[n] + 2y_{(2)}[n-1]$$

where



$$y_{(2)}[n] = \begin{cases} y[n/2], & \text{if } n \text{ is even} \\ 0, & \text{if } n \text{ is odd} \end{cases}$$



Example 5.9

Note that $y[n]=g[n-2]$, where $g[n]$ is a rectangular pulse. Consequently, using the time-shifting property:

$$Y(\Omega) = e^{-j2\Omega} \frac{\sin(5\Omega/2)}{\sin(\Omega/2)}$$

Using the time-expansion property:

$$y_{(2)}[n] \xleftarrow{F} \rightarrow e^{-j4\Omega} \frac{\sin(5\Omega)}{\sin(\Omega)}$$

Using the linearity and the time-shifting properties:

$$2y_{(2)}[n-1] \xleftarrow{F} \rightarrow 2e^{-j5\Omega} \frac{\sin(5\Omega)}{\sin(\Omega)}$$

Combining these two results:

$$X(\Omega) = e^{-j4\Omega} (1 + 2e^{-j\Omega}) \frac{\sin(5\Omega)}{\sin(\Omega)}$$

Example 5.15

Find the Fourier transform of $x[n]$ which is the product of two other signals:

$$x[n] = x_1[n]x_2[n]$$

where $x_1[n] = \frac{\sin(3\pi n / 4)}{\pi n}$ and $x_2[n] = \frac{\sin(\pi n / 2)}{\pi n}$

$X(\Omega)$ is the periodic convolution of $X_1(\Omega)$ and $X_2(\Omega)$:

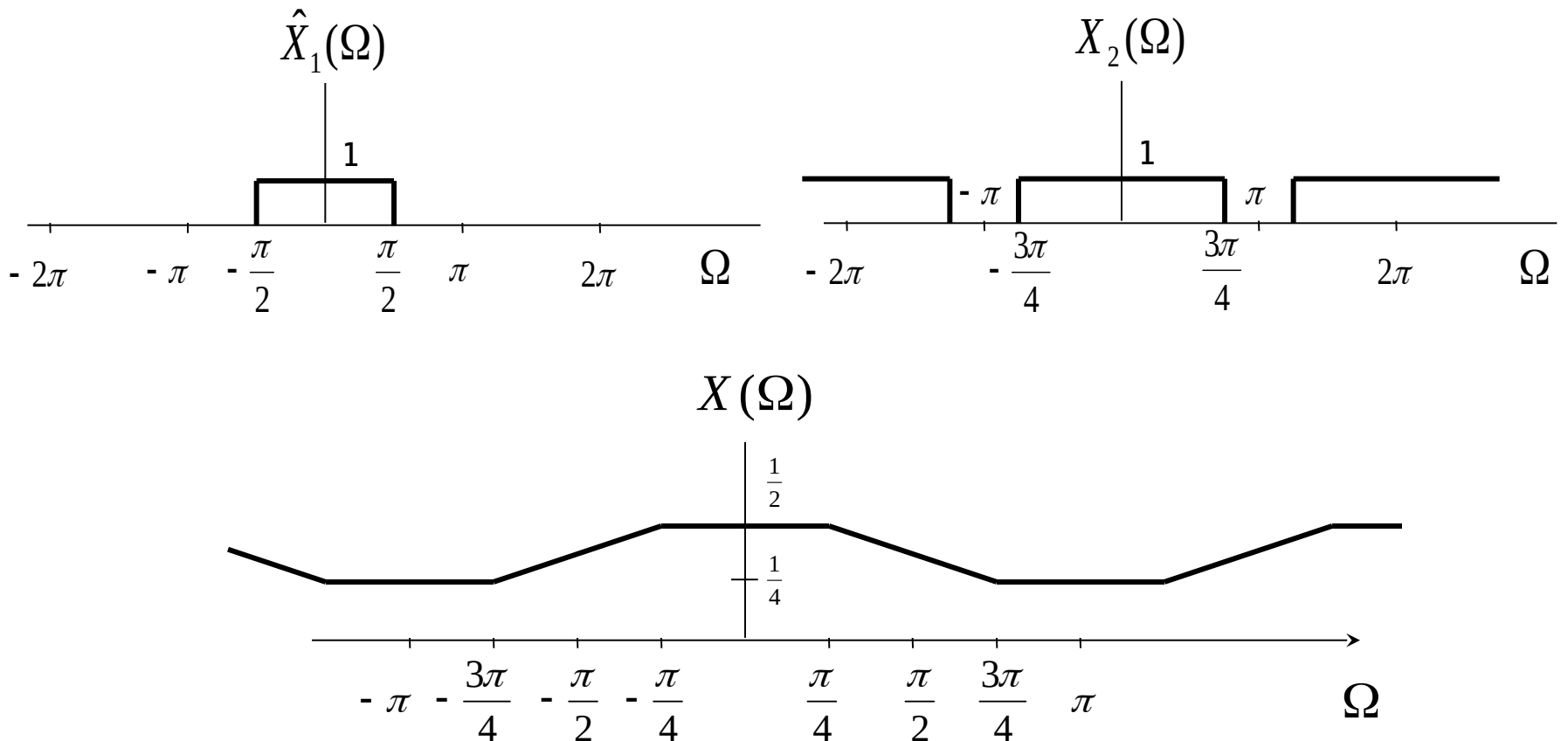
$$X(\Omega) = \frac{1}{2\pi} \int_{-\pi}^{\pi} X_1(\theta) X_2(\Omega - \theta) d\theta$$

We can convert the equation into an ordinary convolution by defining:

$$\hat{X}_1(\Omega) = \begin{cases} X_1(\Omega) & \text{for } -\pi < \Omega \leq \pi \\ 0 & \text{otherwise} \end{cases}$$

Example 5.15

$$X(\Omega) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \hat{X}_1(\theta) X_2(\Omega - \theta) d\theta = \frac{1}{2\pi} \int_{-\infty}^{\infty} \hat{X}_1(\theta) X_2(\Omega - \theta) d\theta$$



Example 5.19

Consider a causal LTI system that is characterized by the difference equation given below. Find its impulse response $h[n]$.

$$y[n] - \frac{3}{4}y[n-1] + \frac{1}{8}y[n-2] = 2x[n].$$

$$Y(\Omega) - \frac{3}{4}e^{-j\Omega}Y(\Omega) + \frac{1}{8}e^{-j2\Omega}Y(\Omega) = 2X(\Omega)$$

The frequency response is

$$H(\Omega) = \frac{2}{1 - \frac{3}{4}e^{-j\Omega} + \frac{1}{8}e^{-j2\Omega}}$$

Example 5.19

As a first step in obtaining the impulse response, we factor the denominator of the frequency response.

$$H(\Omega) = \frac{2}{\left(1 - \frac{1}{2}e^{-j\Omega}\right)\left(1 - \frac{1}{4}e^{-j\Omega}\right)}$$

$H(\Omega)$ can be expanded by the method of partial fractions. The result of this expansion is

$$H(\Omega) = \frac{4}{1 - \frac{1}{2}e^{-j\Omega}} - \frac{2}{1 - \frac{1}{4}e^{-j\Omega}}$$

The inverse transform of each term can be recognized by inspection, with the result that

$$h[n] = 4\left(\frac{1}{2}\right)^n u[n] - 2\left(\frac{1}{4}\right)^n u[n]$$

Problem 3.26

Let $x(t)$ be a periodic signal whose Fourier series coefficients are

$$a_k = \begin{cases} 2, & k = 0 \\ j\left(\frac{1}{2}\right)^{|k|}, & \text{otherwise} \end{cases}$$

Use Fourier series properties to answer the following questions:

- (a) Is $x(t)$ real?
- (b) Is $x(t)$ even?
- (c) Is $dx(t)/dt$ even?

Solution 3.26

(a) If $x(t)$ is real, then $x(t) = x^*(t)$. This implies that for $x(t)$ real $a_k = a_{-k}^*$. Since this is not true in this case, $x(t)$ is not real.

(b) If $x(t)$ is even, then $x(t) = x(-t)$ and $a_k = a_{-k}$. Since this is true for this case, $x(t)$ is even.

(c)

$$g(t) = \frac{dx(t)}{dt} \overset{FS}{\leftrightarrow} b_k = jk \frac{2\pi}{T_0} a_k$$

Therefore,

$$b_k = \begin{cases} 0, & k = 0 \\ -k(1/2)^{|k|}(2\pi/T_0), & \text{otherwise} \end{cases}$$

Since b_k is not even, $g(t)$ is not even.

Problem 3.35

Consider a continuous-time LTI system S whose frequency response is

$$H(j\omega) = \begin{cases} 1, & |\omega| \geq 250 \\ 0, & \text{otherwise} \end{cases}.$$

When the input to this system is a signal $x(t)$ with fundamental period $T = \pi / 7$ and Fourier series coefficients a_k , it is found that the output $y(t)$ is identical to $x(t)$. For what values of k is it guaranteed that $a_k = 0$?

Solution 3.35

The FS coefficients of $y(t)$ are $b_k = H(jk\omega_0)a_k$, where ω_0 is the fundamental frequency of $x(t)$ and a_k are the FS coefficients of $x(t)$.

If $y(t)$ is identical to $x(t)$, then $b_k = a_k$ for all k .

Since $H(j\omega) = 1$ for $|\omega| \geq 250$, $H(jk\omega_0) = 1$ for $|k| \geq 18$ (because $\omega_0 = 14$).

Therefore, a_k must be zero for $|k| < 18$.

Problem 4.5

Use the Fourier transform synthesis equation to determine the inverse Fourier transform of $X(\omega) = |X(\omega)|e^{j\angle X(\omega)}$, where

$$|X(\omega)| = 2\{u(\omega + 3) - u(\omega - 3)\},$$

$$\angle X(\omega) = -\frac{3}{2}\omega + \pi$$

Solution 4.5

$$x(t) = (1/2\pi) \int_{-\infty}^{\infty} X(\omega) e^{j\omega t} d\omega$$

$$= (1/2\pi) \int_{-\infty}^{\infty} |X(\omega)| e^{j\angle\{X(\omega)\}} e^{j\omega t} d\omega$$

$$= (1/2\pi) \int_{-3}^3 2e^{-\frac{3}{2}\omega + \pi} e^{j\omega t} d\omega$$

$$= \frac{2}{\pi(t - 3/2)} \sin[3(t - 3/2)]$$

Problem 4.30 (a)

Suppose $g(t)=x(t)\cos(t)$ and the Fourier transform of the $g(t)$ is

$$G(\omega) = \begin{cases} 1, & |\omega| \leq 2 \\ 0, & \text{otherwise} \end{cases}.$$

Determine $x(t)$.

Problem 4.33

The input and the output of a causal LTI system are related by the differential equation

$$\frac{d^2 y(t)}{dt^2} + 6 \frac{dy(t)}{dt} + 8y(t) = 2x(t)$$

- (a) Find the impulse response of this system.
- (b) What is the response of this system if $x(t) = te^{-2t}u(t)$?

Solution 4.33 (a)

Taking the Fourier transform of both sides of the given differential equation, we obtain the frequency response

$$H(\omega) = \frac{Y(\omega)}{X(\omega)} = \frac{2}{-\omega^2 + 2j\omega + 8}.$$

Using partial fraction expansion, we obtain

$$H(\omega) = \frac{1}{j\omega + 2} - \frac{1}{j\omega + 4}.$$

Taking the inverse Fourier transform,

$$h(t) = e^{-2t}u(t) - e^{-4t}u(t).$$

Solution 4.33 (b)

FT of the given signal $x(t)$

$$X(\omega) = \frac{1}{(2 + j\omega)^2}.$$

Therefore, $Y(\omega) = X(\omega)H(\omega) = \frac{2}{(-\omega^2 + 2j\omega + 8)} \frac{1}{(2 + j\omega)^2}$

Using partial fraction expansion

$$Y(\omega) = \frac{1/4}{j\omega + 2} - \frac{1/2}{(j\omega + 2)^2} + \frac{1}{(j\omega + 2)^3} - \frac{1/4}{j\omega + 4}.$$

Taking the inverse fourier transform,

$$y(t) = \frac{1}{4}e^{-2t}u(t) - \frac{1}{2}te^{-2t}u(t) + t^2e^{-2t}u(t) - \frac{1}{4}e^{-4t}u(t).$$

Problem 5.21 (i)

Compute the Fourier transform of the following signal:

$$x[n] = x[n - 6] \text{ and } x[n] = u[n] - u[n - 5] \text{ for } 0 \leq n \leq 5$$

Solution 5.21 (i)

$x[n]$ is periodic with period 6. The Fourier series coefficients of $x[n]$ are given by

$$a_k = \frac{1}{6} \sum_{n=0}^5 x[n] e^{-j(2\pi/6)kn}$$

$$= \frac{1}{6} \sum_{n=0}^4 e^{-j(2\pi/6)kn}$$

$$= \frac{1}{6} \left[\frac{1 - e^{-j5\pi k/3}}{1 - e^{-j(2\pi/6)k}} \right]$$

Therefore,

$$X(\Omega) = \sum_{l=-\infty}^{\infty} 2\pi \left(\frac{1}{6} \right) \left[\frac{1 - e^{-j5\pi k/3}}{1 - e^{-j(2\pi/6)k}} \right] \delta\left(\Omega - \frac{2\pi}{6} - 2\pi l\right)$$

Problem 5.19

Consider a causal and stable LTI system S whose input $x[n]$ and output $y[n]$ are related through the second-order difference equation

$$y[n] - \frac{1}{6}y[n-1] - \frac{1}{6}y[n-2] = x[n]$$

- (a) Determine the frequency response $H(\Omega)$ for the system S .
- (b) Determine the impulse response $h[n]$ for the system S .

Solution 5. 19 (a)

(a) Taking the Fourier transform of both sides of the difference equation, we have

$$Y(\Omega) \left[1 - \frac{1}{6} e^{-j\Omega} - \frac{1}{6} e^{-2j\Omega} \right] = X(\Omega)$$

Therefore,

$$H(\Omega) = \frac{Y(\Omega)}{X(\Omega)} = \frac{1}{1 - \frac{1}{6} e^{-j\Omega} - \frac{1}{6} e^{-2j\Omega}} = \frac{1}{\left(1 - \frac{1}{2} e^{-j\Omega}\right) \left(1 + \frac{1}{3} e^{-j\Omega}\right)}$$

Solution 5. 19 (b)

(b) Using Partial fraction expansion,

$$H(\Omega) = \frac{3/5}{1 - \frac{1}{2}e^{-j\Omega}} + \frac{2/5}{1 + \frac{1}{3}e^{-j\Omega}}$$

Taking the inverse Fourier transform, we obtain

$$h[n] = \frac{3}{5} \left(\frac{1}{2} \right)^n u[n] + \frac{2}{5} \left(-\frac{1}{3} \right)^n u[n]$$

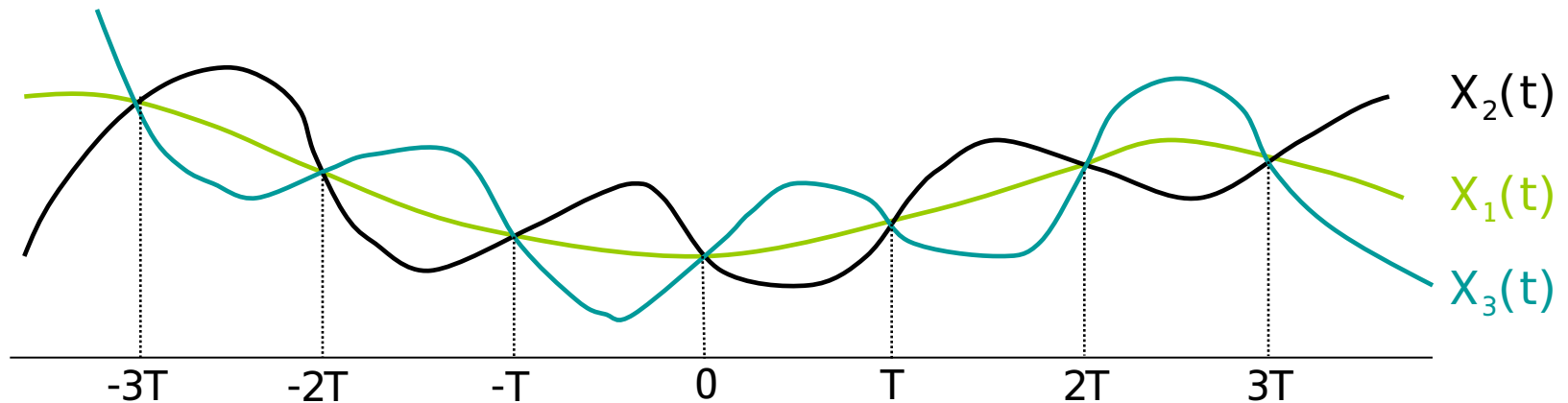
Sampling Theorem

How to process a CT signal in a DT system?

→ Convert the CT signal to DT signal with sampling!

→ Take equally-spaced samples of the CT signal.

Is any sequence of equally-spaced samples enough to uniquely specify the signal?



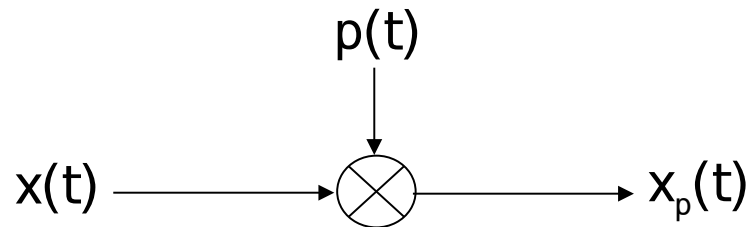
$$x_1(kT) = x_2(kT) = x_3(kT)$$

→ Additional conditions are necessary!

Sampling Theorem

Consider impulse-train sampling:

- CT signal $x(t)$ is multiplied by periodic impulse train $p(t)$ (sampling function).



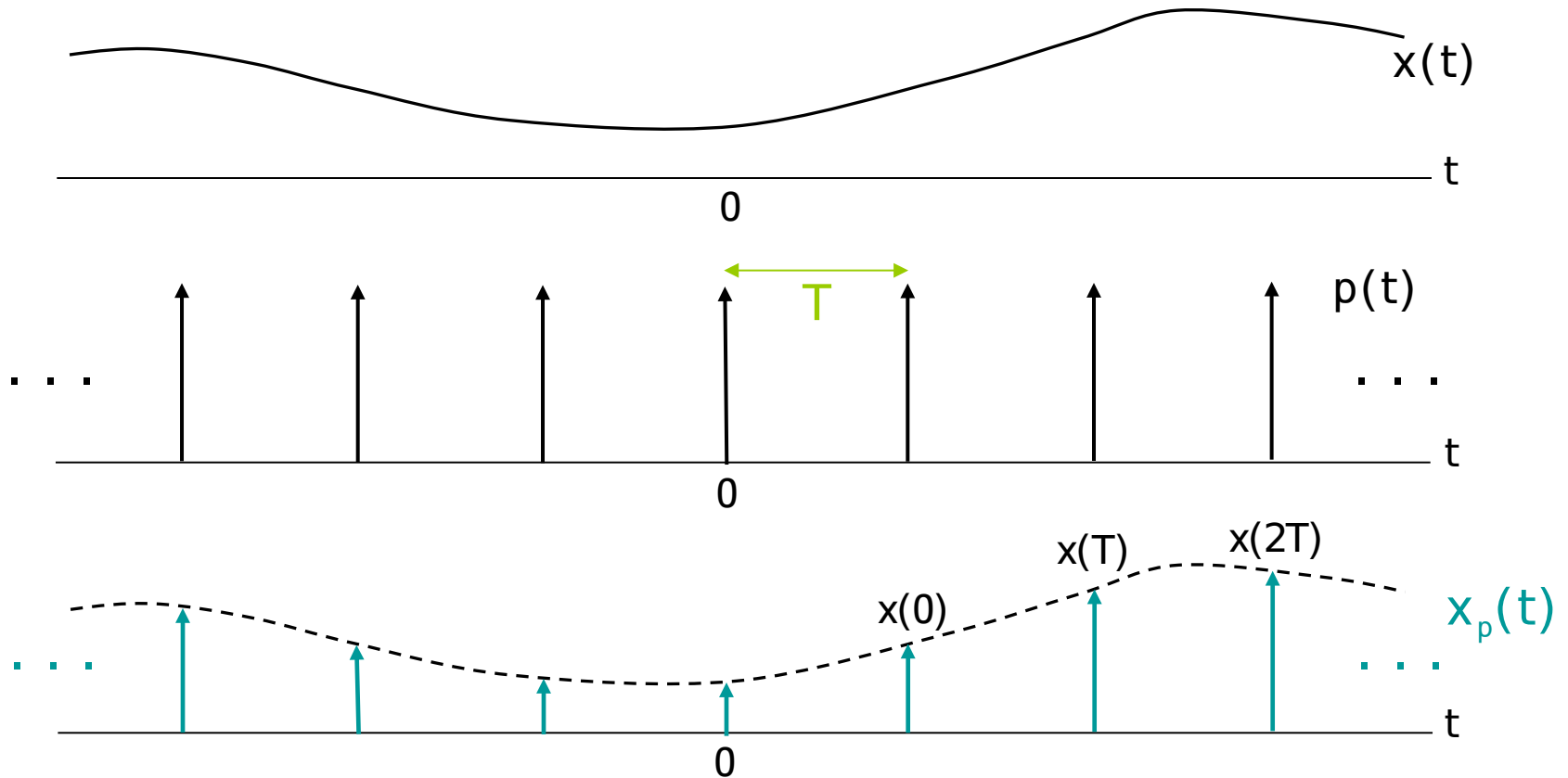
- T : sampling period, $\omega_s = \frac{2\pi}{T}$: sampling frequency

$$x_p(t) = x(t)p(t) \quad p(t) = \sum_{n=-\infty}^{\infty} \delta(t - nT)$$

Sampling Theorem

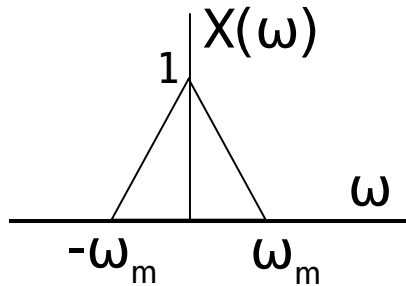
$$x_p(t) = x(t)p(t) \quad p(t) = \sum_{n=-\infty}^{\infty} \delta(t - nT)$$

$$x_p(t) = \sum_{n=-\infty}^{\infty} x(nT)\delta(t - nT)$$

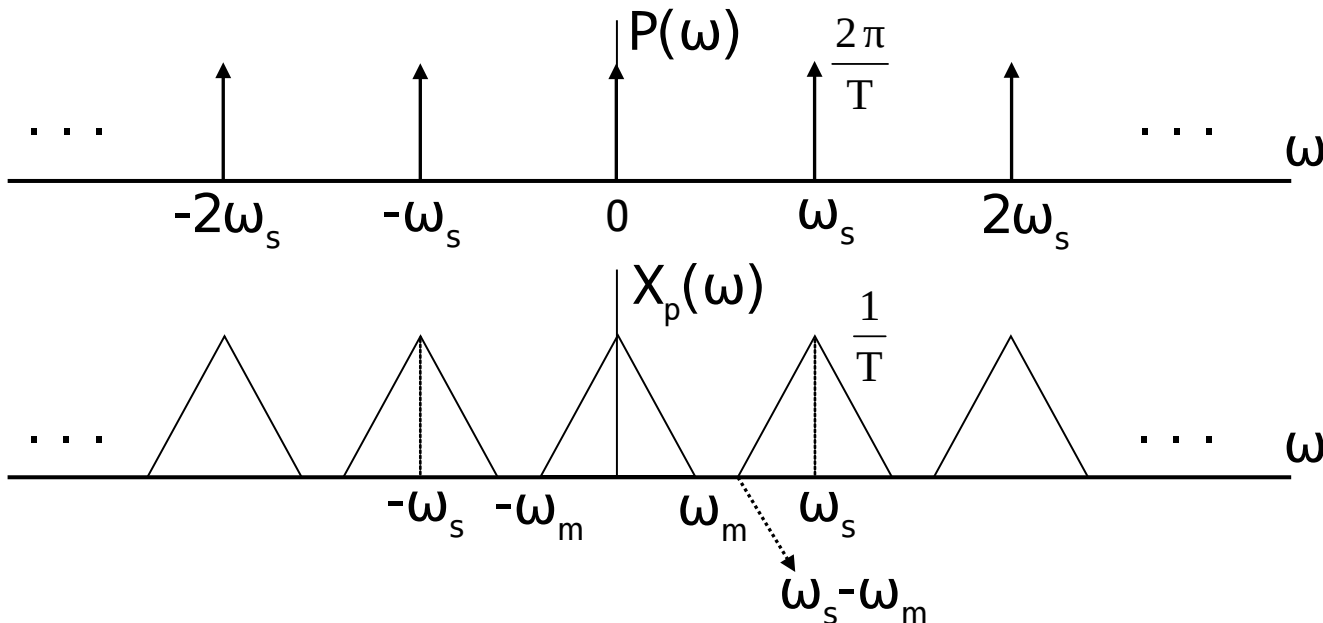


Sampling Theorem

In frequency domain $X_p(\omega) = \frac{1}{2\pi} (X(\omega) * P(\omega))$, $P(\omega) = \frac{2\pi}{T} \sum_{k=-\infty}^{\infty} \delta(\omega - k\omega_s)$



$$\Rightarrow X_p(\omega) = \frac{1}{T} \sum_{k=-\infty}^{\infty} X(\omega - k\omega_s)$$



If $\omega_s - \omega_m > \omega_m$

→ no overlap between shifted replicas.

Hence, $\omega_s - \omega_m > \omega_m$

→ $\omega_s > 2\omega_m$

→ $x(t)$ can be exactly recovered from $x_p(t)$ by low-pass filter (with gain T ,

$\omega_m < \omega_{co} < \omega_s - \omega_m$)

Sampling Theorem

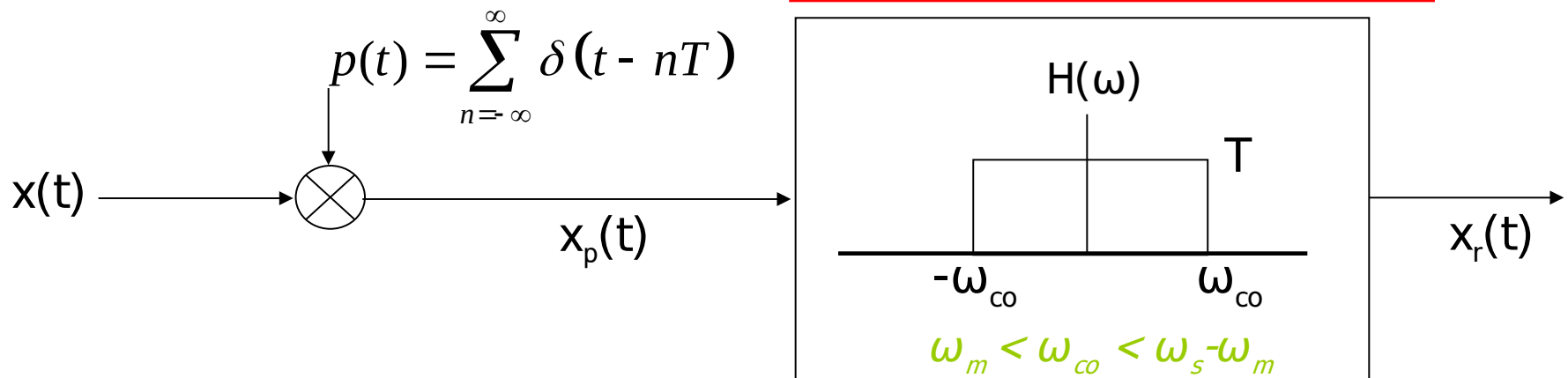
Sampling Theorem: Let $x(t)$ be a band-limited signal, i.e., $X(\omega)=0$ for $|\omega|>\omega_m$.

Then, $x(t)$ is uniquely determined by its samples $x(nT)$, $n=0,\pm1,\pm2,\dots$ if $\omega_s > 2\omega_m$, where

$$\omega_s = \frac{2\pi}{T}$$

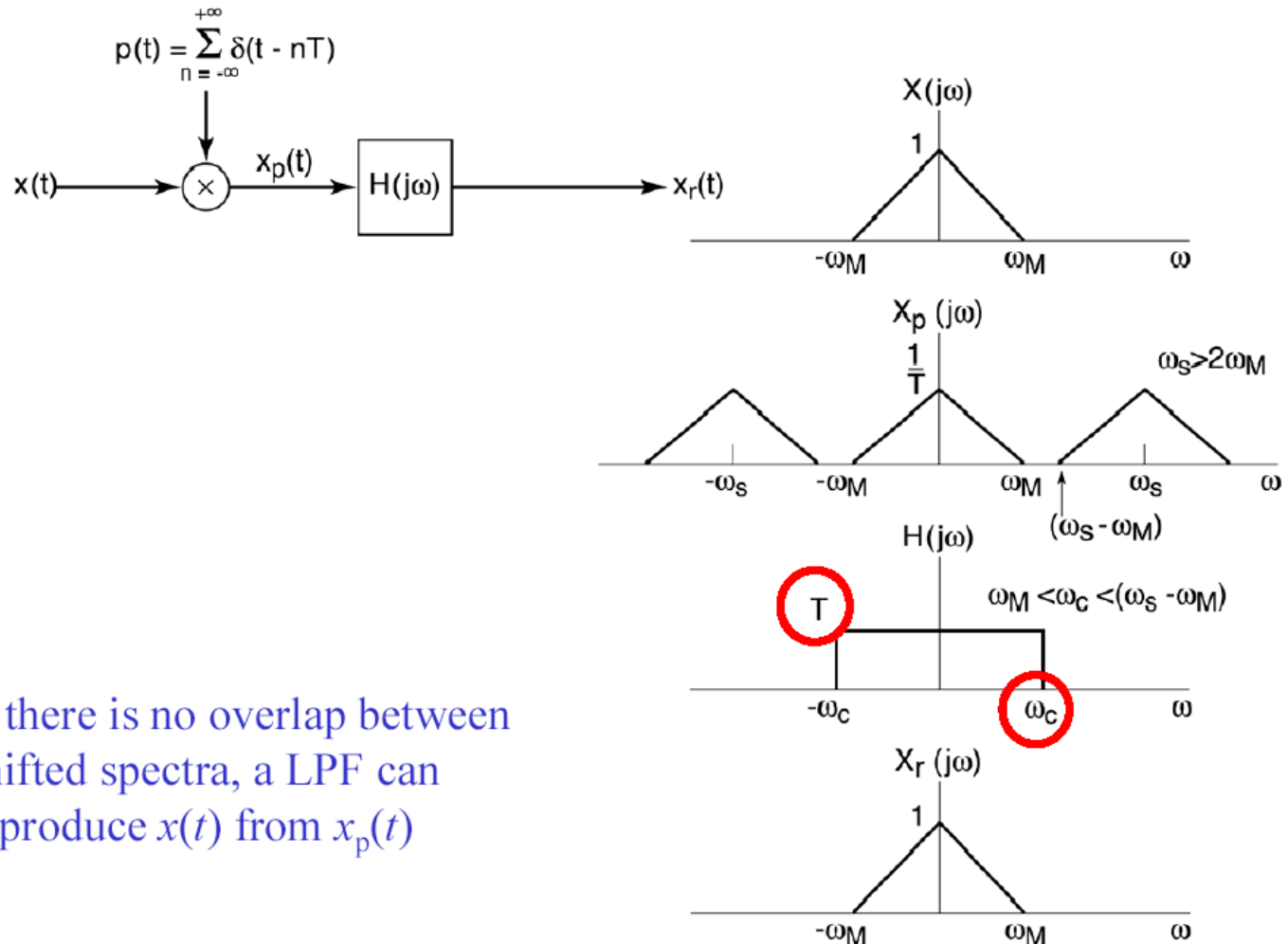
$2\omega_m$: Nyquist rate

ω_m : Nyquist frequency



Reconstruction of $x(t)$

Reconstruction of $x(t)$ from sampled signals



If there is no overlap between shifted spectra, a LPF can reproduce $x(t)$ from $x_p(t)$

Sampling Theorem

Time Domain

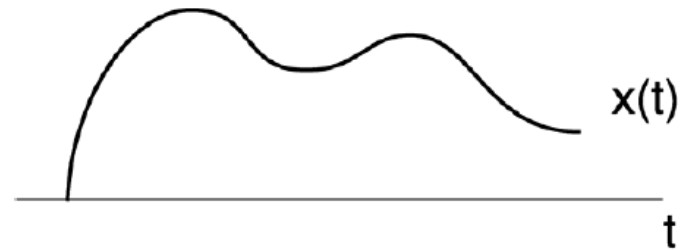
$x_r(t) = x_p(t) \star h(t)$ where $h(t)$ is the impulse response of ideal LPF

$$\begin{aligned} x_r(t) &= \sum_{n=-\infty}^{\infty} x(nT) \delta(t - nT) \star \frac{\omega_c T \sin(\omega_c t)}{\pi \omega_c t} \\ &= \sum_{n=-\infty}^{\infty} x(nT) \frac{T \sin(\omega_c (t - nT))}{\pi (t - nT)} \end{aligned}$$

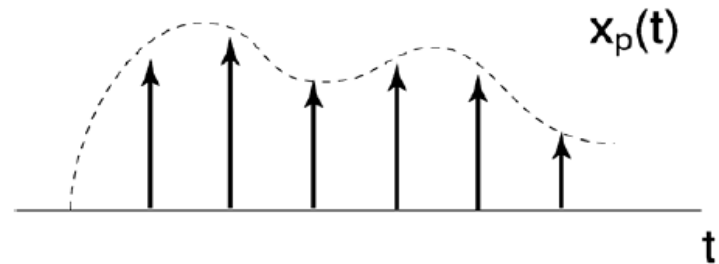
! (interpolation)

Sampling Theorem

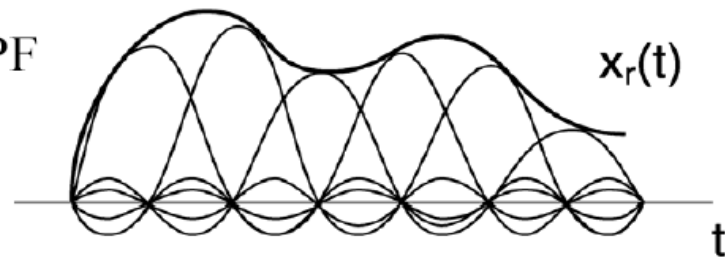
Original
CT signal



After sampling

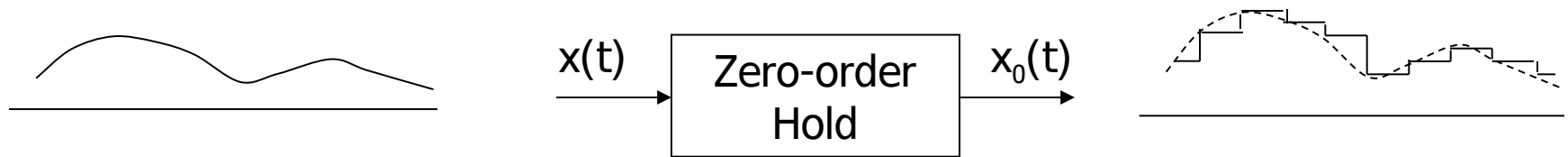


After passing the LPF



Sampling with Zero Order Hold (ZOH)

- Impulses are difficult (impossible) to generate in practice.
- more practical sampling system:

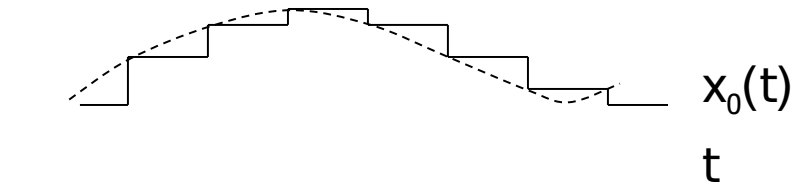
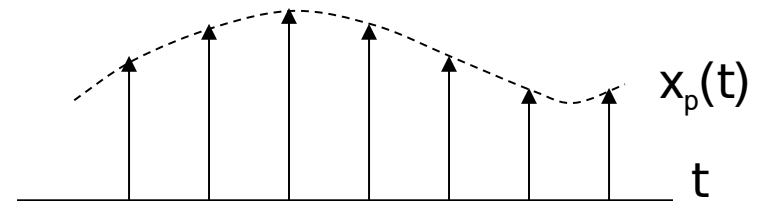
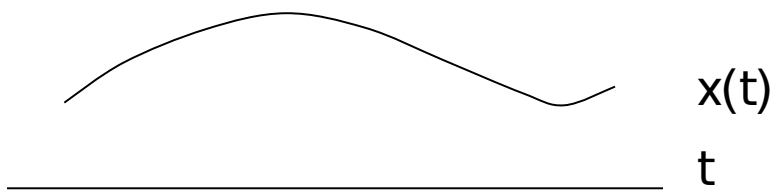
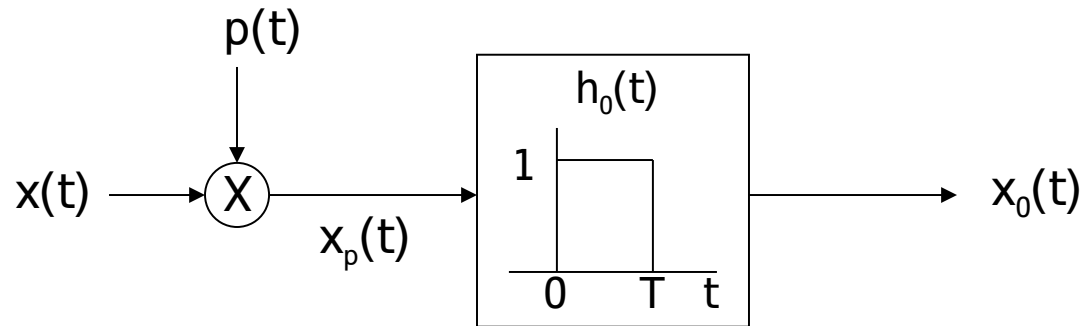


(samples the signal $x(t)$ at any instant and holds the value until next instant)

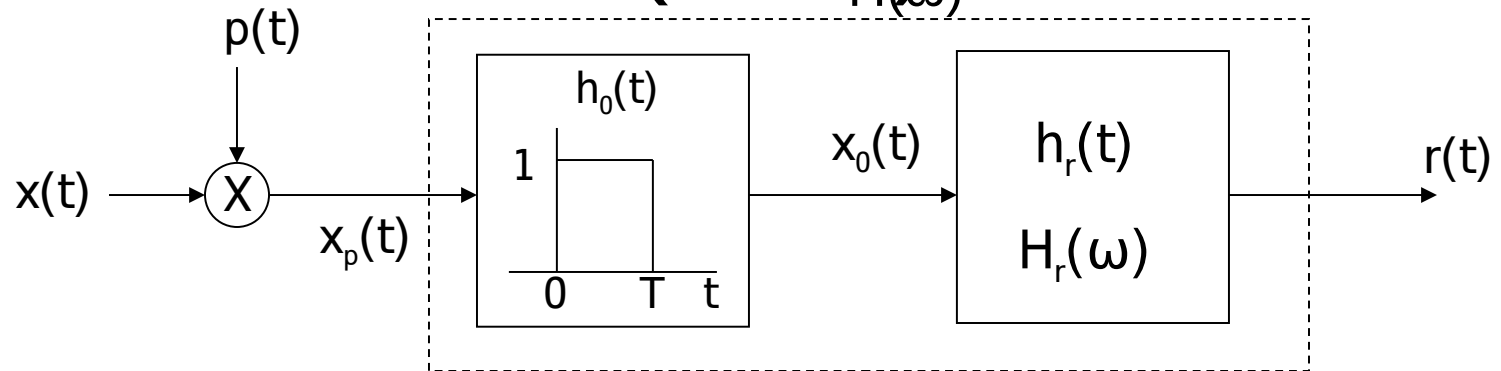
- To reconstruct $x(t)$ from $x_0(t)$, LPF is required.
- What is the frequency response of this filter??

Sampling with Zero Order Hold (ZOH)

- Think of ZOH sampling system as the concatenation of impulse train sampling and an LTI system with rectangular impulse response



Sampling with Zero Order Hold (ZOH)



$$H_0(\omega) H_r(\omega) = H(\omega)$$

$$H_0(\omega) = e^{-j\omega T/2} \left[\frac{2 \sin(\omega T/2)}{\omega} \right]$$

(from time-shifting property)

Sampling with Zero Order Hold (ZOH)

$$\Rightarrow H_r(\omega) = \frac{H(\omega)}{e^{-j\omega T/2} \frac{2 \sin(\omega T/2)}{\omega}} = \frac{\omega e^{j\omega T/2}}{2 \sin(\omega T/2)} H(\omega)$$

For $H(\omega) = \begin{cases} T, & |\omega| < \omega_s/2 \\ 0, & \text{o/w} \end{cases}$ ideal LPF

then, $H_r(\omega) = \begin{cases} \frac{\omega e^{j\omega T/2} T}{2 \sin(\omega T/2)}, & |\omega| < \omega_s/2 \\ 0, & \text{o/w} \end{cases}$

Sampling with Zero Order Hold (ZOH)

Remarks:

- ideal LPF is impossible to implement
- ZOH is very rough approximation
- Higher-order holds may be used (smoother interpolating filter) for better recovery! (read Opp. Section 7.2, 7.3).

$\omega_s > 2\omega_m$ (Nyquist theorem) for reconstruction:

if $\omega_s < 2\omega_m \longrightarrow$ undersampling $\longrightarrow$ aliasing occurs

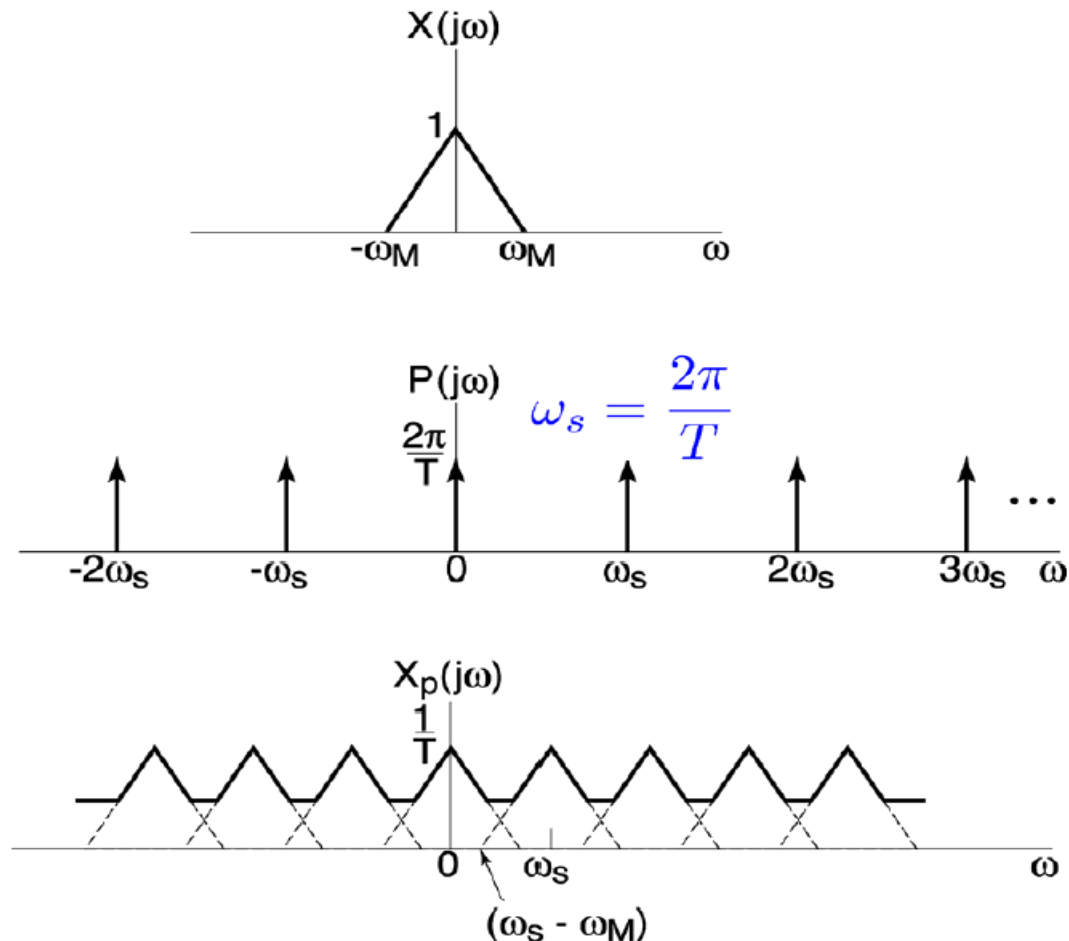
(unable to reconstruct the signal from its samples using LPF)

study Figs. 7.15, 7.16 in Oppenheim.

Undersampling and Aliasing

Undersampling and Aliasing

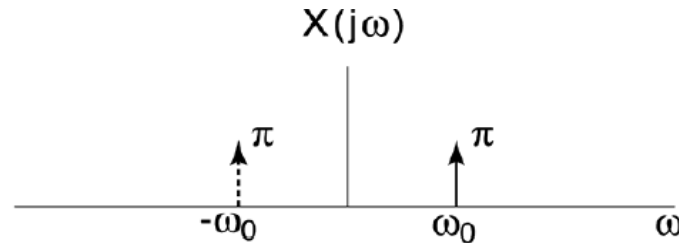
When $\omega_s \leq 2 \omega_M \Rightarrow$ Undersampling



Undersampling and Aliasing

A Simple Example

$$x(t) = \cos(\omega_0 t + \phi)$$



Picture would be
Modified...

