

DT FOURIER TRANSFORM

Derivation of DT Fourier Transform pair

DTFT of Periodic Signals

Properties of DTFT

Convolution Property

DT LTI Systems

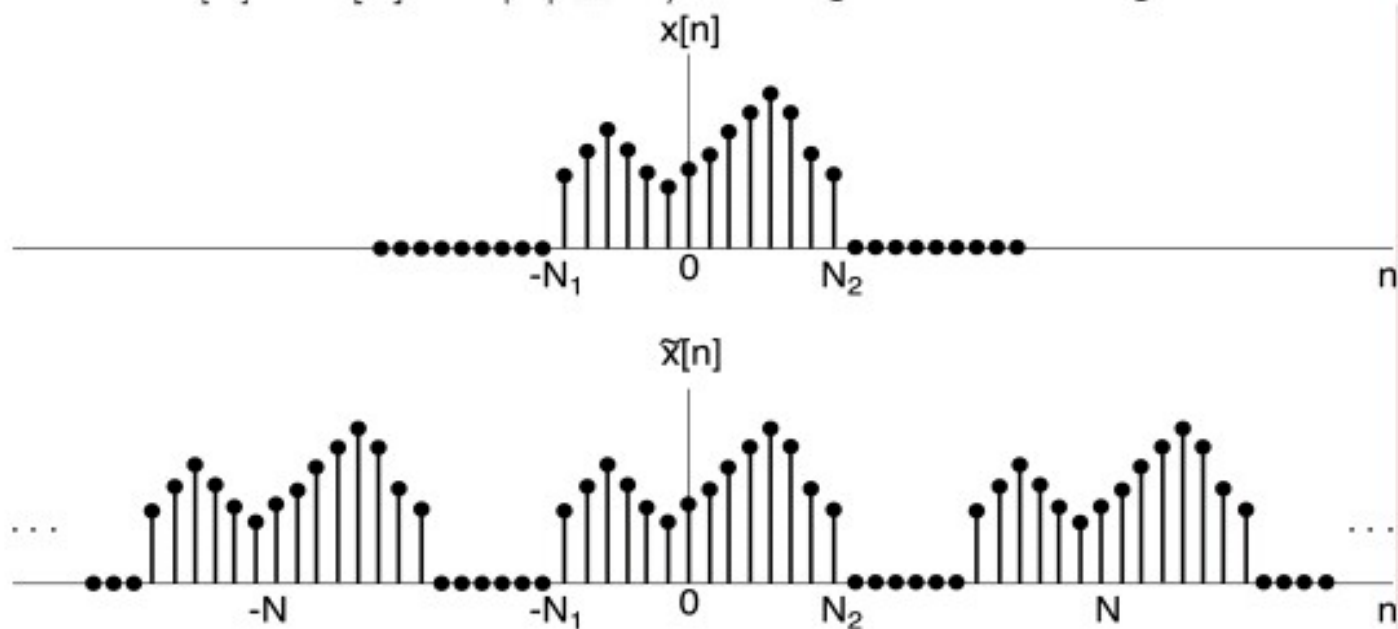
Duality in Fourier Analysis

Magnitude/Phase of FT
and Frequency Response

The Discrete-Time Fourier Transform

Derivation: (Analogous to CTFT except $e^{j\omega n} = e^{j(\omega+2\pi)n}$)

- $x[n]$ - aperiodic and (for simplicity) of finite duration
- N is large enough so that $x[n] = 0$ if $|n| \geq N/2$
- $\tilde{x}[n] = x[n]$ for $|n| \leq N/2$ and periodic with period N



$$\tilde{x}[n] = x[n] \text{ for any } n \text{ as } N \rightarrow \infty$$

DTFT Derivation (Continued)

$$\tilde{x}[n] = \sum_{k=\langle N \rangle} a_k e^{jk\omega_0 n}, \quad \omega_0 = \frac{2\pi}{N} \quad \text{DTFS synthesis eq.}$$

$$a_k = \frac{1}{N} \sum_{n=\langle N \rangle} \tilde{x}[n] e^{-jk\omega_0 n} \quad \text{DTFS analysis eq.}$$

$$= \frac{1}{N} \sum_{n=-N_1}^{N_2} \tilde{x}[n] e^{-jk\omega_0 n} = \frac{1}{N} \sum_{n=-\infty}^{\infty} x[n] e^{-jk\omega_0 n}$$

Define

$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n] e^{-j\omega n} \quad \boxed{\text{-- periodic in } \omega \text{ with period } 2\pi}$$

$$\Downarrow$$
$$a_k = \frac{1}{N} X(e^{jk\omega_0})$$

DTFT Derivation

$$\tilde{x}[n] = \sum_{k=\langle N \rangle} \underbrace{\frac{1}{N} X(e^{jk\omega_0})}_{a_k} e^{jk\omega_0 n} = \frac{1}{2\pi} \sum_{k=\langle N \rangle} X(e^{jk\omega_0}) e^{jk\omega_0 n} \omega_0 \quad (*)$$

As $N \rightarrow \infty$: $\tilde{x}[n] \rightarrow x[n]$ for every n

$$\omega_0 \rightarrow 0, \quad \sum \omega_0 \rightarrow \int d\omega$$

The sum in $(*) \rightarrow$ an integral

$\Downarrow$ The DTFT Pair

$$x[n] = \frac{1}{2\pi} \int_{2\pi} X(e^{j\omega}) e^{j\omega n} d\omega \quad \text{Synthesis equation}$$

Any 2π
interval in ω

$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n] e^{-j\omega n} \quad \text{Analysis equation}$$

DT Fourier Transform Pair

$$x[n] \longleftrightarrow X(e^{j\omega})$$

$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n} \quad \begin{array}{l} \text{-- Analysis Equation} \\ \text{-- FT} \end{array}$$

$$x[n] = \frac{1}{2\pi} \int_{2\pi} X(e^{j\omega})e^{j\omega n} d\omega \quad \begin{array}{l} \text{-- Synthesis Equation} \\ \text{-- Inverse FT} \end{array}$$

Observations

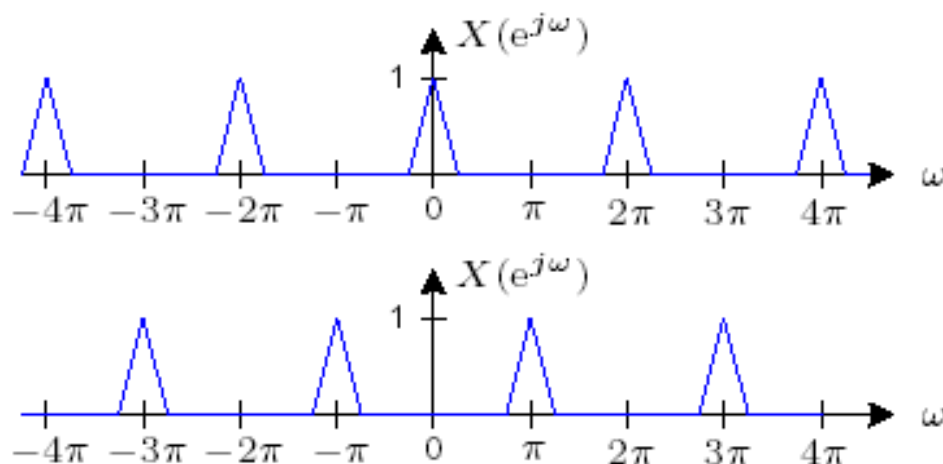
$$X(e^{j\omega}) = \sum_{n=-\infty}^{+\infty} x[n] e^{-j\omega n} \quad x[n] = \frac{1}{2\pi} \int_{2\pi} X(e^{j\omega}) e^{j\omega n} d\omega$$

- Called the **analysis** and **synthesis** equations, respectively
- Recall that $e^{j\omega n} = e^{j(\omega+\ell 2\pi)n}$, for any pair of integers ℓ and n
- Thus, $X(e^{j\omega})$ is a periodic function of ω with a fundamental period of 2π
- Unlike the DT Fourier series, the frequency ω is continuous
- Recall that the CT Fourier transform $X(j\omega)$ was not periodic
- Thus the DT *synthesis* integral can be taken over any continuous interval of length 2π
- This is similar to the CT Fourier series analysis equation

Discrete-Time Frequency Concepts

- Recall that $e^{j(\omega+\ell 2\pi)n} = e^{j\omega n}$
- If seemingly very high-frequency discrete-time signals, $\cos((\omega + \ell 2\pi)n)$, are equal to low-frequency discrete-time signals, $\cos(\omega n)$, what does low- and high-frequency mean in discrete-time?
- Note that the units of ω are *radians per sample*
- A sinusoid with a frequency of 0.1 radians per sample is the same as one with a frequency of $(0.1 + 2\pi)$ radians per sample
- Recall that $\cos(\pi n) = (-1)^n$
- No DT signal can oscillate “faster” between two samples
- No DT signal can oscillate “slower” than 0 radians per sample
- Thus
 - $\omega = \pi = \ell(\pi + 2\pi)$ is the highest perceivable DT frequency
 - $\omega = 0 = \ell(2\pi)$ is the lowest perceivable frequency

Discrete-Time Frequency Concepts Continued



- Low frequencies are those that are near 0
- High frequencies are those near $\pm\pi$
- Intermediate frequencies are those in between
- Note that the highest frequency, π radians per sample is equal to 0.5 cycles per sample
- We will encounter this concept again when we discuss sampling

Convergence Issues

Synthesis Equation: None, since integrating over a finite interval

Analysis Equation: Need conditions analogous to CTFT, e.g.

$$\sum_{n=-\infty}^{\infty} |x[n]|^2 < \infty \quad \text{— Finite energy}$$

or

$$\sum_{n=-\infty}^{\infty} |x[n]| < \infty \quad \text{— Absolutely summable}$$

Examples

Parallel with the CT examples

1) $x[n] = \delta[n]$

$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} \delta[n] e^{-j\omega n} = 1$$

2) $x[n] = \delta[n - n_0]$ - shifted unit sample

$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} \delta[n - n_0] e^{-j\omega n} = e^{-j\omega n_0}$$

– Same amplitude (=1) as above, but with a *linear* phase $-\omega n_0$

More Examples

3) $x[n] = a^n u[n]$, $|a| < 1$ - Exponentially decaying function

$$X(e^{j\omega}) = \sum_{n=0}^{\infty} a^n e^{-j\omega n} = \sum_{n=0}^{\infty} \underbrace{(ae^{-j\omega})^n}_{|ae^{-j\omega}| < 1}$$

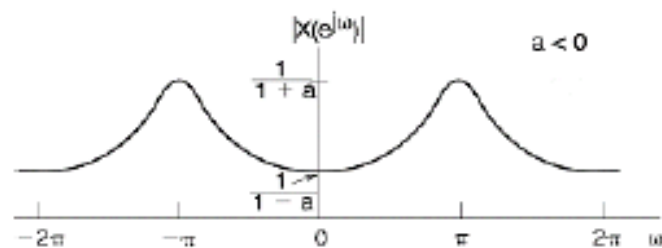
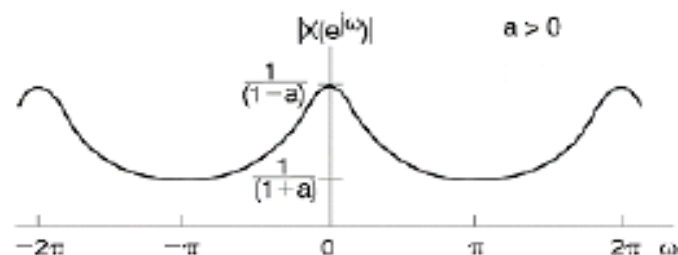
Infinite sum formula

$$= \frac{1}{1 - ae^{-j\omega}} = \frac{1}{(1 - a \cos \omega) + ja \sin \omega}$$

$$|X(e^{j\omega})| = \frac{1}{\sqrt{1 - 2a \cos \omega + a^2}}$$

$$\omega = 0 : X(e^{j\omega}) = \frac{1}{\sqrt{1 - 2a + a^2}} = \frac{1}{1 - a}$$

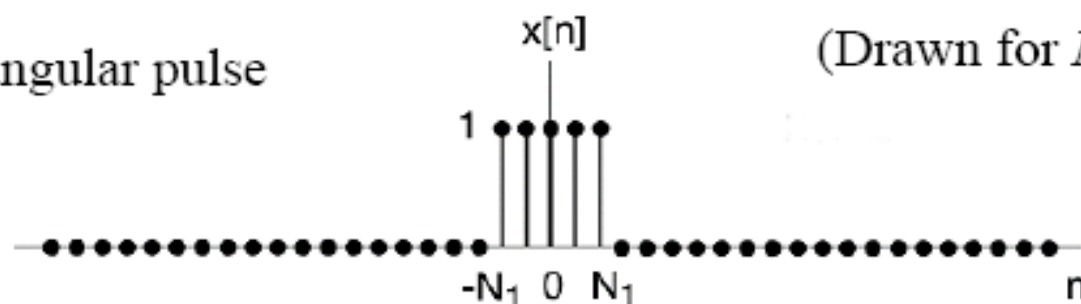
$$\omega = \pi : X(e^{j\omega}) = \frac{1}{\sqrt{1 + 2a + a^2}} = \frac{1}{1 + a}$$



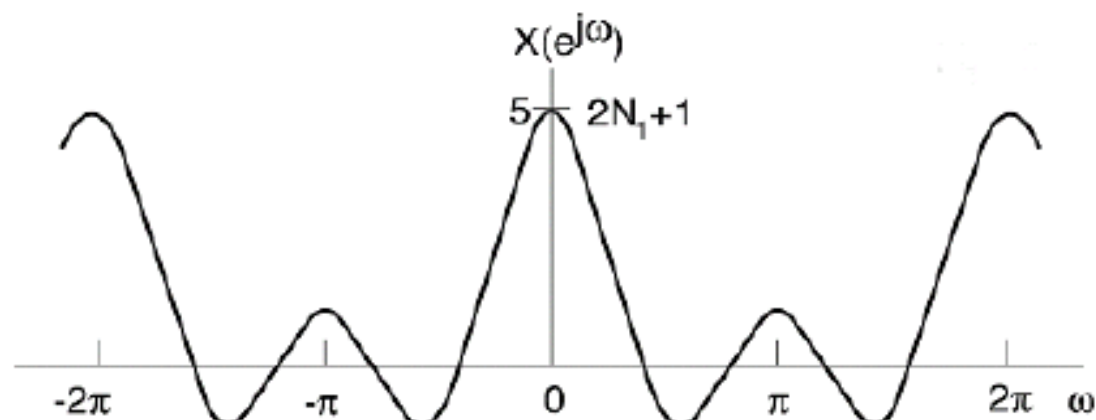
Still More

4) DT Rectangular pulse

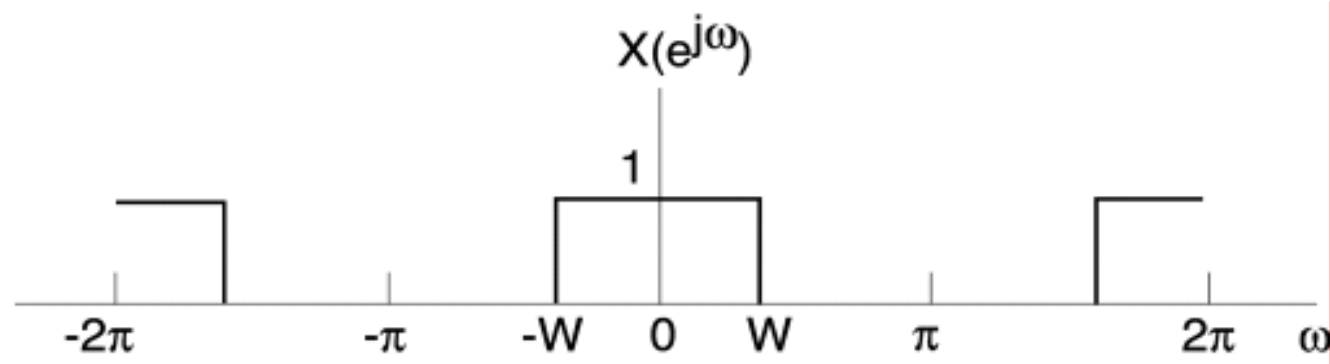
(Drawn for $N_1 = 2$)



$$X(e^{j\omega}) = \sum_{n=-N_1}^{N_1} e^{-j\omega n} = \sum_{n=-N_1}^{N_1} (e^{-j\omega})^n = \frac{\sin \omega (N_1 + \frac{1}{2})}{\sin(\omega/2)} = X(e^{j(\omega-2\pi)})$$

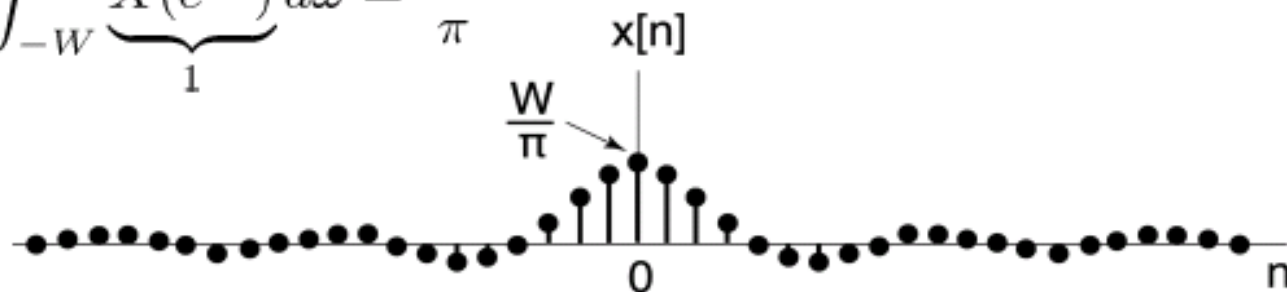


5)



$$x[n] = \frac{1}{2\pi} \int_{-W}^W e^{j\omega n} d\omega = \frac{\sin Wn}{\pi n}$$

$$x[0] = \frac{1}{2\pi} \int_{-W}^W \underbrace{X(e^{j\omega})}_1 d\omega = \frac{W}{\pi}$$



DTFTs of Sums of Complex Exponentials

Recall CT result: $x(t) = e^{j\omega_0 t} \longleftrightarrow X(j\omega) = 2\pi\delta(\omega - \omega_0)$

What about DT: $x[n] = e^{j\omega_0 n} \longleftrightarrow X(e^{j\omega}) = ?$

- a) We expect an impulse (of area 2π) at $\omega = \omega_0$
- b) But $X(e^{j\omega})$ must be periodic with period 2π

In fact

$$X(e^{j\omega}) = 2\pi \sum_{m=-\infty}^{\infty} \delta(\omega - \omega_0 - 2\pi m)$$

Note: The integration in the synthesis equation is over 2π period, only need $X(e^{j\omega})$ in *one* 2π period. Thus,

$$x[n] = \frac{1}{2\pi} \int_{-\pi}^{\pi} \underbrace{2\pi \sum_{m=-\infty}^{\infty} \delta(\omega - \omega_0 - 2\pi m)}_{X(e^{j\omega})} e^{j\omega n} d\omega = e^{j\omega_0 n}$$

DTFT of Periodic Signals

$$x[n] = x[n + N]$$

$$x[n] = \sum_{k=\langle N \rangle} a_k e^{jk\omega_0 n}, \quad \omega_0 = \frac{2\pi}{N} \quad \text{DTFS synthesis eq.}$$

From the last page: $e^{jk\omega_0 n} \longleftrightarrow 2\pi \sum_{m=-\infty}^{\infty} \delta(\omega - k\omega_0 - 2\pi m)$

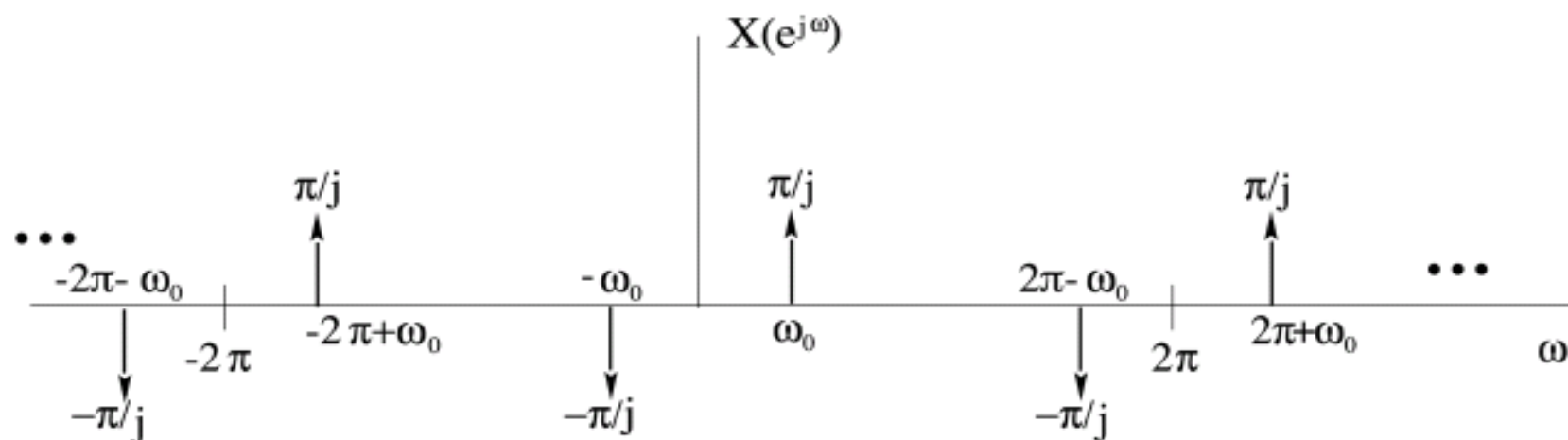
$$X(e^{j\omega}) = \sum_{k=\langle N \rangle} a_k \left[2\pi \sum_{m=-\infty}^{\infty} \delta(\omega - k\omega_0 - 2\pi m) \right] \quad \text{Linearity of DTFT}$$

$$= 2\pi \sum_{k=-\infty}^{\infty} a_k \delta(\omega - k\omega_0) = 2\pi \sum_{k=-\infty}^{\infty} a_k \delta\left(\omega - \frac{2\pi k}{N}\right)$$

Example : DT sine function

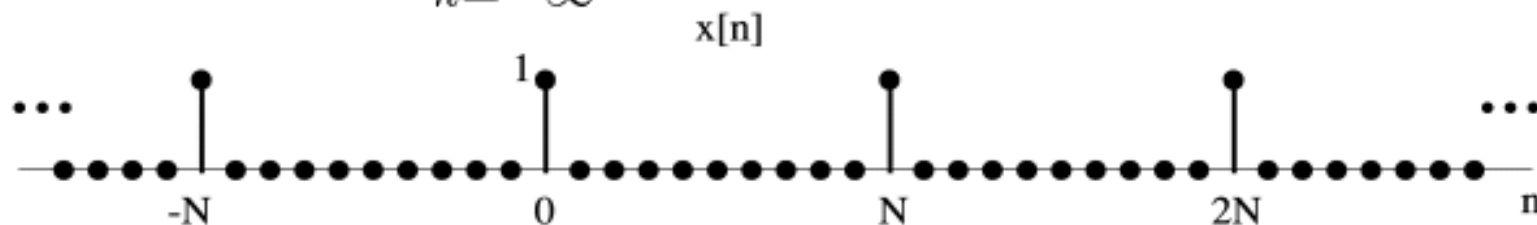
$$x[n] = \sin \omega_0 n = \frac{1}{2j} e^{j\omega_0 n} - \frac{1}{2j} e^{-j\omega_0 n}$$

$$X(e^{j\omega}) = \frac{\pi}{j} \sum_{m=-\infty}^{\infty} \delta(\omega - \omega_0 - 2\pi m) - \frac{\pi}{j} \sum_{m=-\infty}^{\infty} \delta(\omega + \omega_0 - 2\pi m)$$



Example : DT periodic impulse train

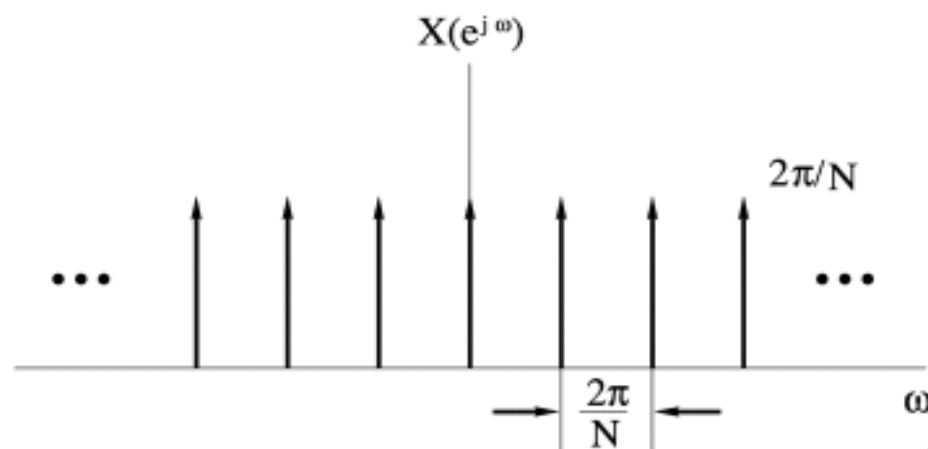
$$x[n] = \sum_{k=-\infty}^{\infty} \delta[n - kN] \quad \omega_0 = 2\pi/N$$



$$\begin{aligned} a_k &= \frac{1}{N} \sum_{n=\langle N \rangle} x[n] e^{-jk\omega_0 n} \\ &= \frac{1}{N} \sum_{n=0}^{N-1} \underbrace{x[n]}_{=\delta[n]} e^{-jk\omega_0 n} = \frac{1}{N} \end{aligned}$$

$$\Downarrow$$

$$X(e^{j\omega}) = \frac{2\pi}{N} \sum_{k=-\infty}^{\infty} \delta\left(\omega - \frac{2\pi k}{N}\right)$$



— Also periodic impulse train – in the frequency domain!

Properties of the DT Fourier Transform

$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n} \quad - \text{Analysis equation}$$

$$x[n] = \frac{1}{2\pi} \int_{2\pi} X(e^{j\omega})e^{j\omega n} d\omega \quad - \text{Synthesis equation}$$

1) Periodicity: $X(e^{j(\omega+2\pi)}) = X(e^{j\omega})$ — Different from CTFT

2) Linearity: $ax_1[n] + bx_2[n] \longleftrightarrow aX_1(e^{j\omega}) + bX_2(e^{j\omega})$

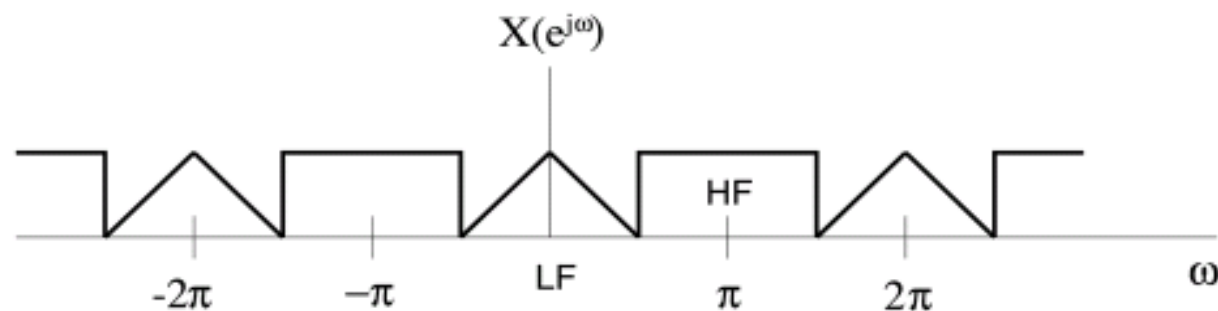
More Properties

3) Time Shifting: $x[n - n_0] \longleftrightarrow e^{-j\omega n_0} X(e^{j\omega})$

4) Frequency Shifting: $e^{j\omega_0 n} x[n] \longleftrightarrow X(e^{j(\omega - \omega_0)})$

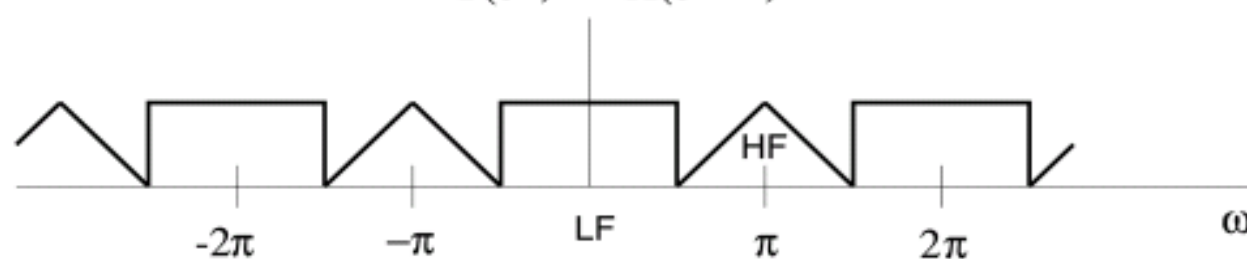
— Important implications in DT because of periodicity

Example



$$\omega_0 = \pi, \quad y[n] = e^{j\pi n} x[n] = (-1)^n x[n]$$

$$Y(e^{j\omega}) = X(e^{j(\omega - \pi)})$$



More Properties

5) Time Reversal:

$$x[-n] \longleftrightarrow X(e^{-j\omega})$$

6) Conjugate Symmetry:

$$x[n] \text{ real} \Rightarrow X(e^{j\omega}) = X^*(e^{-j\omega})$$

$\Downarrow$

$|X(e^{j\omega})|$ and $\Re \{X(e^{j\omega})\}$ are even functions
 $\angle X(e^{j\omega})$ and $\Im \{X(e^{j\omega})\}$ are odd functions

and

$x[n]$ real and even $\Leftrightarrow X(e^{j\omega})$ real and even
 $x[n]$ real and odd $\Leftrightarrow X(e^{j\omega})$ purely imaginary and odd

More Properties

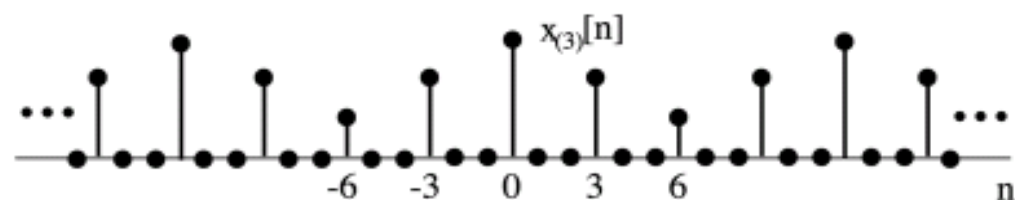
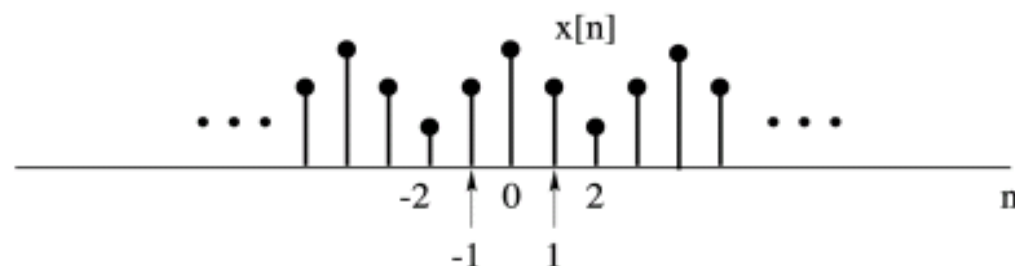
- 7) Time Expansion
 Recall CT property: $x(at) \longleftrightarrow \frac{1}{|a|} X\left(j\left(\frac{\omega}{a}\right)\right)$ Time scale in CT is infinitely fine

But in DT: $x[n/2]$ makes no sense
 $x[2n]$ misses odd values of $x[n]$

But we can “slow” a DT signal down by inserting zeros:

k — an integer ≥ 1

$x_{(k)}[n]$ — insert $(k - 1)$ zeros between successive values

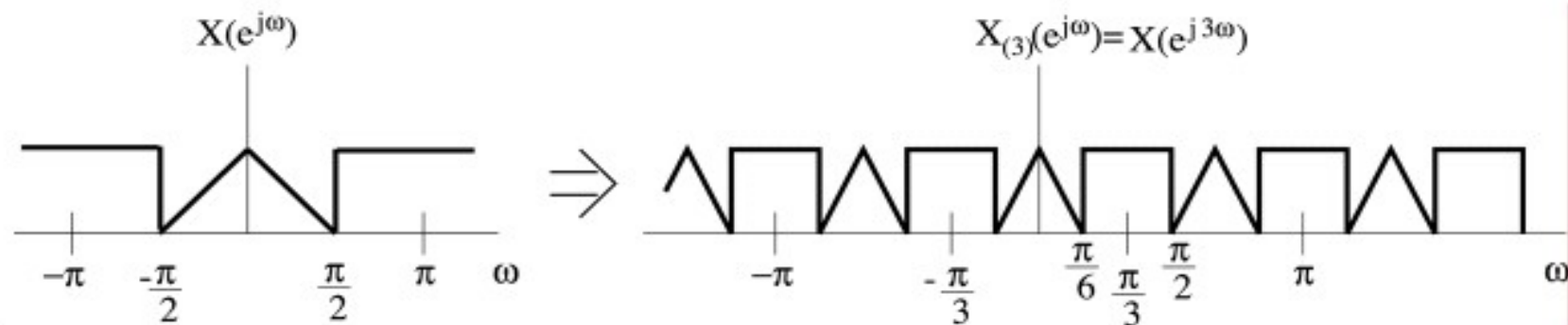


Insert two zeros
 in this example
 ($k=3$)

Time Expansion (continued)

$$x_{(k)}[n] = \begin{cases} x[n/k] & \text{if } n \text{ is an integer multiple of } k \\ 0 & \text{otherwise} \end{cases} \quad \text{— Stretched by a factor of } k \text{ in time domain}$$

$$\begin{aligned} X_{(k)}(e^{j\omega}) &= \sum_{n=-\infty}^{\infty} x_{(k)}[n] e^{-j\omega n} \stackrel{n=mk}{=} \sum_{m=-\infty}^{\infty} x_{(k)}[mk] e^{-j\omega mk} \\ &= \sum_{m=-\infty}^{\infty} x[m] e^{-j(k\omega)m} = X(e^{jk\omega}) \quad \text{-compressed by a factor of } k \text{ in frequency domain} \end{aligned}$$



8) Differentiation in Frequency

$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n}$$

$$\frac{d}{d\omega}X(e^{j\omega}) = -j \sum_{n=-\infty}^{\infty} nx[n]e^{-j\omega n}$$

↓ multiply by j on both sides

$$\text{Multiplication by } n \quad nx[n] \leftrightarrow j \frac{d}{d\omega} X(e^{j\omega}) \quad \text{Differentiation in frequency}$$

9) Parseval's Relation

$$\underbrace{\sum_{n=-\infty}^{\infty} |x[n]|^2}_{\text{Total energy in time domain}} = \underbrace{\frac{1}{2\pi} \int_{2\pi} |X(e^{j\omega})|^2 d\omega}_{\text{Total energy in frequency domain}}$$

Properties Continued

Differencing

$$x[n] - x[n-1] \xleftrightarrow{\mathcal{F}} (1 - e^{-j\omega})X(e^{j\omega})$$

This is similar to a continuous-time derivative.

Accumulation

$$\sum_{m=-\infty}^n x[m] \xleftrightarrow{\mathcal{F}} \frac{1}{1 - e^{-j\omega}}X(e^{j\omega}) + \pi X(e^{j0}) \sum_{\ell=-\infty}^{\infty} \delta(\omega - 2\pi\ell)$$

This is similar to a continuous-time integration.

The Convolution Property



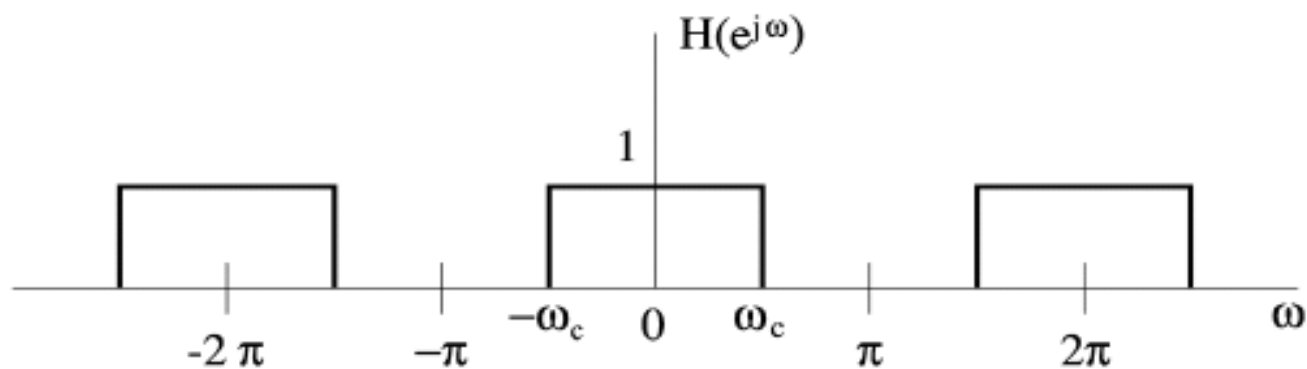
$$Y(e^{j\omega}) = H(e^{j\omega})X(e^{j\omega})$$

$\Rightarrow$ Frequency response $H(e^{j\omega}) = \text{DTFT of the unit sample response}$

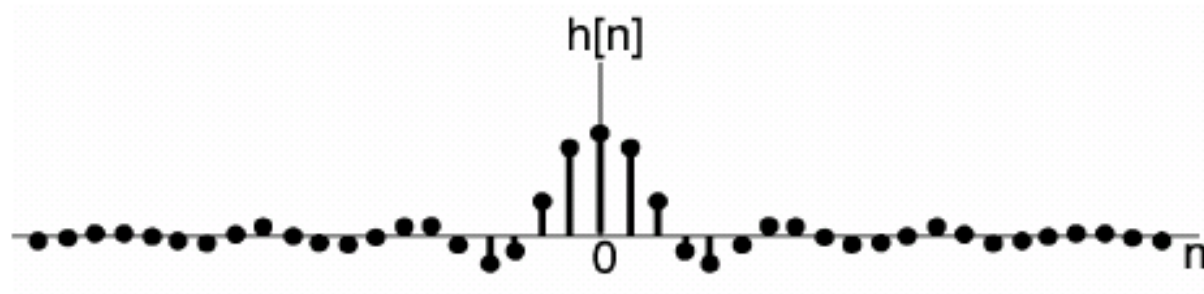
Example :

$$\begin{aligned}
 x[n] = e^{j\omega_0 n} &\longleftrightarrow X(e^{j\omega}) &= & 2\pi \sum_{k=-\infty}^{\infty} \delta(\omega - \omega_0 - 2\pi k) \\
 Y(e^{j\omega}) &= & H(e^{j\omega}) 2\pi \sum_{k=-\infty}^{\infty} \delta(\omega - \omega_0 - 2\pi k) \\
 &= & 2\pi \sum_{k=-\infty}^{\infty} H(e^{j(\omega_0 + 2\pi k)}) \delta(\omega - \omega_0 - 2\pi k) \\
 &\stackrel{H \text{ Periodic}}{=} & H(e^{j\omega_0}) 2\pi \sum_{k=-\infty}^{\infty} \delta(\omega - \omega_0 - 2\pi k) \\
 &\Downarrow & \\
 y[n] &= & H(e^{j\omega_0}) e^{j\omega_0 n}
 \end{aligned}$$

Example : Ideal Lowpass Filter

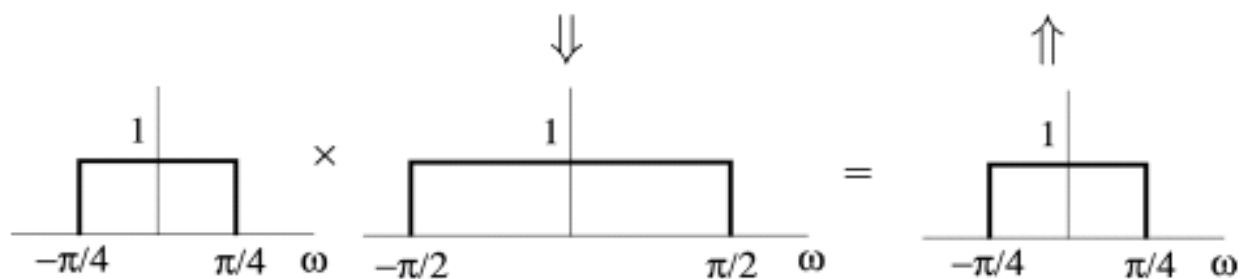


$$h[n] = \frac{1}{2\pi} \int_{-\omega_c}^{\omega_c} e^{j\omega n} d\omega = \frac{\sin \omega_c n}{\pi n}$$



Example

$$\frac{\sin(\pi n/4)}{\pi n} * \frac{\sin(\pi n/2)}{\pi n} = \frac{\sin(\pi n/4)}{\pi n}$$



Convolution Property Example

$$h[n] = \alpha^n u[n], \quad x[n] = \beta^n u[n] \quad |\alpha|, |\beta| < 1$$

$$H(e^{j\omega}) = \frac{1}{1 - \alpha e^{-j\omega}}, \quad X(e^{j\omega}) = \frac{1}{1 - \beta e^{-j\omega}}$$

$$\Downarrow$$
$$y[n] = h[n] * x[n] \longleftrightarrow Y(e^{j\omega}) = \left(\frac{1}{1 - \alpha e^{-j\omega}} \right) \left(\frac{1}{1 - \beta e^{-j\omega}} \right)$$

- ratio of polynomials in $e^{-j\omega}$

$$\beta \neq \alpha : Y(e^{j\omega}) \stackrel{\text{PFE}}{=} \frac{A}{1 - \alpha e^{-j\omega}} + \frac{B}{1 - \beta e^{-j\omega}}$$

A, B - determined by partial fraction expansion

$$y[n] = A\alpha^n u[n] + B\beta^n u[n]$$

$$\beta = \alpha : Y(e^{j\omega}) = \left(\frac{1}{1 - \alpha e^{-j\omega}} \right)^2$$

$$y[n] = \overbrace{\frac{1}{n\alpha^n}}^{nx[n] \leftrightarrow j \frac{dX(e^{j\omega})}{d\omega}} \quad y[n] = (n+1)\alpha^n u[n]$$

DT LTI System Described by LCCDE's

$$\sum_{k=0}^N a_k y[n-k] = \sum_{k=0}^M b_k x[n-k]$$

From time-shifting property: $x[n-k] \longleftrightarrow e^{-jk\omega} X(e^{j\omega})$

$\Downarrow$

$$\sum_{k=0}^N a_k e^{-jk\omega} Y(e^{j\omega}) = \sum_{k=0}^M b_k e^{-jk\omega} X(e^{j\omega})$$

$\Downarrow$

$$Y(e^{j\omega}) = \underbrace{\left[\frac{\sum_{k=0}^M b_k e^{-jk\omega}}{\sum_{k=0}^N a_k e^{-jk\omega}} \right]}_{H(e^{j\omega})} X(e^{j\omega})$$

— Rational function of $e^{j\omega}$,
use PFE to get $h[n]$

DT transfer function

Example: First-order recursive system

$$y[n] - \alpha y[n-1] = x[n], \quad |\alpha| < 1$$

with the condition of initial rest $\Leftrightarrow$ *causal*

$$(1 - \alpha e^{-j\omega})Y(e^{j\omega}) = X(e^{j\omega})$$

$$H(e^{j\omega}) = \frac{Y(e^{j\omega})}{X(e^{j\omega})} = \frac{1}{1 - \alpha e^{-j\omega}}$$

$\Downarrow$

$$h[n] = \alpha^n u[n]$$

DTFT Multiplication Property

$$\begin{aligned}y[n] = x_1[n] \cdot x_2[n] &\longleftrightarrow Y(e^{j\omega}) = \frac{1}{2\pi} \int_{2\pi} X_1(e^{j\theta}) X_2(e^{j(\omega-\theta)}) d\theta \\&= \frac{1}{2\pi} X_1(e^{j\omega}) \otimes X_2(e^{j\omega}) \\&\hookrightarrow \text{Periodic Convolution}\end{aligned}$$

Derivation:

$$\begin{aligned}Y(e^{j\omega}) &= \sum_{n=-\infty}^{\infty} x_1[n] \cdot x_2[n] e^{-j\omega n} \\&= \sum_{n=-\infty}^{\infty} \left(\frac{1}{2\pi} \int_{2\pi} X_1(e^{j\theta}) e^{j\theta n} d\theta \right) x_2[n] e^{-j\omega n} \\&= \frac{1}{2\pi} \int_{2\pi} X_1(e^{j\theta}) \underbrace{\sum_{n=-\infty}^{\infty} x_2[n] e^{-j(\omega-\theta)n}}_{X_2(e^{j(\omega-\theta)})} d\theta \\&= \frac{1}{2\pi} \int_{2\pi} X_1(e^{j\theta}) X_2(e^{j(\omega-\theta)}) d\theta\end{aligned}$$

Calculating Periodic Convolutions

Suppose we integrate from $-\pi$ to π :

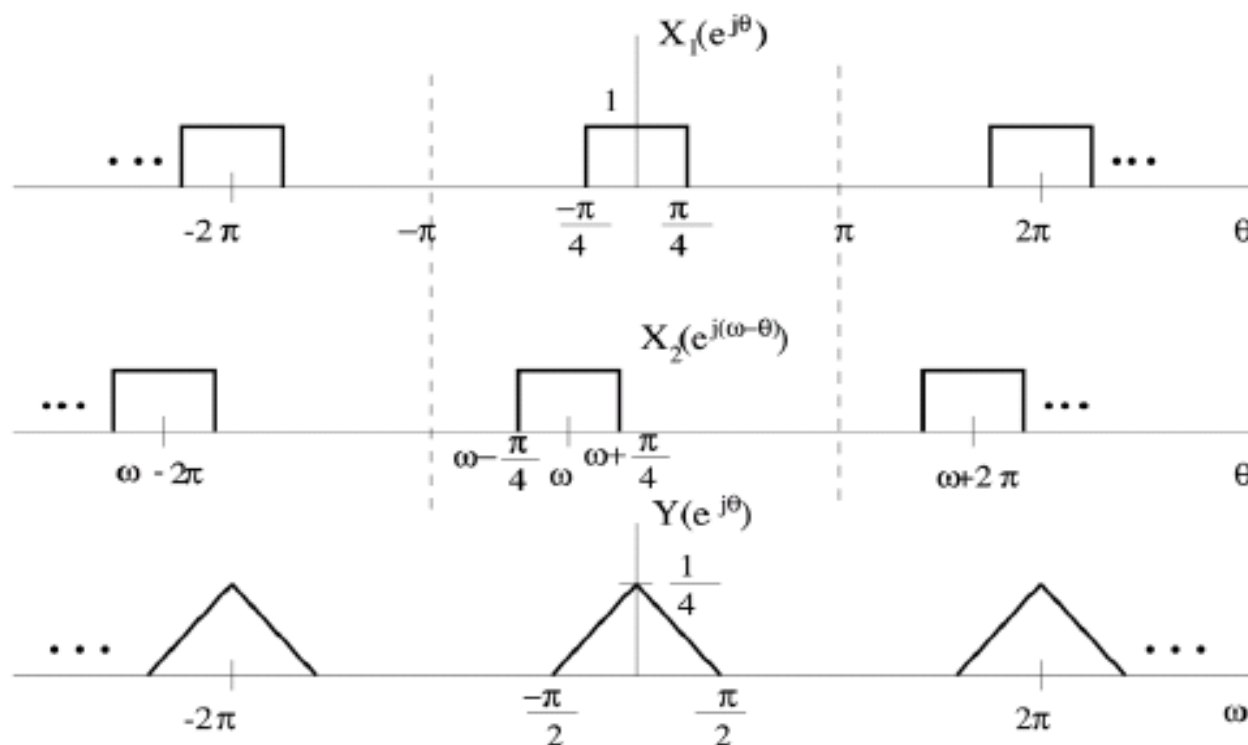
$$\begin{aligned} Y(e^{j\omega}) &= \frac{1}{2\pi} \int_{-\pi}^{\pi} X_1(e^{j\theta}) X_2(e^{j(\omega-\theta)}) d\theta \\ &= \frac{1}{2\pi} \int_{-\infty}^{\infty} \hat{X}_1(e^{j\theta}) X_2(e^{j(\omega-\theta)}) d\theta \end{aligned}$$

where

$$\hat{X}_1(e^{j\theta}) = \begin{cases} X_1(e^{j\theta}), & |\theta| \leq \pi \\ 0, & \text{otherwise} \end{cases}$$

Example: $y[n] = \left(\frac{\sin(\pi n/4)}{\pi n} \right)^2 = x_1[n] \cdot x_2[n], \quad x_1[n] = x_2[n] = \frac{\sin(\pi n/4)}{\pi n}$

$$Y(e^{j\omega}) = \frac{1}{2\pi} X_1(e^{j\omega}) \otimes X_2(e^{j\omega})$$



Duality in Fourier Analysis

Fourier Transform is highly symmetric

CTFT: *Both* time and frequency are continuous and in general *aperiodic*

$$x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(j\omega) e^{j\omega t} d\omega$$

Same except for
these differences

$$X(j\omega) = \int_{-\infty}^{\infty} x(t) e^{-j\omega t} dt$$

$\Downarrow$

Suppose $f(\cdot)$ and $g(\cdot)$ are two functions related by

$$f(r) = \int_{-\infty}^{\infty} g(\tau) e^{-jr\tau} d\tau$$

Then

Let $\tau = t$ and $r = \omega$:

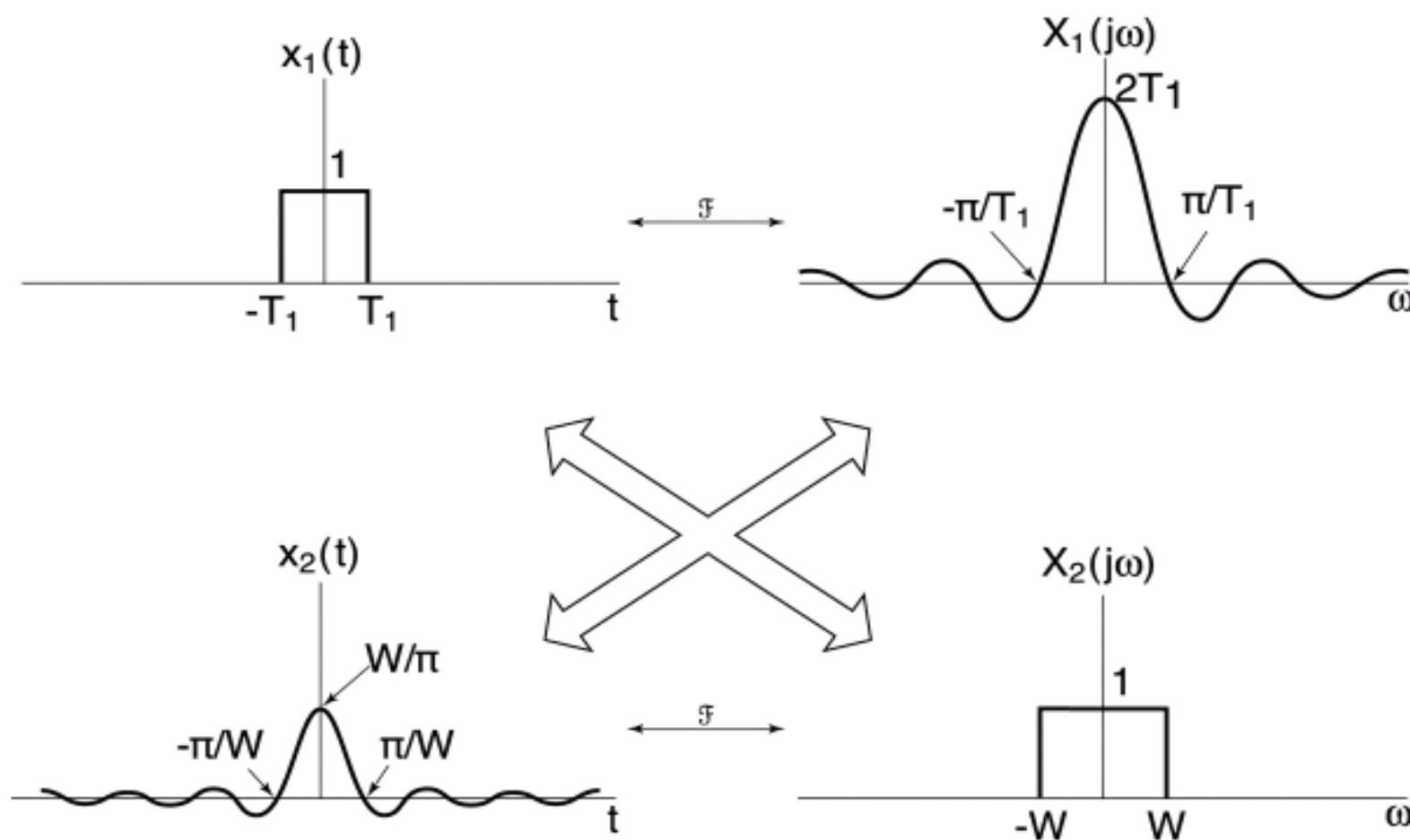
$$x_1(t) = g(t) \longleftrightarrow X_1(j\omega) = f(\omega)$$

Let $\tau = -\omega$ and $r = t$:

$$x_2(t) = f(t) \longleftrightarrow X_2(j\omega) = 2\pi g(-\omega)$$

Example of CTFT duality

Square pulse in either time or frequency domain



DTFS

Discrete & periodic in time $\longleftrightarrow$ *Periodic & discrete in frequency*

$$x[n] = \sum_{k=\langle N \rangle} a_k e^{jk\omega_0 n} = x[n+N], \quad \omega_0 = \frac{2\pi}{N}$$

$$a_k = \frac{1}{N} \sum_{n=\langle N \rangle} x[n] e^{-jk\omega_0 n} = a_{k+N}$$

Duality in DTFS

Suppose $f[\cdot]$ and $g[\cdot]$ are two functions related by

$$f[m] = \frac{1}{N} \sum_{r=\langle N \rangle} g[r] e^{-jr\omega_0 m}$$

$$\Rightarrow g[r] = \sum_{m=\langle N \rangle} f[m] e^{jr\omega_0 m}$$

Then

$$\text{Let } m = n \text{ and } r = -k: \quad x_1[n] = f[n] \longleftrightarrow a_k = \frac{1}{N} g[-k]$$

$$\text{Let } r = n \text{ and } m = k: \quad x_2[n] = g[n] \longleftrightarrow a_k = f[k]$$

Duality between CTFS and DTFT

CTFS $x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t} = x(t + T), \quad \omega_0 = \frac{2\pi}{T}$

$$a_k = \frac{1}{T} \int_T x(t) e^{-jk\omega_0 t} dt$$

Periodic in time $\leftrightarrow$ Discrete in frequency

DTFT

$$x[n] = \frac{1}{2\pi} \int_{2\pi} X(e^{j\omega}) e^{j\omega n} d\omega$$

$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n] e^{-j\omega n} = X(e^{j(\omega+2\pi)})$$

Discrete in time $\leftrightarrow$ Periodic in frequency

CTFS-DTFT Duality

Suppose $f(\cdot)$ is a CT signal and $g[\cdot]$ a DT sequence related by

$$f(\tau) = \sum_{m=-\infty}^{+\infty} g[m]e^{jm\tau} = f(\tau + 2\pi)$$

Then

$$x(t) = f(t) \longleftrightarrow a_k = g[k] \\ \text{(periodic with period } 2\pi)$$

$$x[n] = g[n] \longleftrightarrow X(e^{j\omega}) = f(-\omega)$$

Magnitude and Phase of FT, and Parseval Relation

CT:
$$x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(j\omega) e^{j\omega t} d\omega$$

$$X(j\omega) = |X(j\omega)| e^{j\angle X(j\omega)}$$

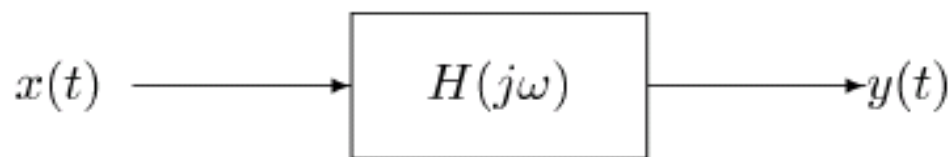
Parseval Relation:
$$\int_{-\infty}^{\infty} |x(t)|^2 dt = \int_{-\infty}^{\infty} \frac{1}{2\pi} \underbrace{|X(j\omega)|^2}_{\text{Energy density in } \omega} d\omega$$

DT:
$$x[n] = \frac{1}{2\pi} \int_{2\pi} X(e^{j\omega}) e^{j\omega n} d\omega$$

$$X(e^{j\omega}) = |X(e^{j\omega})| e^{j\angle X(e^{j\omega})}$$

Parseval Relation:
$$\sum_{n=-\infty}^{\infty} |x[n]|^2 = \int_{2\pi} \frac{1}{2\pi} |X(e^{j\omega})|^2 d\omega$$

Log-Magnitude and Phase

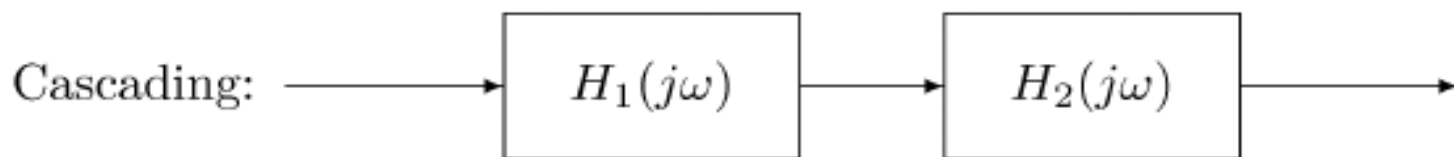


$\Downarrow$

$$|Y(j\omega)| = |H(j\omega)| \cdot |X(j\omega)|$$

or $\log |Y(j\omega)| = \log |H(j\omega)| + \log |X(j\omega)|$

and $\angle Y(j\omega) = \angle H(j\omega) + \angle X(j\omega)$



$$\left. \begin{aligned} \log |H(j\omega)| &= \log |H_1(j\omega)| + \log |H_2(j\omega)| \\ \angle H(j\omega) &= \angle H_1(j\omega) + \angle H_2(j\omega) \end{aligned} \right\} \text{Easy to add}$$

Plotting Log-Magnitude and Phase

- a) For real-valued signals and systems

$$\left. \begin{aligned} |H(-j\omega)| &= |H(j\omega)| \\ \angle H(-j\omega) &= -\angle H(j\omega) \end{aligned} \right\} \Rightarrow \text{Plot for } \omega \geq 0, \text{ often with a } \textit{logarithmic} \text{ scale for frequency in CT}$$

- b) In DT, need only plot for $0 \leq \omega \leq \pi$ (with *linear* scale)

- c) For historical reasons, log-magnitude is usually plotted in units of *decibels* (dB): $(1 \text{ bel} = 10 \text{ decibels} = \frac{\text{output power}}{\text{input power}} = 10)$

$$10 \log_{10} |H(j\omega)|^2 \overset{\text{power}}{=} 20 \log_{10} |H(j\omega)| \overset{\text{magnitude}}{=}$$

$$|H(j\omega)| = 1 \longrightarrow 0 \text{ dB}$$

$$|H(j\omega)| = \sqrt{2} \longrightarrow \sim 3 \text{ dB}$$

$$|H(j\omega)| = 2 \longrightarrow \sim 6 \text{ dB}$$

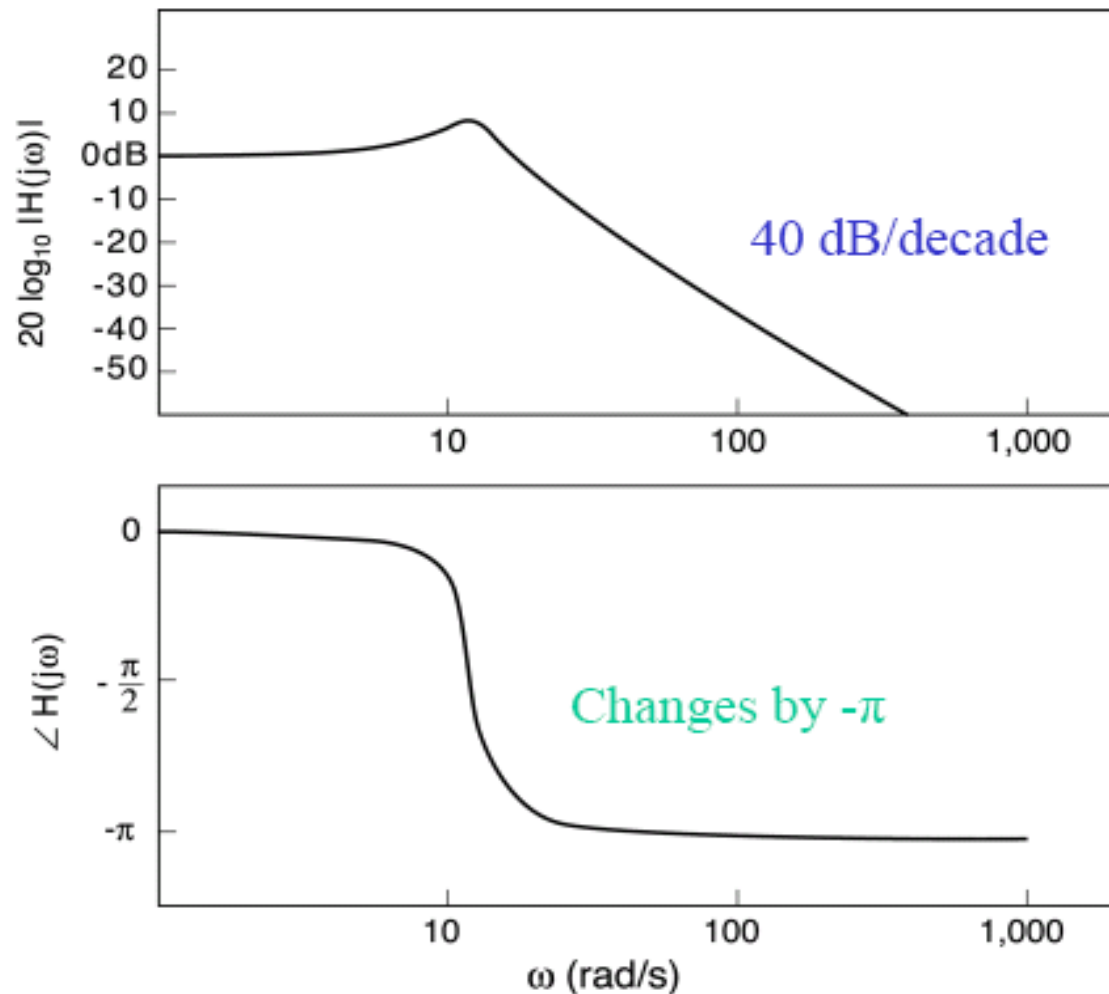
$$|H(j\omega)| = 10 \longrightarrow 20 \text{ dB}$$

$$|H(j\omega)| = 100 \longrightarrow 40 \text{ dB}$$

So... 20 dB or 2 bels:
 = 10 amplitude gain
 = 100 power gain

A Typical Bode plot for a second-order CT system

$20 \log|H(j\omega)|$ and $\angle H(j\omega)$ vs. $\log \omega$



A typical plot of the magnitude and phase of a second-order DT frequency response

$20\log|H(e^{j\omega})|$ and $\angle H(e^{j\omega})$ vs. ω

