

DT FOURIER SERIES

Fourier Series Representation
Comparison of DTFS and CTFS
Fourier Series and LTI Systems
Frequency Response
Frequency Selective Filters

Fourier Series Representation of DT Periodic Signals

- $x[n]$ - periodic with fundamental period N , fundamental frequency

$$x[n + N] = x[n] \quad \text{and} \quad \omega_0 = \frac{2\pi}{N}$$

- Only $e^{j\omega n}$ which are periodic with period N will appear in the FS

$$\omega N = k2\pi \Leftrightarrow \omega = k\omega_0 \quad , \quad k = 0, \pm 1, \pm 2, \dots$$

- There are only N distinct signals of this form

$$e^{j(k+N)\omega_0 n} = e^{jk\omega_0 n} \overbrace{e^{jN\omega_0 n}}^{2\pi n} = e^{jk\omega_0 n}$$

- So we *could* just use $e^{j0\omega_0 n}, e^{j1\omega_0 n}, e^{j2\omega_0 n}, \dots, e^{j(N-1)\omega_0 n}$
- However, it is often useful to allow the choice of N consecutive values of k to be *arbitrary*.



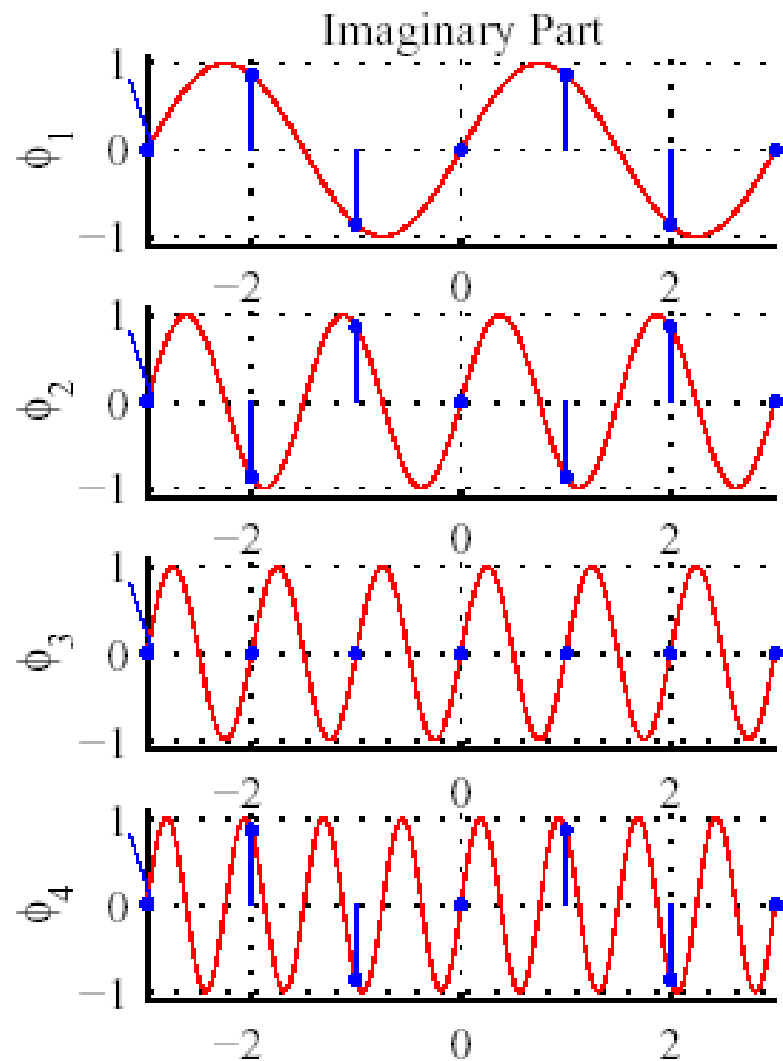
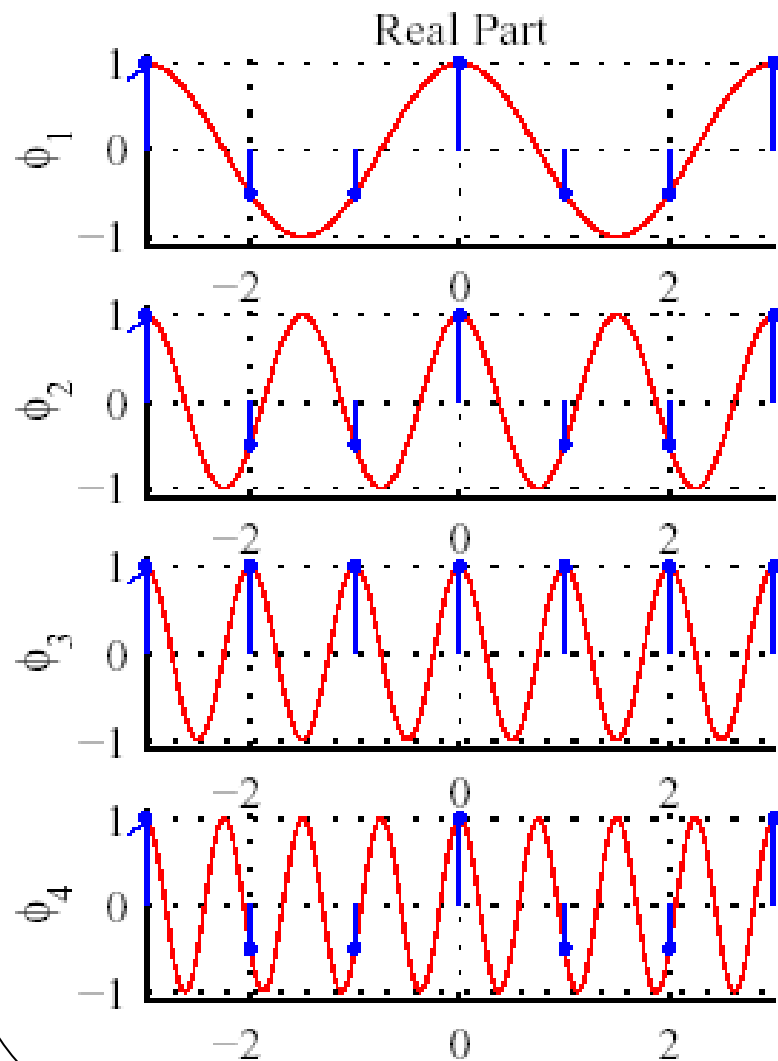
Discrete-Time Exponential Harmonics

- The set of all discrete-time complex exponential signals that are periodic with period N can be expressed as

$$\phi_k[n] = e^{jk\omega n} = e^{jk\frac{2\pi}{N}n}, \quad k = 0, \pm 1, \pm 2, \dots$$

- $e^{j\omega n}$ is called the **fundamental component**
- $e^{j2\omega n}$ is called the **2nd harmonic component**
- $e^{jk\omega n}$ is called the **k th harmonic component**

Example : Discrete-Time Harmonics ($N = 3$)



DT Fourier Series Representation

$$x[n] = \sum_{k=\langle N \rangle} a_k e^{jk(2\pi/N)n}$$

$\sum_{k=\langle N \rangle}$ = Sum over *any* N consecutive values of k

— This is a *finite* series

$\{a_k\}$ - Fourier (series) coefficients

Questions:

- 1) What DT periodic signals have such a representation?
- 2) How do we find a_k ?

Answer to Question #1:

Any DT periodic signal has a Fourier series representation

$$\begin{aligned}x[n] &= \sum_{k=\langle N \rangle} a_k e^{jk\omega_0 n} \\&\Downarrow \\x[0] &= \sum_{k=\langle N \rangle} a_k \\x[1] &= \sum_{k=\langle N \rangle} a_k e^{jk\omega_0} \\x[2] &= \sum_{k=\langle N \rangle} a_k e^{j2k\omega_0} \\&\vdots \\x[N-1] &= \sum_{k=\langle N \rangle} a_k e^{j(N-1)k\omega_0}\end{aligned}$$

N equations for N unknowns, $a_0, a_1, \dots, a_{N-1}$

Equivalent Expressions for Exponential Sums

$$x[n] = \sum_k a_k e^{jk\omega n}$$

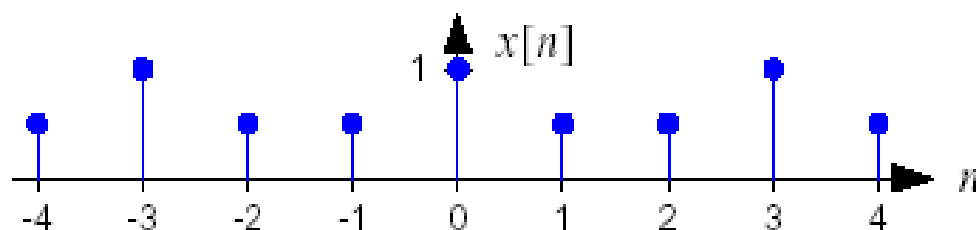
Since only N of the harmonics are distinct, we may truncate the sum so that it only contains N terms

$$x[n] = \sum_{k=0}^{N-1} a_k e^{jk\omega n} = \sum_{k=\ell}^{N-1+\ell} a_k e^{jk\omega n} = \sum_{k=\langle N \rangle} a_k e^{jk\omega n}$$

Note the new notation:

$$\sum_{k=\langle N \rangle} a_k e^{jk\omega n} \triangleq \sum_{k=1+\ell}^{N+\ell} a_k e^{jk\omega n} \text{ for any } \ell$$

Discrete-Time Fourier Series



$$x[n] = \begin{cases} 1.0 & n = 3k \\ 0.5 & \text{Otherwise} \end{cases} = \sum_{k=\langle N \rangle} a_k \phi_k[n] = \sum_{k=\langle N \rangle} a_k e^{jk\omega n}$$

- *Harmonically* related exponentials can be combined linearly to form a **Fourier series**
- Any finite-valued discrete-time periodic signal can be written in this form
- The trick (again) is how to find the coefficients a_k

A More Direct Way to Solve for a_k

Finite geometric series

$$\sum_{n=0}^{N-1} \alpha^n = \begin{cases} N & , \alpha = 1 \\ \frac{1 - \alpha^N}{1 - \alpha} & , \alpha \neq 1 \end{cases}$$

$$\Downarrow \quad \alpha = e^{jk\omega_0}$$

$$\sum_{n=\langle N \rangle} e^{jk\omega_0 n} = \sum_{n=0}^{N-1} (e^{jk\omega_0})^n = \sum_{n=0}^{N-1} \left(e^{jk2\pi/N} \right)^n$$

$$= \begin{cases} N & , k = 0, \pm N, \pm 2N, \dots \\ \frac{1 - e^{jk(2\pi/N)N}}{1 - e^{jk\omega_0}} = 0 & , \text{otherwise} \end{cases}$$

So, from $x[n] = \sum_{k=\langle N \rangle} a_k e^{jk\omega_0 n}$



multiply both sides by $e^{-jm\omega_0 n}$
and then $\sum_{n=\langle N \rangle}$

$$\begin{aligned} \sum_{n=\langle N \rangle} x[n] e^{-jm\omega_0 n} &= \sum_{n=\langle N \rangle} \left(\sum_{k=\langle N \rangle} a_k e^{jk\omega_0 n} \right) e^{-jm\omega_0 n} \\ &= \sum_{k=\langle N \rangle} a_k \underbrace{\left(\sum_{n=\langle N \rangle} e^{j(k-m)\omega_0 n} \right)}_{=N\delta[k-m] - \text{orthogonality}} \\ &= Na_m \\ &\Downarrow \end{aligned}$$

DT Fourier Series Pair $\left(\omega_0 = \frac{2\pi}{N}\right)$

$$x[n] = \sum_{k=\langle N \rangle} a_k e^{jk\omega_0 n} \quad (\text{Synthesis equation})$$

$$a_k = \frac{1}{N} \sum_{n=\langle N \rangle} x[n] e^{-jk\omega_0 n} \quad (\text{Analysis equation})$$

Note: It is convenient to think of a_k as being defined for *all* integers k . So:

- 1) $a_{k+N} = a_k$ — Special property of DT Fourier Coefficients.
- 2) We only use N consecutive values of a_k in the synthesis equation. (Since $x[n]$ is periodic, it is specified by N numbers, either in the time or frequency domain)
- 3) a_k are called the **spectral** or **FS coefficients**

Example #1: Sum of a pair of sinusoids

$$x[n] = \cos(\pi n/8) + \cos(\pi n/4 + \pi/4)$$

– periodic with period $N = 16 \Rightarrow \omega_0 = \pi/8$

$$x[n] = \frac{1}{2} [e^{j\omega_0 n} + e^{-j\omega_0 n}] + \frac{1}{2} [e^{j\pi/4} e^{j2\omega_0 n} + e^{-j\pi/4} e^{-j2\omega_0 n}]$$

$$a_0 = 0$$

$$a_1 = 1/2$$

$$a_{-1} = 1/2$$

$$a_2 = e^{j\pi/4}/2$$

$$a_{-2} = e^{j\pi/4}/2$$

$$a_3 = 0$$

$$a_{-3} = 0$$

$$\vdots$$
$$\Downarrow$$

$$a_{15} = a_{-1+16} = a_{-1} = 1/2$$

$$a_{66} = a_{2+4 \times 16} = a_2 = e^{j\pi/4}/2$$

Example #2: DT Square Wave



$$a_0 = \frac{1}{N} \sum_{n=-N_1}^{N_1} x[n] = \frac{2N_1 + 1}{N} = a_N = a_{-N} = a_{6N} = \dots$$

For $k \neq$ multiple of N :

Using $n = m - N_1$

$$a_k = \frac{1}{N} \sum_{n=-N_1}^{N_1} e^{-jk\omega_0 n} = \frac{1}{N} \sum_{m=0}^{2N_1} e^{-jk\omega_0 (m - N_1)}$$

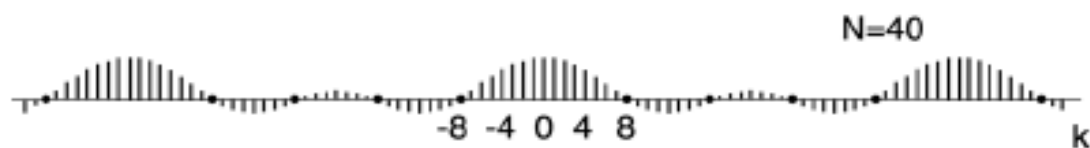
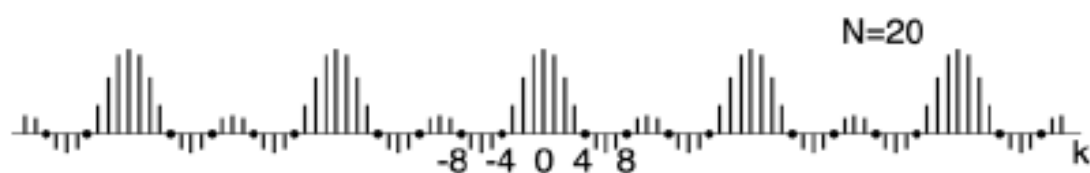
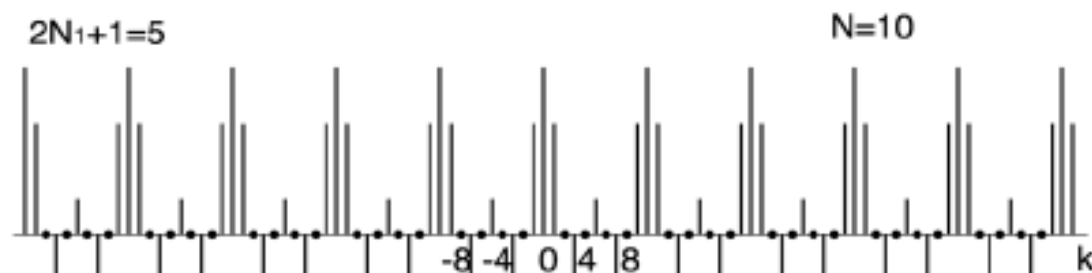
$$= \frac{1}{N} e^{jk\omega_0 N_1} \sum_{m=0}^{2N_1} (e^{-jk\omega_0})^m = \frac{1}{N} e^{jk\omega_0 N_1} \frac{1 - e^{-jk\omega_0 (2N_1 + 1)}}{1 - e^{jk\omega_0}}$$

$$= \frac{1}{N} \frac{\sin[k(N_1 + 1/2)\omega_0]}{\sin(k\omega_0/2)} = \frac{1}{N} \frac{\sin[2\pi k(N_1 + 1/2)/N]}{\sin(\pi k/N)}$$

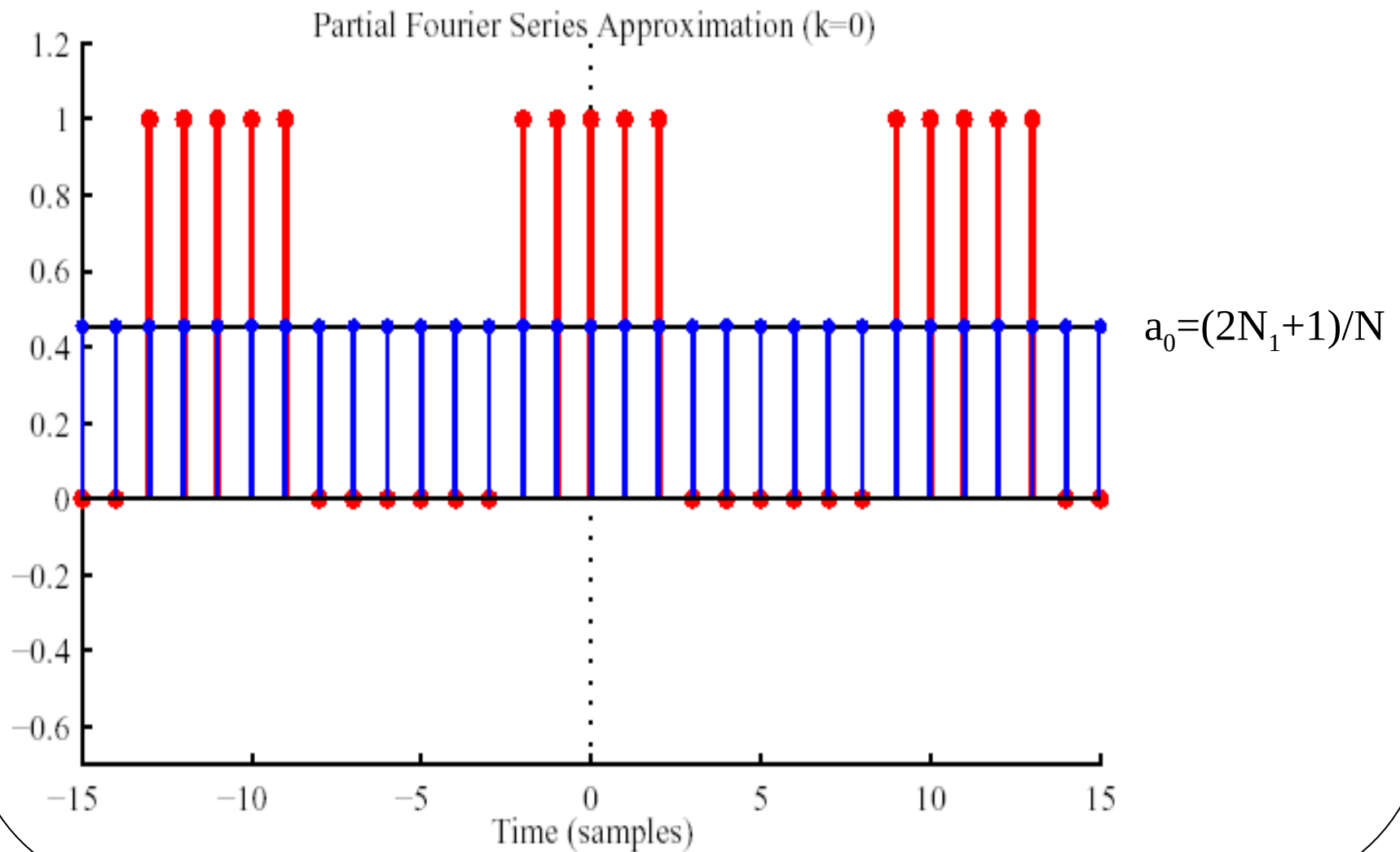
$\Downarrow$

Example #2: DT Square wave (continued)

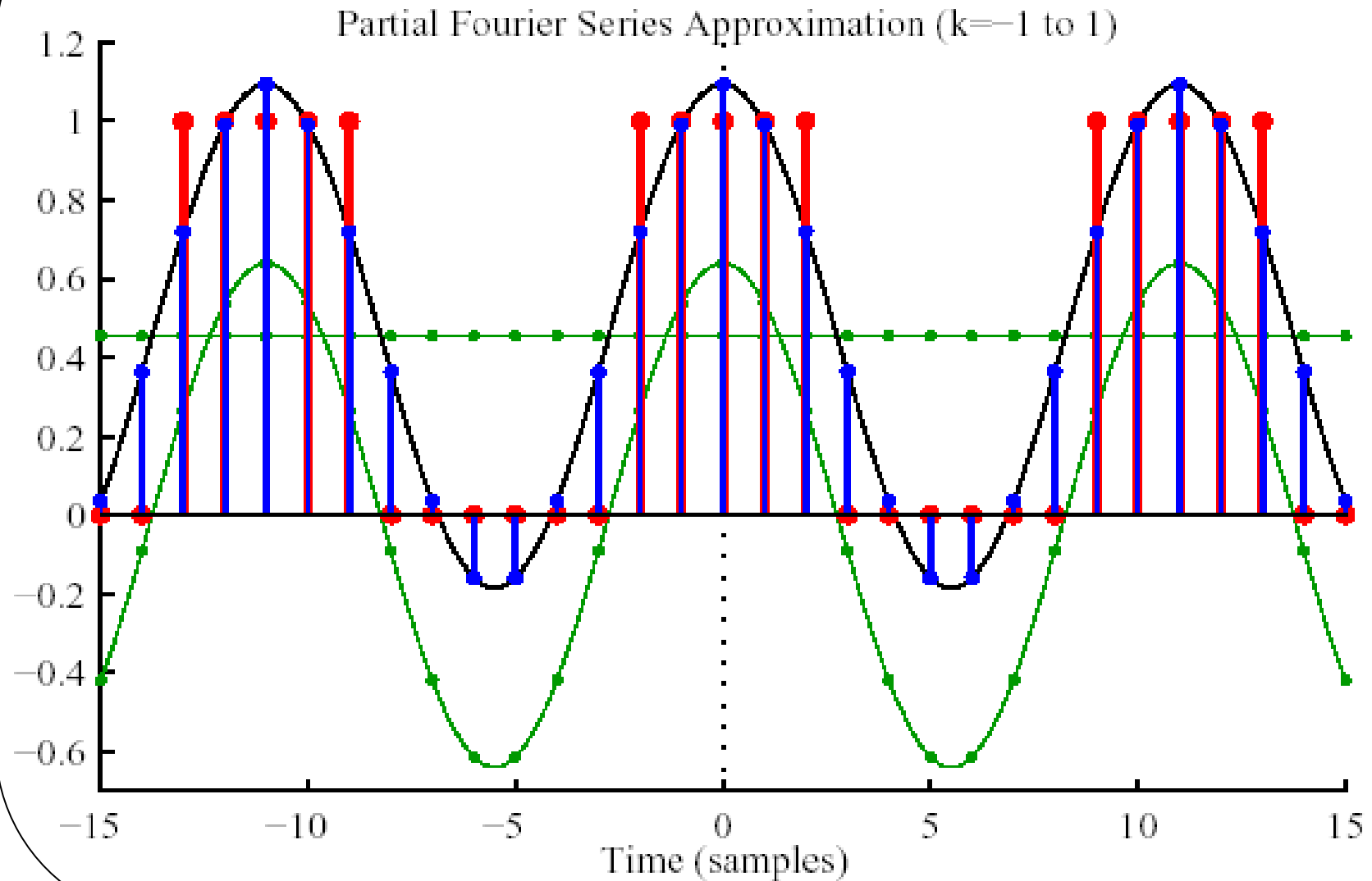
$$a_k = \frac{1}{N} \frac{\sin [2\pi k(N_1 + 1/2)/N]}{\sin(\pi k/N)}$$



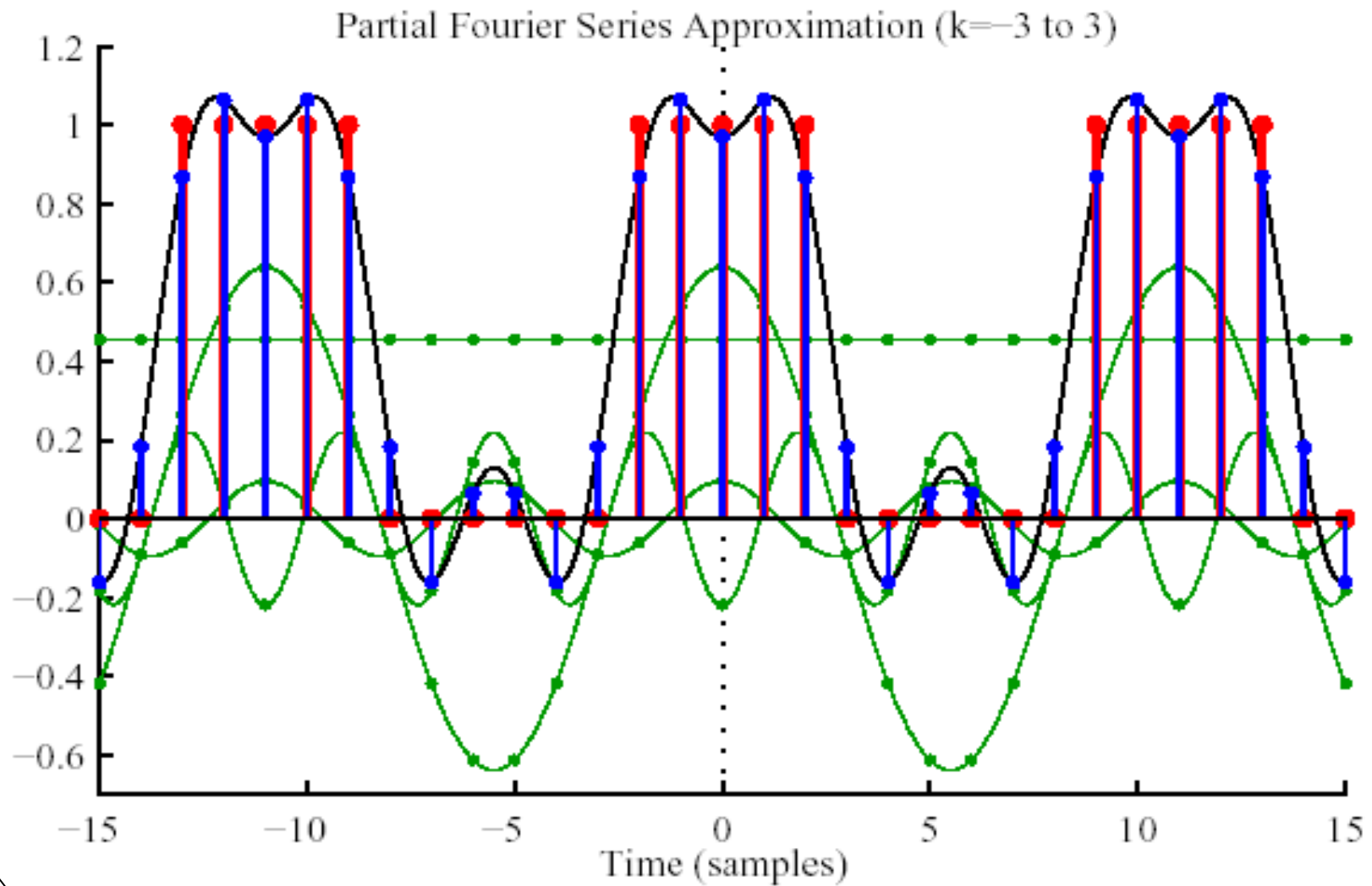
Example : Partial Fourier Series



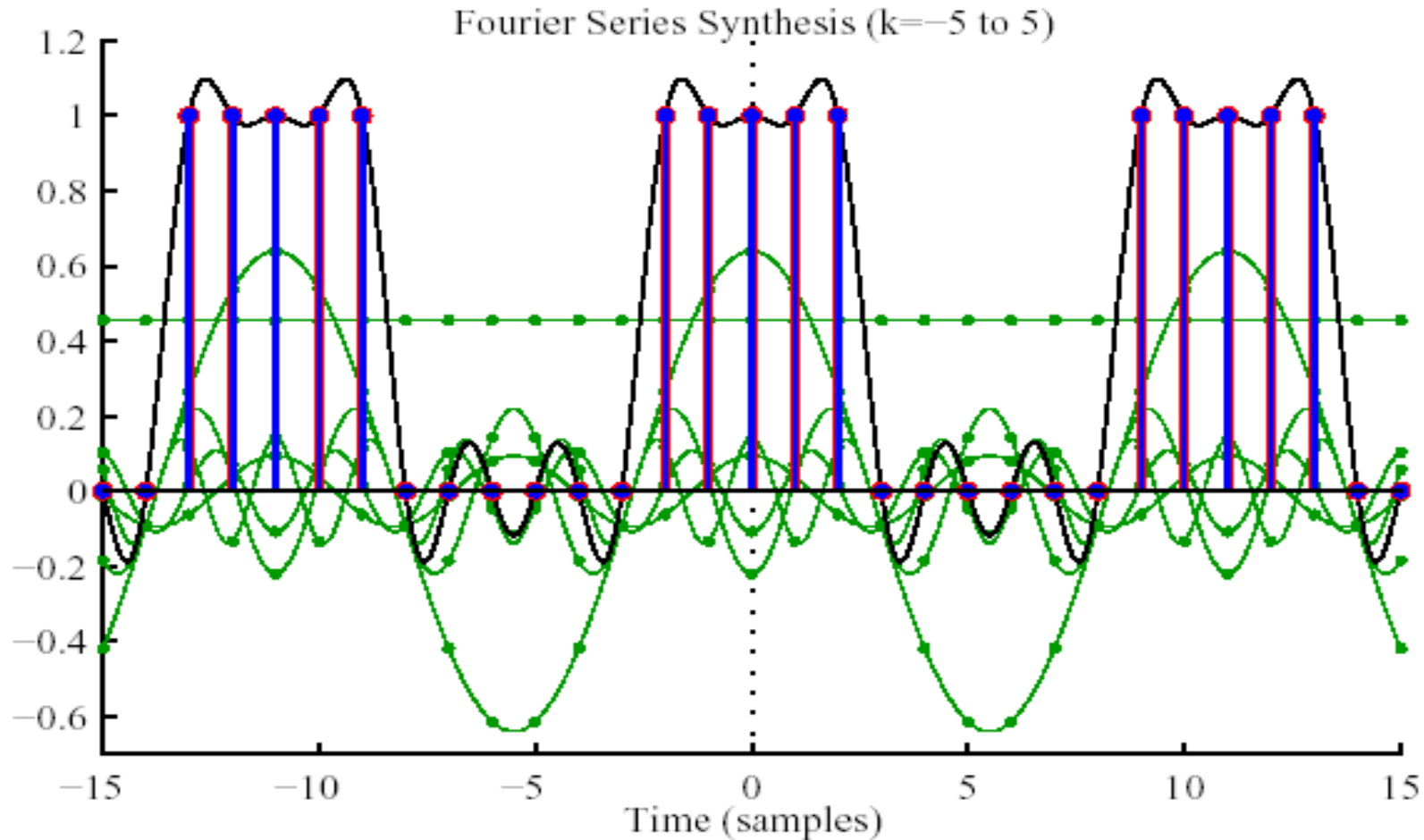
Example : Partial Fourier Series



Example : Partial Fourier Series



Example : Partial Fourier Series



Note that when the no. of terms is equal to N , the series give the exact periodic signal where N taken as 11 and $N_1=2$ in these plots

Compare and Contrast DTFS with CTFS

$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega t}$$

$$a_k = \frac{1}{T} \int_T x(t) e^{-jk\omega t} dt$$

$$x[n] = \sum_{k=\langle N \rangle} a_k e^{jk\omega n}$$

$$a_k = \frac{1}{N} \sum_{n=\langle N \rangle} x[n] e^{-jk\omega n}$$

Unlike the CTFS

- There are only N distinct DT exponential harmonics
- The *finite* DTFS sum is over *any* N consecutive terms
- Thus, the DTFS coefficients a_k form a discrete-time periodic signal (complex valued)
- The DTFS always converges for all periodic signals such that $|x[n]| < \infty$
- No Gibb's Phenomenon

Convergence Issues for DT Fourier Series:

Not an issue, since all series are finite sums.

Properties of DT Fourier Series: Lots, just as with CT Fourier Series

Example:

$$\begin{aligned} x[n] &\leftrightarrow a_k \\ e^{jM\omega_0 n} x[n] &\leftrightarrow b_k = ? \end{aligned}$$

$$x[n]e^{jM\omega_0 n} = \sum_{r=\langle N \rangle} a_r e^{jr\omega_0 n} e^{jM\omega_0 n}$$

$$\stackrel{k=r+M}{=} \sum_{k=\langle N \rangle} a_{k-M} e^{jk\omega_0 n}$$

$\Downarrow$

$$b_k = a_{k-M}$$

Frequency shift

$$jk\omega_0 \rightarrow j(k-M)\omega_0$$

Fourier Series Key Concepts

$$x[n] = \sum_{k=\langle N \rangle} a_k e^{jk\omega n}$$

$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega t}$$

- Class of signals that can be represented as a Fourier series
 - Any practical periodic CT signal
 - Any periodic DT signal
- There is a one-to-one mapping between a signal and its FS coefficients
 - If you know the signal, you can calculate the coefficients
 - If you know the coefficients, you can generate the signal
- The coefficients a_k contain all of the information in the signal
- The squared magnitude of the coefficients $|a_k|^2$ represent the average power of the signal at the k th harmonic

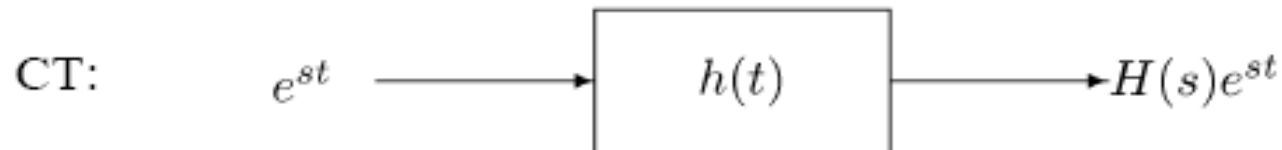
Parseval's relation for periodic DT signals:

$$\frac{1}{N} \sum_{n=\langle N \rangle} |x(n)|^2 = \sum_{k=\langle N \rangle} |a_k|^2$$

where $a_k \leftrightarrow x(n)$. Similarly, the average power in one period of the periodic $x(n)$ is equal to the sum of the average powers in all of its harmonic components.

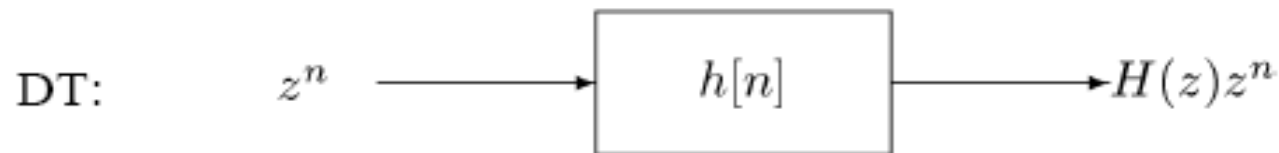
Note that the properties of DTFS are given in Table 3.2

The Eigenfunction Property of Complex Exponentials



CT
"System Function"

$$H(s) = \int_{-\infty}^{\infty} h(t)e^{-st}dt$$



DT
"System Function"

$$H(z) = \sum_{n=-\infty}^{\infty} h[n]z^{-n}$$

where $h(t)$ and $h(n)$ are impulse responses

Fourier Series: Periodic Signals and LTI Systems

$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t} \longrightarrow \boxed{h(t)} \longrightarrow y(t) = \sum_{k=-\infty}^{\infty} H(jk\omega_0) a_k e^{jk\omega_0 t}$$

$$a_k \longrightarrow \underbrace{H(jk\omega_0)}_{\text{"gain"}} a_k$$

So $|a_k| \longrightarrow |H(jk\omega_0)| |a_k|$
or powers of signals get
modified through filter/system

$$H(jk\omega_0) = |H(jk\omega_0)| e^{j\angle H(jk\omega_0)}$$

includes both amplitude & phase

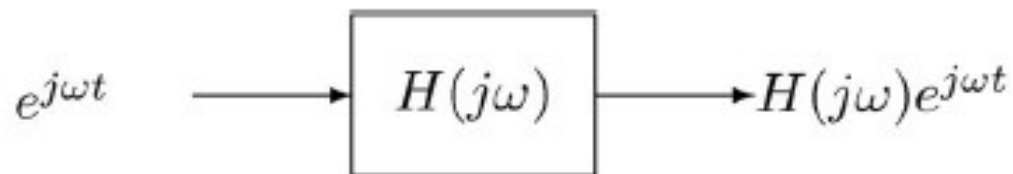
$$x[n] = \sum_{k=\langle N \rangle} a_k e^{jk\omega_0 n} \longrightarrow \boxed{h[n]} \longrightarrow y[n] = \sum_{k=\langle N \rangle} H(e^{jk\omega_0}) a_k e^{jk\omega_0 n}$$

$$a_k \longrightarrow \underbrace{H(e^{jk\omega_0})}_{\text{"gain"}} a_k$$

$$H(e^{jk\omega_0}) = |H(e^{jk\omega_0})| e^{j\angle H(e^{jk\omega_0})}$$

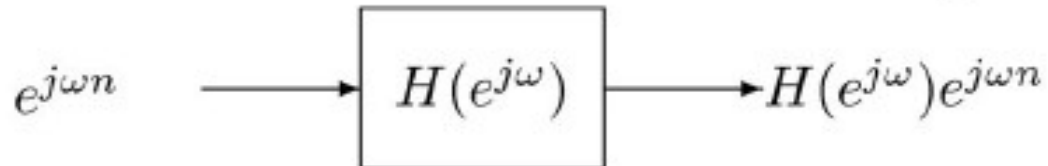
includes both amplitude & phase

The Frequency Response of an LTI System



$$(s = j\omega)$$

CT Frequency response: $H(j\omega) = \int_{-\infty}^{\infty} h(t)e^{-j\omega t} dt$



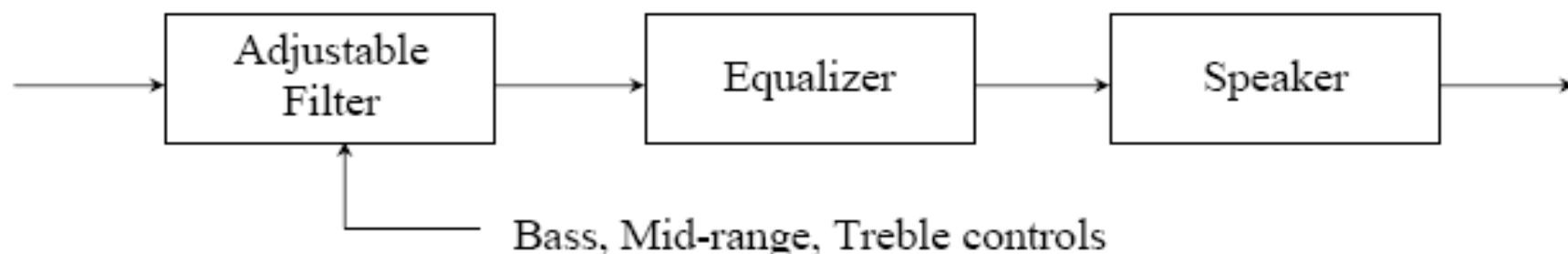
$$(z = e^{j\omega})$$

DT Frequency response: $H(e^{j\omega}) = \sum_{n=-\infty}^{\infty} h[n]e^{-j\omega n}$

Frequency Shaping and Filtering

- By choice of $H(j\omega)$ (or $H(e^{j\omega})$) as a function of ω , we can *shape* the frequency composition of the output
 - Preferential amplification
 - Selective filtering of some frequencies

Example #1: Audio System

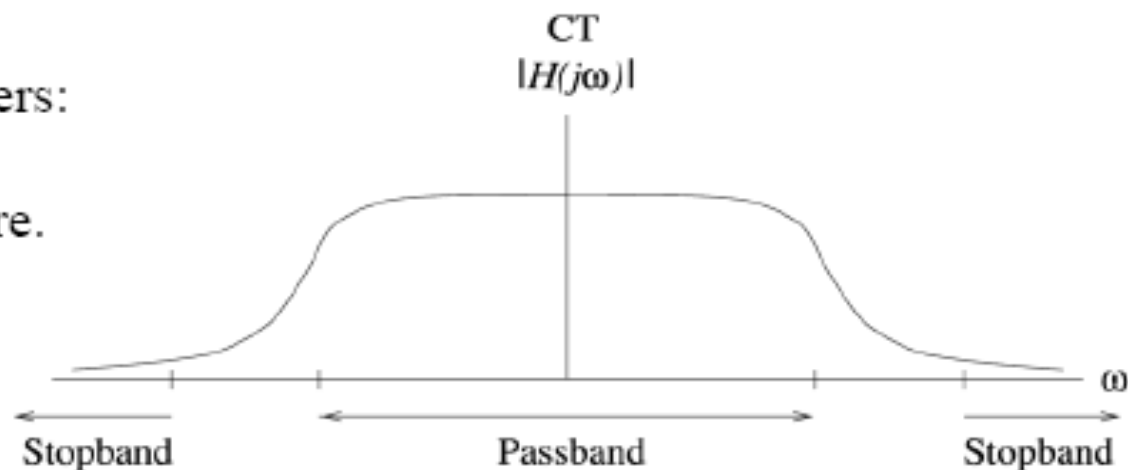


For audio signals, the amplitude is much more important than the phase.

Example #2: Frequency Selective Filters

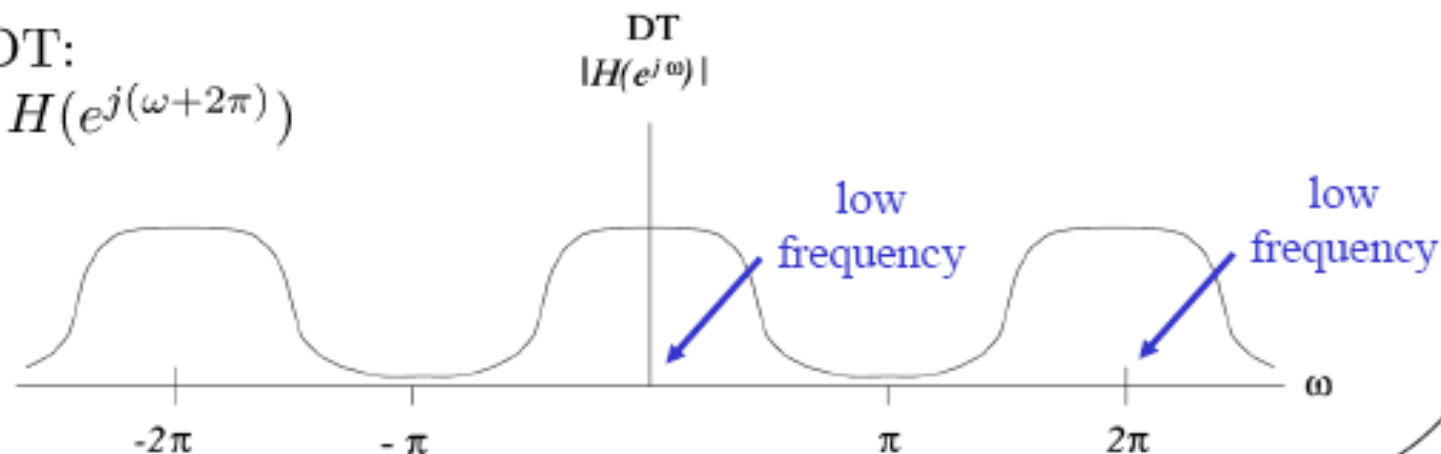
- Filter out signals outside of the frequency range of interest

Lowpass Filters:
Only show
amplitude here.



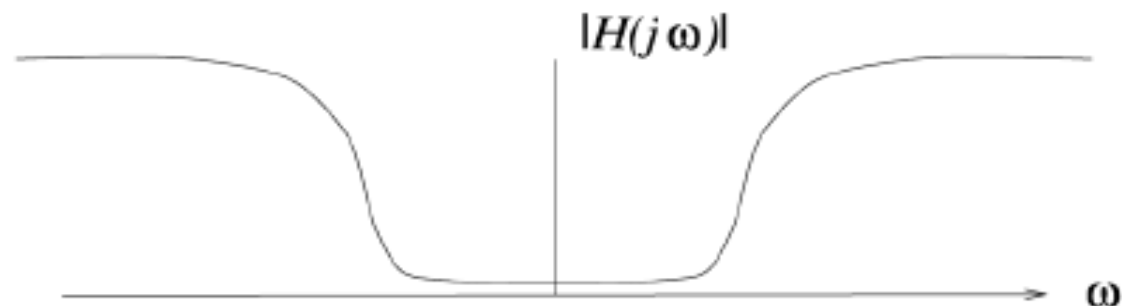
Note for DT:

$$H(e^{j\omega}) = H(e^{j(\omega+2\pi)})$$



Highpass Filters

CT

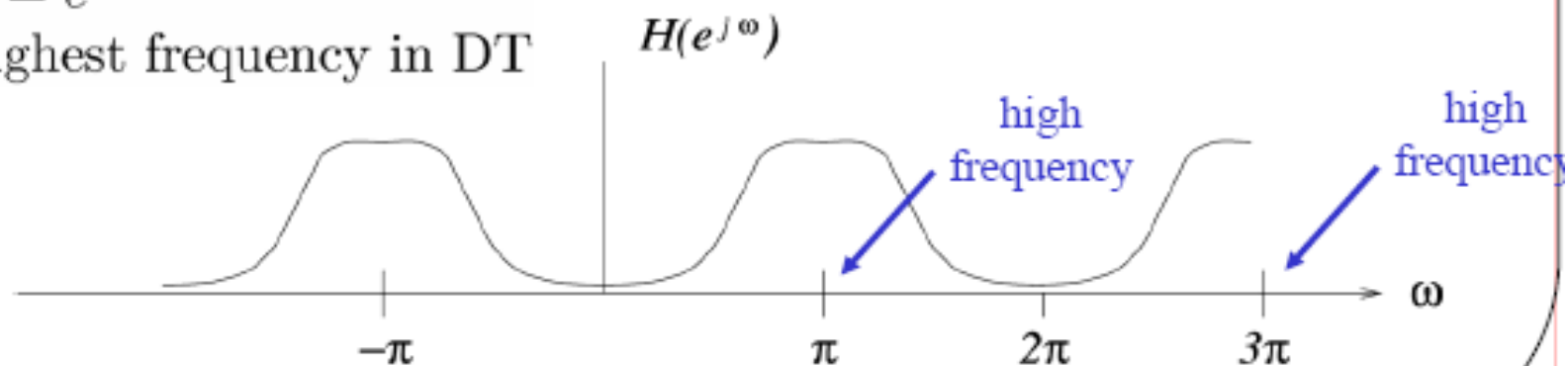


Remember:

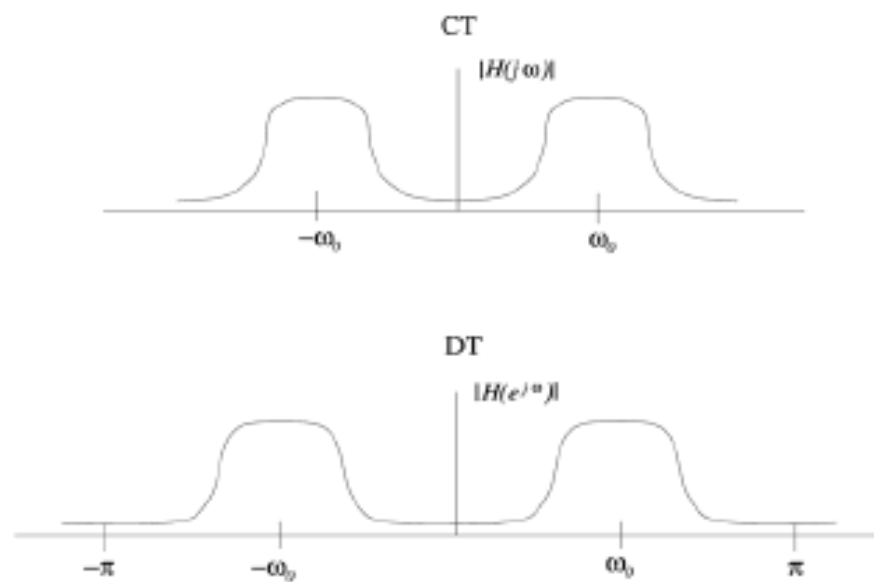
$$(-1)^n = e^{j\pi n}$$

π = highest frequency in DT

DT

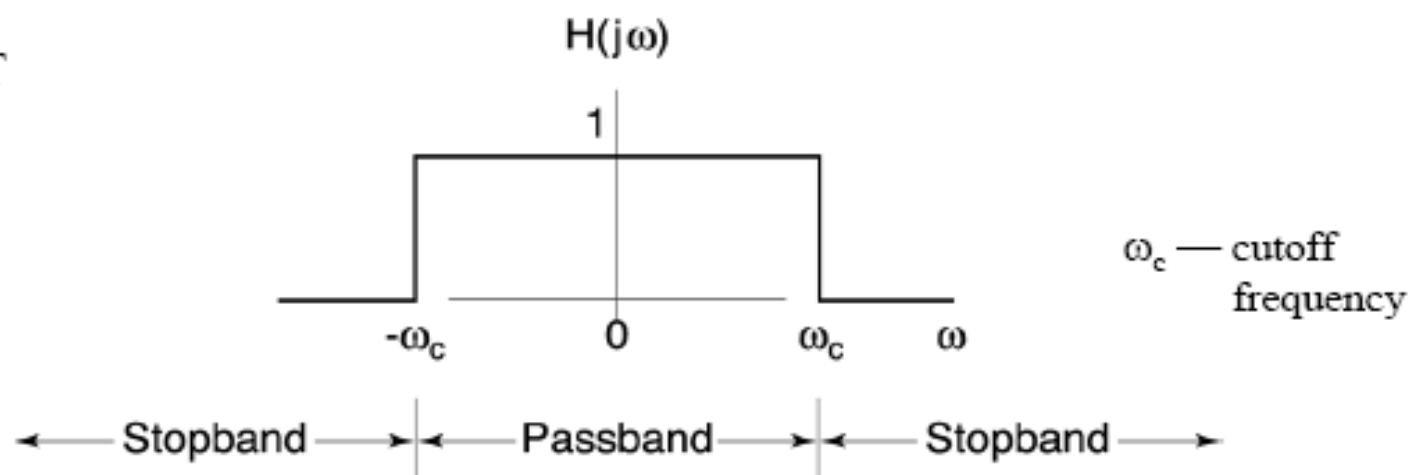


Bandpass Filters

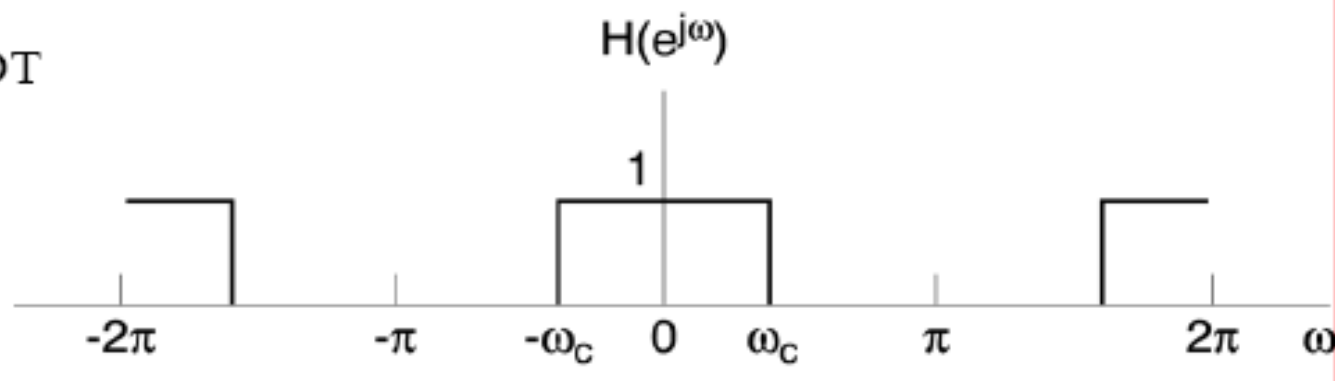


Idealized Filters

CT



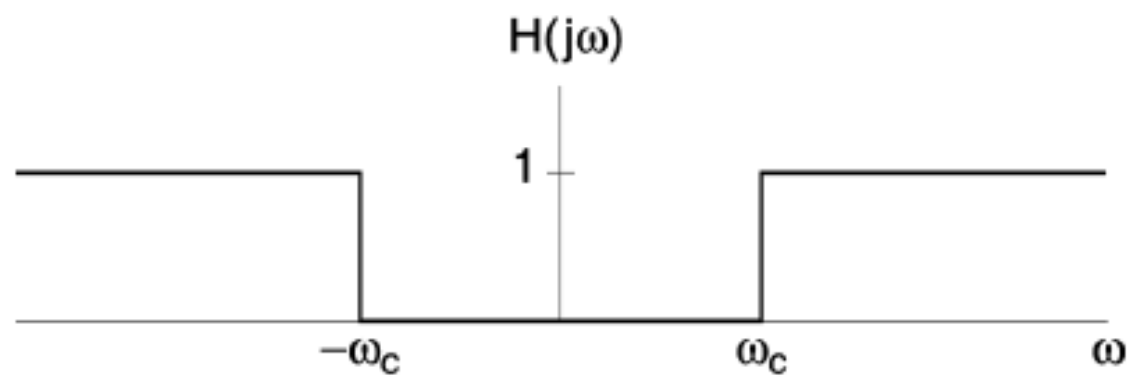
DT



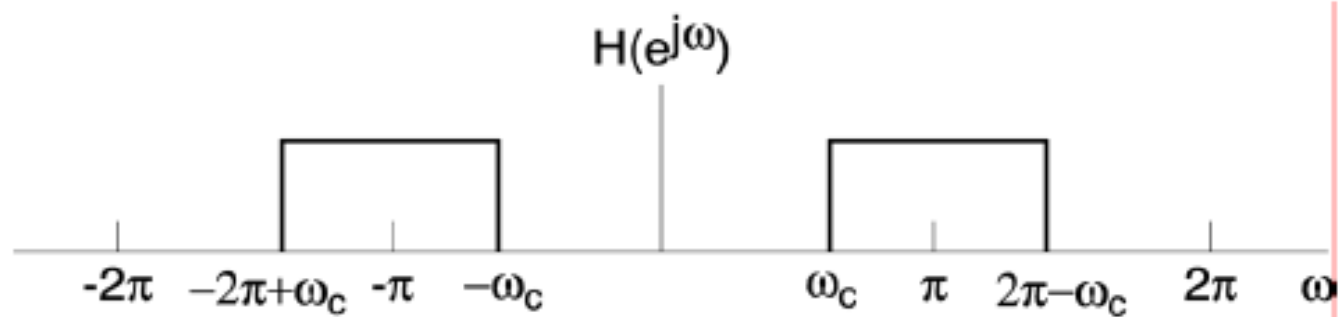
Note: $|H| = 1$ and $\angle H = 0$ for the ideal filters in the passbands, no need for the phase plot.

Highpass

CT

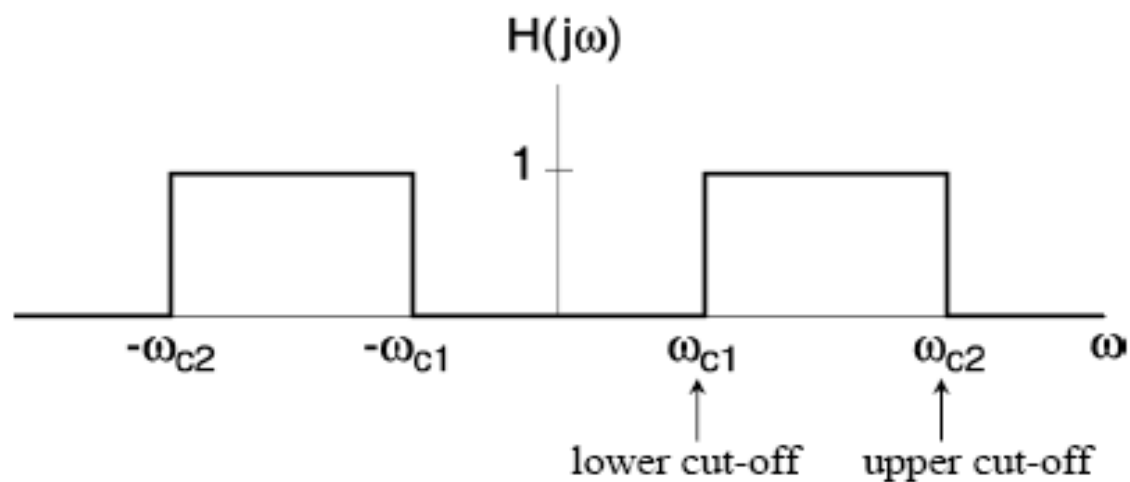


DT

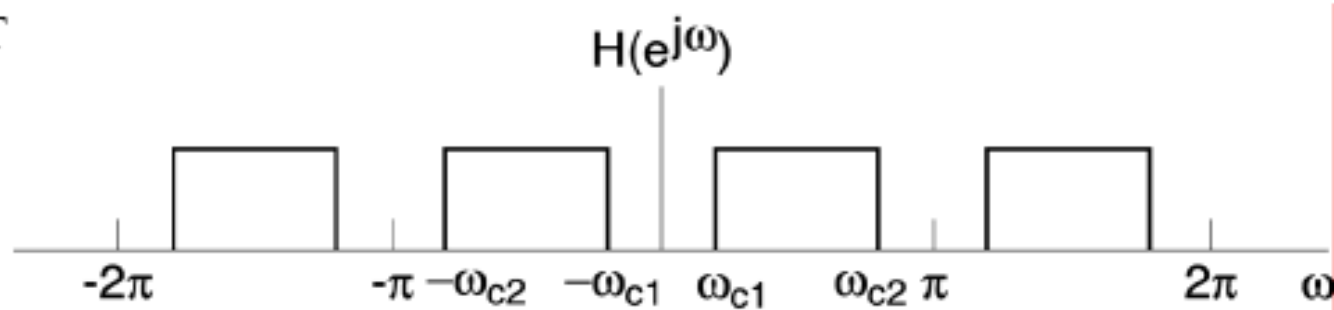


Bandpass

CT



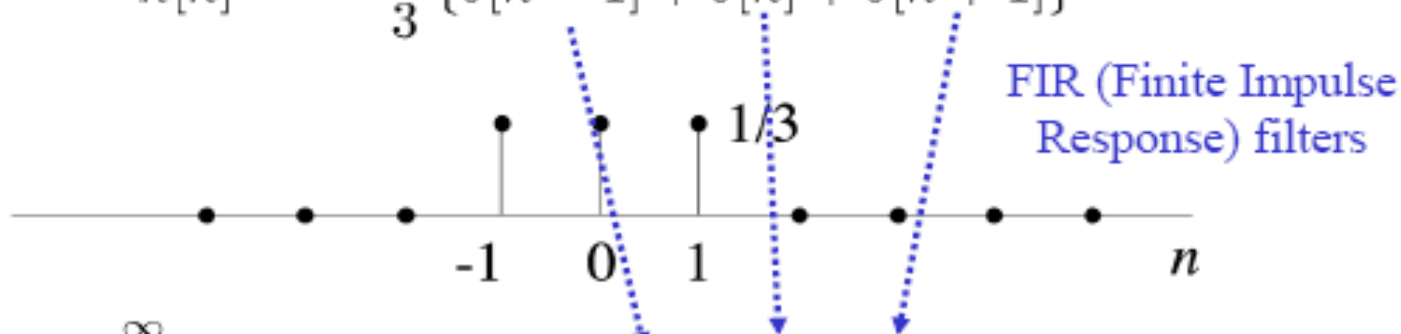
DT



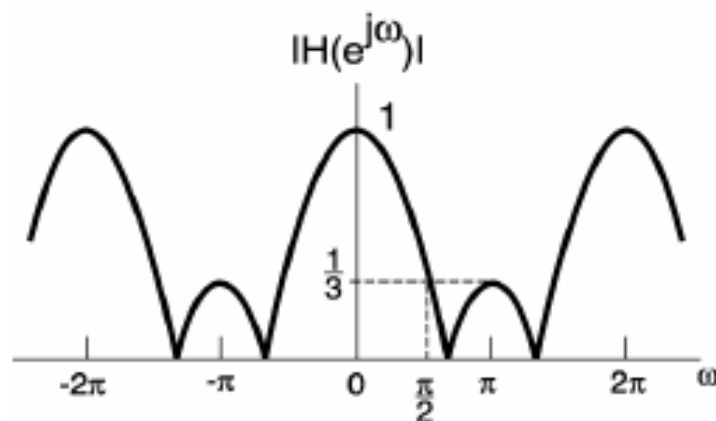
Example #3: DT Averager/Smoothing

$$y[n] = \frac{1}{3} \{x[n-1] + x[n] + x[n+1]\}$$

$$h[n] = \frac{1}{3} \{\delta[n-1] + \delta[n] + \delta[n+1]\}$$



$$H(e^{j\omega}) = \sum_{n=-\infty}^{\infty} h[n]e^{-j\omega n} = \frac{1}{3} [e^{-j\omega} + 1 + e^{j\omega}] = \frac{1}{3} + \frac{2}{3} \cos \omega$$

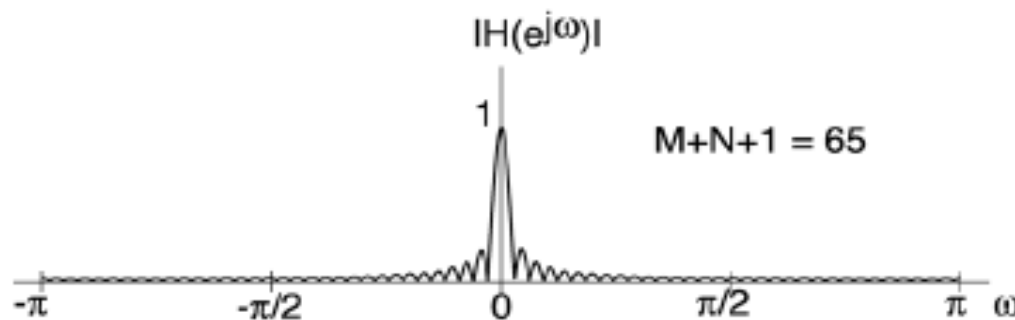
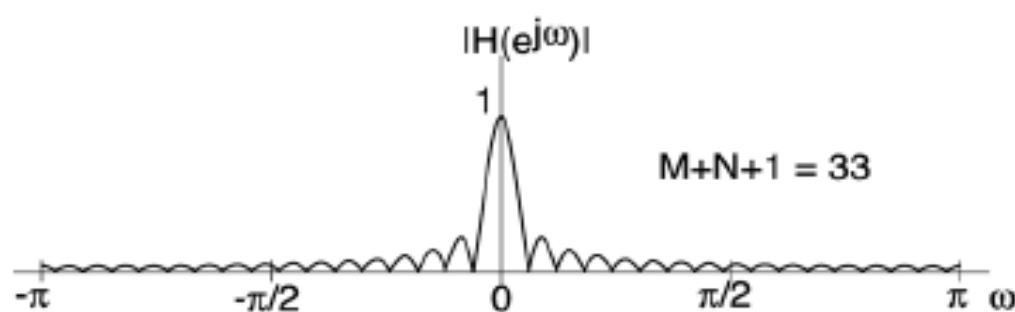


LPF

Example #4: Nonrecursive DT (FIR) filters

$$y[n] = \frac{1}{N+M+1} \sum_{k=-N}^M x[n-k] \longrightarrow h[n] = \frac{1}{N+M+1} \sum_{k=-N}^M \delta[n-k]$$

$$H(e^{j\omega}) = \frac{1}{N+M+1} \sum_{k=-N}^M e^{-jk\omega} = \frac{1}{N+M+1} e^{j\omega[(N-M)/2]} \frac{\sin[\omega(M+N+1)/2]}{\sin(\omega/2)}$$

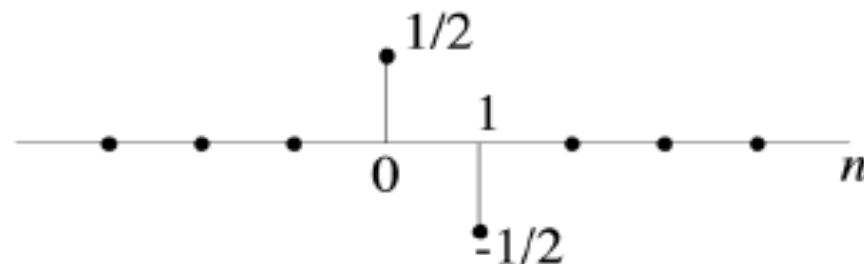


Rolls off at lower ω as $M+N+1$ increases

Example #5: Simple DT “Edge” Detector
 — DT 2-point “differentiator”

$$y[n] = \frac{1}{2} \{x[n] - x[n-1]\}$$

$$h[n] = \frac{1}{2} \{\delta[n] - \delta[n-1]\}$$

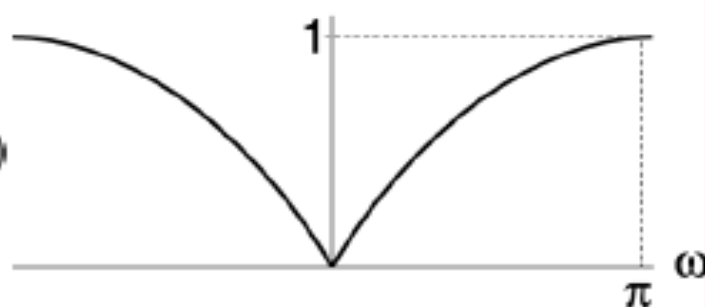


$$\frac{j}{2j} e^{-j\omega/2} (e^{j\omega/2} - e^{-j\omega/2}) = j e^{-j\omega/2} \sin(\omega/2) \quad |H(e^{j\omega})|$$

$\Downarrow$

$$H(e^{j\omega}) = \frac{1}{2} (1 - e^{-j\omega}) = j e^{-j\omega/2} \sin(\omega/2)$$

$$|H(e^{j\omega})| = |\sin(\omega/2)|$$



Passes high-frequency components

