

FOURIER TRANSFORM

Derivation of CT Fourier Transform pair

Fourier Transforms of Periodic Signals

Properties of CT Fourier Transform

Convolution Property

Frequency Response and LTI Systems

Multiplication Property and Parseval' Relation

Duality

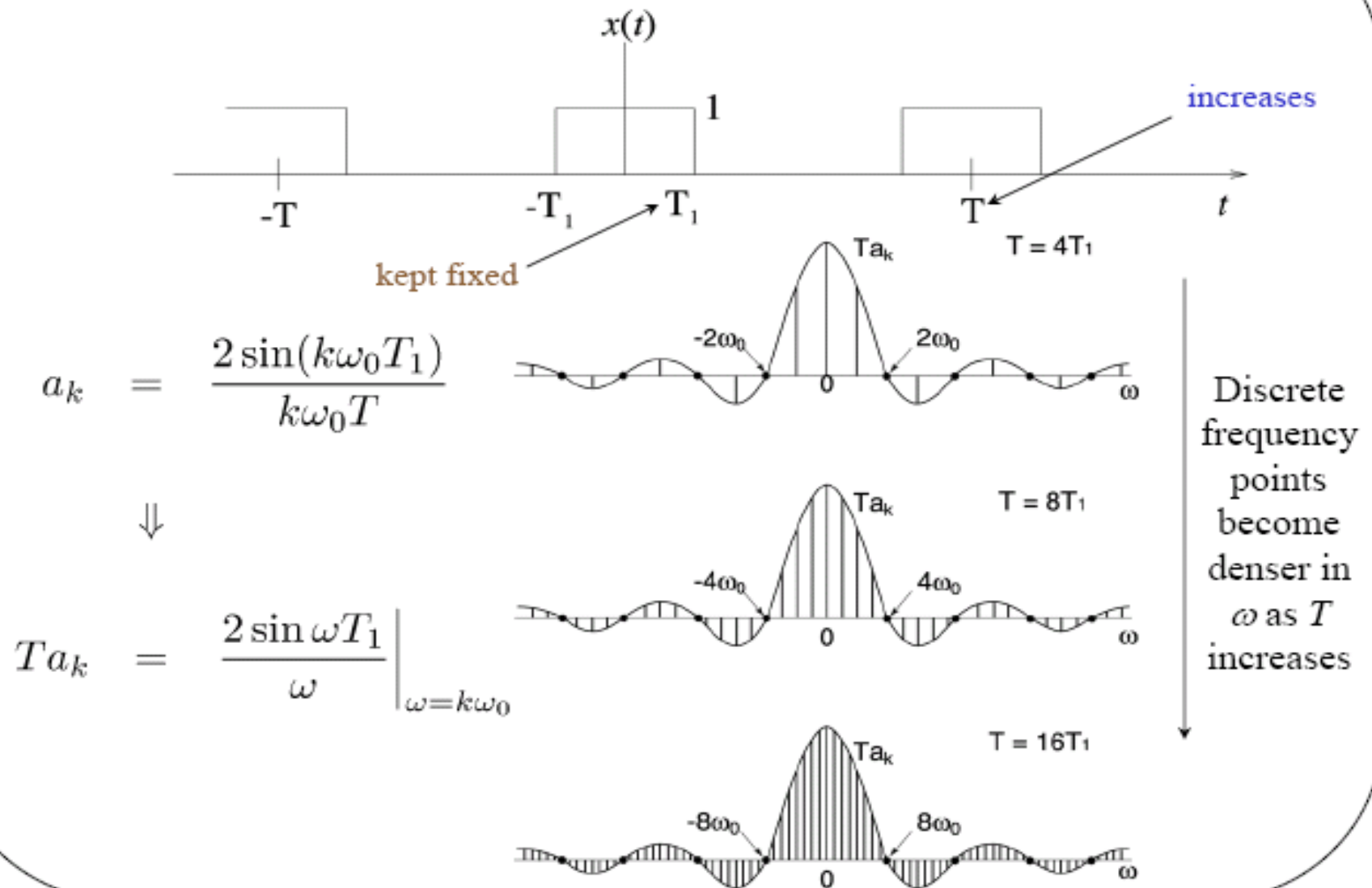
Fourier's Derivation of the CT Fourier Transform

- $x(t)$ - an aperiodic signal
 - view it as the limit of a periodic signal as $T \rightarrow \infty$
- For a periodic signal, the harmonic components are spaced $\omega_0 = 2\pi/T$ apart ...
- As $T \rightarrow \infty$, $\omega_0 \rightarrow 0$, and harmonic components are spaced closer and closer in frequency



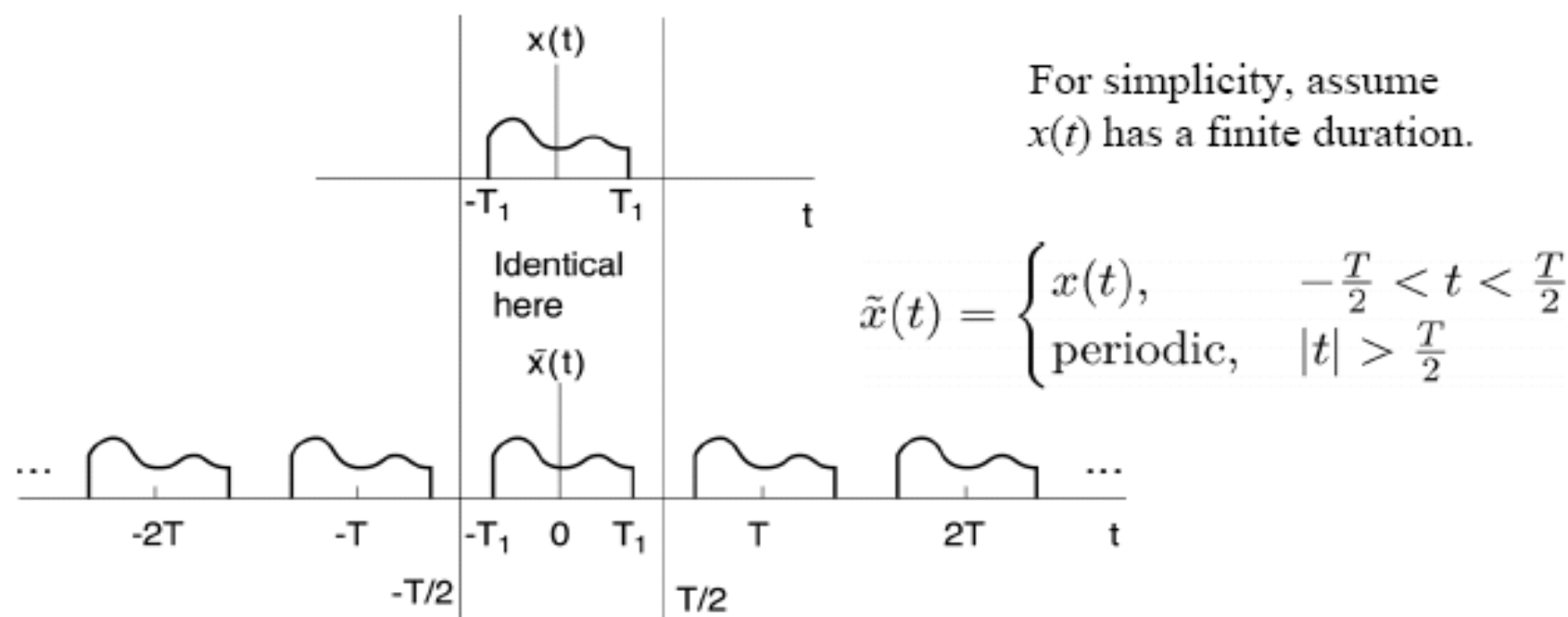
Fourier series $\longrightarrow$ Fourier integral

Motivating Example: Square wave



So, on with the derivation ...

For simplicity, assume $x(t)$ has a finite duration.



As $T \rightarrow \infty$, $\tilde{x}(t) = x(t)$ for all t

Derivation (continued)

$$\tilde{x}(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t} \quad \left(\omega_0 = \frac{2\pi}{T} \right)$$

$$\begin{aligned} a_k &= \frac{1}{T} \int_{-\frac{T}{2}}^{\frac{T}{2}} \tilde{x}(t) e^{-jk\omega_0 t} dt = \frac{1}{T} \int_{-\frac{T}{2}}^{\frac{T}{2}} x(t) e^{-jk\omega_0 t} dt \\ &\quad \uparrow \\ &\quad \tilde{x}(t) = x(t) \text{ in this interval} \\ &= \frac{1}{T} \int_{-\infty}^{\infty} x(t) e^{-jk\omega_0 t} dt \quad (1) \end{aligned}$$

If we define

$$X(j\omega) = \int_{-\infty}^{\infty} x(t) e^{-j\omega t} dt$$

then Eq.(1) $\Rightarrow$

$$a_k = \frac{X(jk\omega_0)}{T}$$

Derivation (continued)

Thus, for $-\frac{T}{2} < t < \frac{T}{2}$

$$\begin{aligned}x(t) = \tilde{x}(t) &= \sum_{k=-\infty}^{\infty} \underbrace{\frac{1}{T} X(jk\omega_0)}_{a_k} e^{jk\omega_0 t} \\&= \frac{1}{2\pi} \sum_{k=-\infty}^{\infty} \omega_0 X(jk\omega_0) e^{jk\omega_0 t} \\&\quad \Downarrow\end{aligned}$$

As $T \rightarrow \infty$, $\sum \omega_0 \rightarrow \int d\omega$, we get the CT Fourier Transform pair

$$x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(j\omega) e^{j\omega t} d\omega \quad \text{Synthesis equation}$$

$$X(j\omega) = \int_{-\infty}^{\infty} x(t) e^{-j\omega t} dt \quad \text{Analysis equation}$$

For what kinds of signals can we do this?

(1) It works also even if $x(t)$ is infinite duration, but satisfies:

a) Finite energy $\int_{-\infty}^{\infty} |x(t)|^2 dt < \infty$

In this case, there is *zero* energy in the error

$$e(t) = x(t) - \frac{1}{2\pi} \int_{-\infty}^{\infty} X(j\omega) e^{j\omega t} d\omega \quad \text{Then} \quad \int_{-\infty}^{\infty} |e(t)|^2 dt = 0$$

b) Dirichlet conditions

(i) $\frac{1}{2\pi} \int_{-\infty}^{\infty} X(j\omega) e^{j\omega t} d\omega = x(t)$ at points of continuity

(ii) $\frac{1}{2\pi} \int_{-\infty}^{\infty} X(j\omega) e^{j\omega t} d\omega = \text{midpoint at discontinuity}$

(iii) Gibbs's phenomenon

c) By allowing impulses in $x(t)$ or in $X(j\omega)$, we can represent even *more* signals

E.g. It allows us to consider *FT* for *periodic* signals

Example

(a) $x(t) = \delta(t)$

$$X(j\omega) = \int_{-\infty}^{\infty} \delta(t) e^{-j\omega t} dt = 1$$

$\Downarrow$

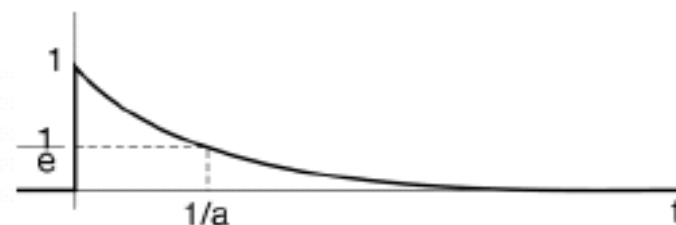
$$\delta(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{j\omega t} d\omega \quad \text{— Synthesis equation for } \delta(t)$$

(b) $x(t) = \delta(t - t_0)$

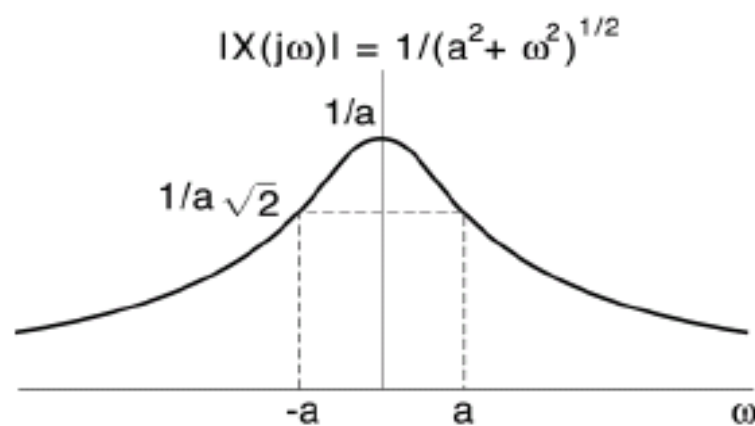
$$\begin{aligned} X(j\omega) &= \int_{-\infty}^{\infty} \delta(t - t_0) e^{-j\omega t} dt \\ &= e^{-j\omega t_0} \end{aligned}$$

Example : Exponential function

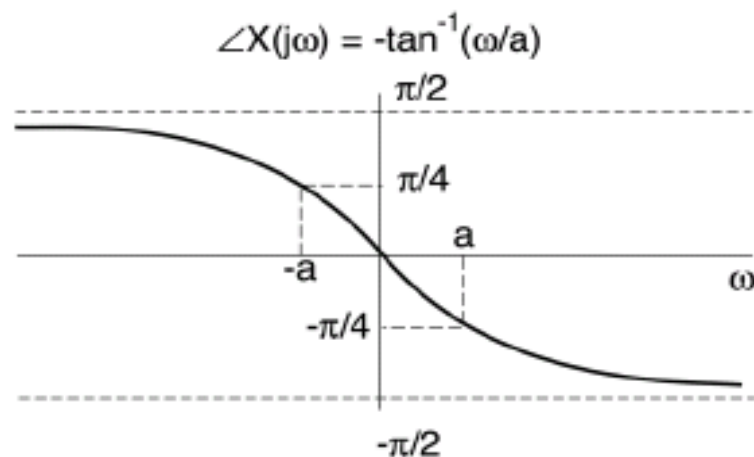
$$x(t) = e^{-at}u(t), a > 0$$



$$\begin{aligned} X(j\omega) &= \int_{-\infty}^{\infty} x(t)e^{-j\omega t} dt = \int_0^{\infty} \underbrace{e^{-at}e^{-j\omega t}}_{e^{-(a+j\omega)t}} dt \\ &= -\left(\frac{1}{a+j\omega}\right) e^{-(a+j\omega)t} \Big|_0^{\infty} = \frac{1}{a+j\omega} \end{aligned}$$



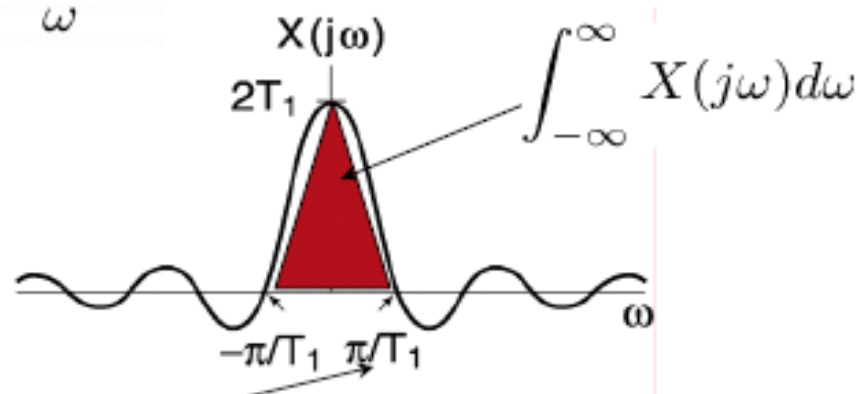
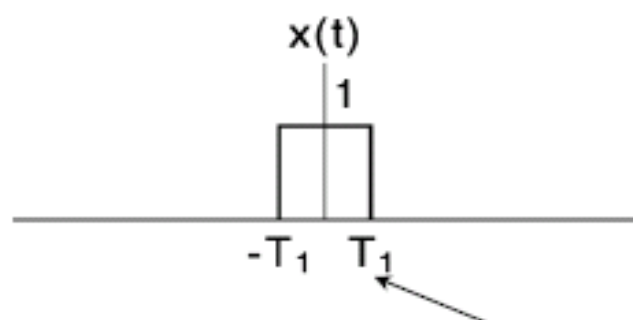
Even symmetry



Odd symmetry

Example : A square pulse in the time-domain

$$X(j\omega) = \int_{-T_1}^{T_1} e^{-j\omega t} dt = \frac{2 \sin \omega T_1}{\omega}$$



Note the inverse relation between the two widths $\Rightarrow$ Uncertainty principle

Useful facts about CTFT's

$$X(0) = \int_{-\infty}^{\infty} x(t) dt$$

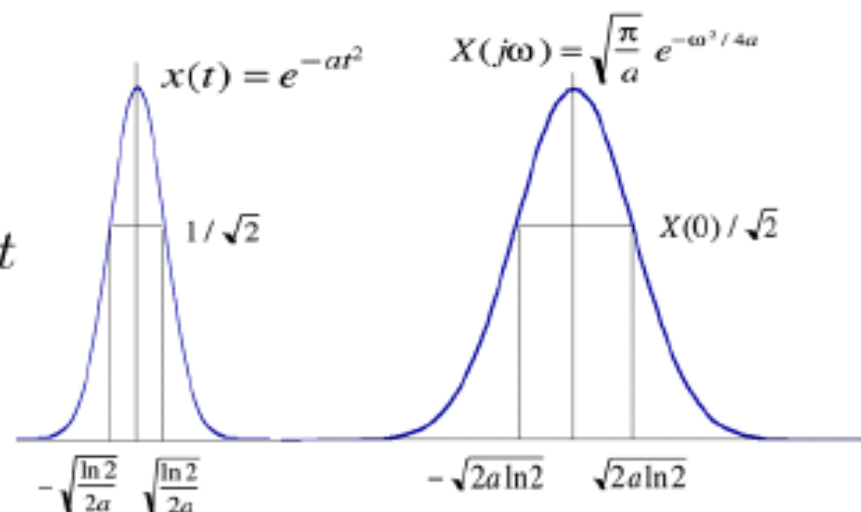
Example above: $\int_{-\infty}^{\infty} x(t) dt = 2T_1 = X(0)$

$$x(0) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(j\omega) d\omega$$

Ex. above: $x(0) = 1 = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(j\omega) d\omega$
 $= \frac{1}{2\pi} \times (\text{Area of the triangle})$

Example : $x(t) = e^{-at^2}$ — A Gaussian, important in probability, optics, etc.

$$\begin{aligned}
 &X(j\omega) \\
 = &\int_{-\infty}^{\infty} e^{-at^2} e^{-j\omega t} dt \\
 = &\int_{-\infty}^{\infty} e^{-a\left[t^2 + j\frac{\omega}{a}t + \left(\frac{j\omega}{2a}\right)^2\right] + a\left(\frac{j\omega}{2a}\right)^2} dt \\
 = &\underbrace{\left[\int_{-\infty}^{\infty} e^{-a\left(t + \frac{j\omega}{2a}\right)^2} dt\right]}_{\sqrt{\pi}/a} \cdot e^{-\frac{\omega^2}{4a}} \\
 = &\sqrt{\frac{\pi}{a}} e^{-\frac{\omega^2}{4a}}
 \end{aligned}$$



(Pulse width in t) • (Pulse width in ω)
 $\Rightarrow \Delta t \cdot \Delta \omega \sim (1/a^{1/2}) \cdot (a^{1/2}) = 1$

Also a Gaussian!

Uncertainty Principle! Cannot make *both* Δt and $\Delta \omega$ arbitrarily small.

Symmetry

$$x_e(t) = \frac{1}{2} (x(t) + x(-t)) \quad \xleftrightarrow{\mathcal{F}} \quad \text{Re} \{X(j\omega)\}$$

$$x_o(t) = \frac{1}{2} (x(t) - x(-t)) \quad \xleftrightarrow{\mathcal{F}} \quad \text{Im} \{X(j\omega)\}$$

- The Fourier transform exhibits the same symmetry as the Fourier series
- If $x(t)$ is real, then
 - $\text{Re} \{X(j\omega)\}$ is even
 - $\text{Im} \{X(j\omega)\}$ is odd
- Because of this symmetry
 - $X(j\omega)$ is often only plotted for positive frequencies
 - $X(j\omega)$ for $\omega < 0$ can be inferred from these plots

CT Fourier Transforms of Periodic Signals

Suppose

$$X(j\omega) = \delta(\omega - \omega_0)$$

$\Downarrow$

$$x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \delta(\omega - \omega_0) e^{j\omega t} d\omega = \frac{1}{2\pi} e^{j\omega_0 t} \text{ — periodic in } t \text{ with frequency } \omega_0$$

That is

$$e^{j\omega_0 t} \leftrightarrow 2\pi\delta(\omega - \omega_0)$$

— All the energy is concentrated in one frequency — ω_0

More generally

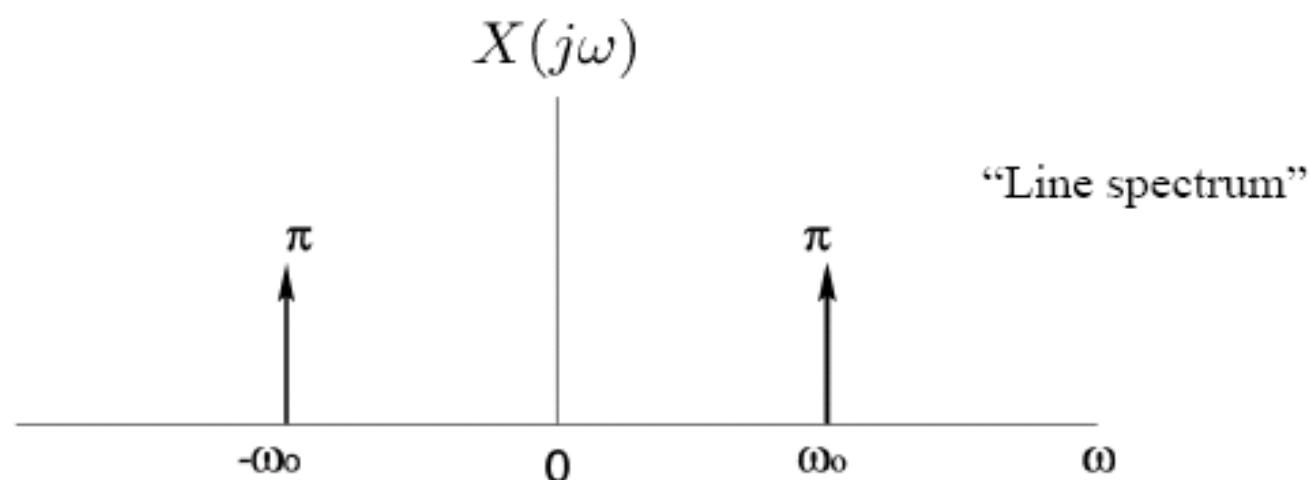
$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t} \leftrightarrow X(j\omega) = \sum_{k=-\infty}^{\infty} 2\pi a_k \delta(\omega - k\omega_0)$$

Example

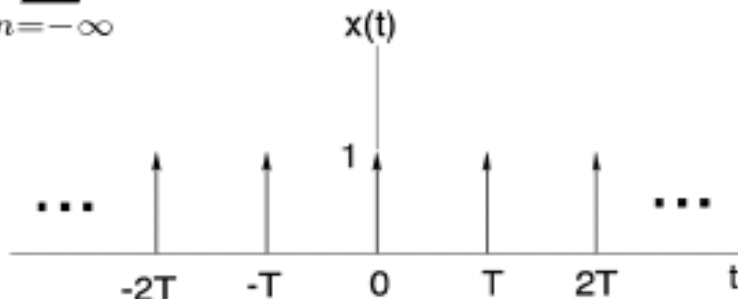
$$x(t) = \cos \omega_0 t = \frac{1}{2}e^{j\omega_0 t} + \frac{1}{2}e^{-j\omega_0 t}$$

$\Downarrow$

$$X(j\omega) = \pi\delta(\omega - \omega_0) + \pi\delta(\omega + \omega_0)$$



Example : $x(t) = \sum_{n=-\infty}^{\infty} \delta(t - nT)$ — Sampling function

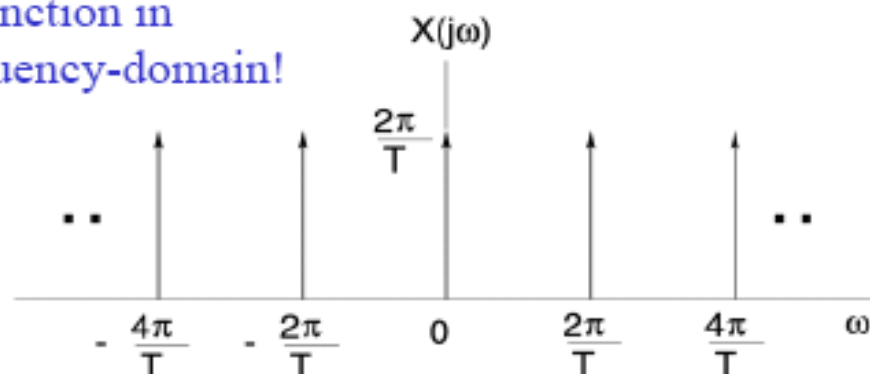


$$x(t) \leftrightarrow a_k = \frac{1}{T} \int_{-T/2}^{T/2} x(t) e^{-jk\omega_0 t} dt = \frac{1}{T}$$

$\Downarrow$

$$X(j\omega) = \sum_{n=-\infty}^{\infty} \underbrace{\frac{2\pi}{T}}_{2\pi a_k} \delta\left(\omega - \underbrace{\frac{k2\pi}{T}}_{k\omega_0}\right)$$

Same function in
the frequency-domain!



Note: (period in t) T
 $\Leftrightarrow$ (period in ω) $2\pi/T$
Inverse relationship again!

Properties of the CT Fourier Transform

1) Linearity $ax(t) + by(t) \leftrightarrow aX(j\omega) + bY(j\omega)$

2) Time Shifting $x(t - t_0) \leftrightarrow e^{-j\omega t_0} X(j\omega)$

Proof:
$$\int_{-\infty}^{\infty} x(\underbrace{t - t_0}_{t'}) e^{-j\omega t} dt = e^{-j\omega t_0} \underbrace{\int_{-\infty}^{\infty} x(t') e^{-j\omega t'} dt'}_{X(j\omega)}$$

FT magnitude unchanged

$$|e^{-j\omega t_0} X(j\omega)| = |X(j\omega)|$$

Linear change in *FT* phase

$$\angle(e^{-j\omega t_0} X(j\omega)) = \angle X(j\omega) - \omega t_0$$

Properties (continued)

3) Conjugate Symmetry

$$x(t) \text{ real} \leftrightarrow X(-j\omega) = X^*(j\omega)$$

$$\Downarrow$$

$$|X(-j\omega)| = |X(j\omega)| \quad \text{Even}$$

$$\angle X(-j\omega) = -\angle X(j\omega) \quad \text{Odd}$$

$$\operatorname{Re}\{X(-j\omega)\} = \operatorname{Re}\{X(j\omega)\} \quad \text{Even}$$

$$\operatorname{Im}\{X(-j\omega)\} = -\operatorname{Im}\{X(j\omega)\} \quad \text{Odd}$$

The Properties Keep on Coming ...

- 4) Time-Scaling $x(at) \longleftrightarrow \frac{1}{|a|} X\left(j\frac{\omega}{a}\right)$
 $\Downarrow a = -1$
 $x(-t) \longleftrightarrow X(-j\omega)$
 $\Downarrow$
- E.g. $a > 1 \rightarrow at > t$
 compressed in time $\Leftrightarrow$
 stretched in frequency
- a) $x(t)$ real and even $x(t) = x(-t)$
 $\Rightarrow X(j\omega) = X(-j\omega) = X^*(j\omega)$ – Real & even
- b) $x(t)$ real and odd $x(t) = -x(-t)$
 $\Rightarrow X(j\omega) = -X(-j\omega) = -X^*(j\omega)$ – Purely imaginary & odd
- c) $X(j\omega) = \underset{\uparrow}{\text{Re}}\{X(j\omega)\} + j\underset{\uparrow}{\text{Im}}\{X(j\omega)\}$
For real $x(t)$ $= \underset{\uparrow}{\text{Ev}}\{x(t)\} + \underset{\uparrow}{j\text{Od}}\{x(t)\}$

The CT Fourier Transform Pair

$$x(t) \longleftrightarrow X(j\omega)$$

$$X(j\omega) = \int_{-\infty}^{\infty} x(t)e^{-j\omega t} dt \quad \begin{array}{l} \text{-- FT} \\ \text{(Analysis Equation)} \end{array}$$

$$x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(j\omega)e^{j\omega t} d\omega \quad \begin{array}{l} \text{-- Inverse FT} \\ \text{(Synthesis Equation)} \end{array}$$

Frequency Translation (modulation)

$$e^{j\omega_o t} x(t) \xleftrightarrow{\mathcal{F}} X(j(\omega - \omega_o))$$

- Multiplication by a complex exponential shifts the Fourier transform origin to the specified frequency
- This is the basis of amplitude modulation (think AM radio band)
- This is a convenient method for merging multiple bandlimited signals into a single signal
- The signal can then be recovered by bandpass filtering and AM demodulation

Example : Transform of a Modulated Pulse

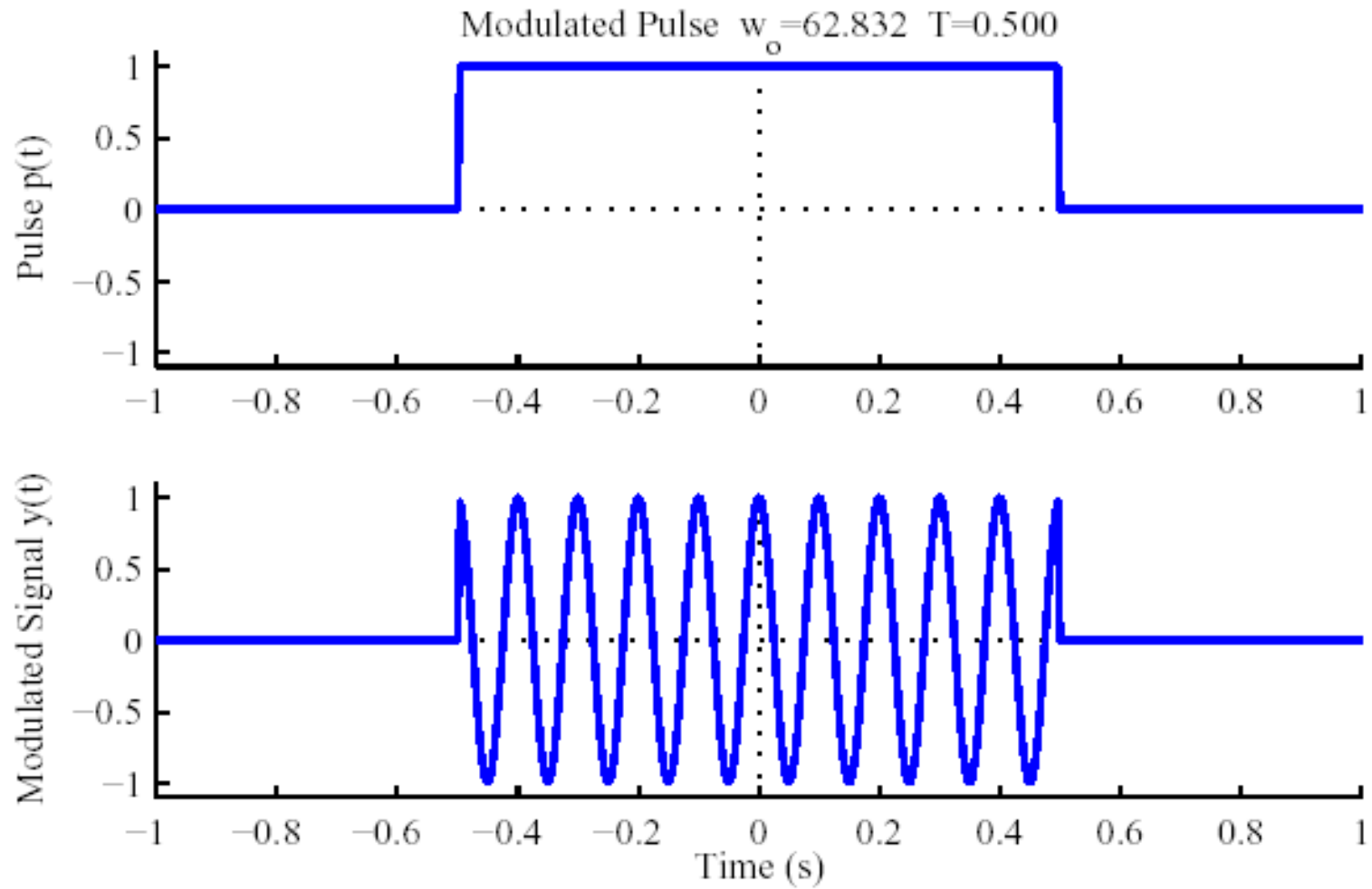
We found the Fourier transform of a pulse earlier:

$$p(t) = u(t+T) - u(t-T) = \begin{cases} 1 & |t| < T \\ 0 & |t| > T \end{cases}$$

$$P(j\omega) = 2T \frac{\sin(\omega T)}{\omega T} = 2T \operatorname{sinc}\left(\frac{\omega T}{\pi}\right)$$

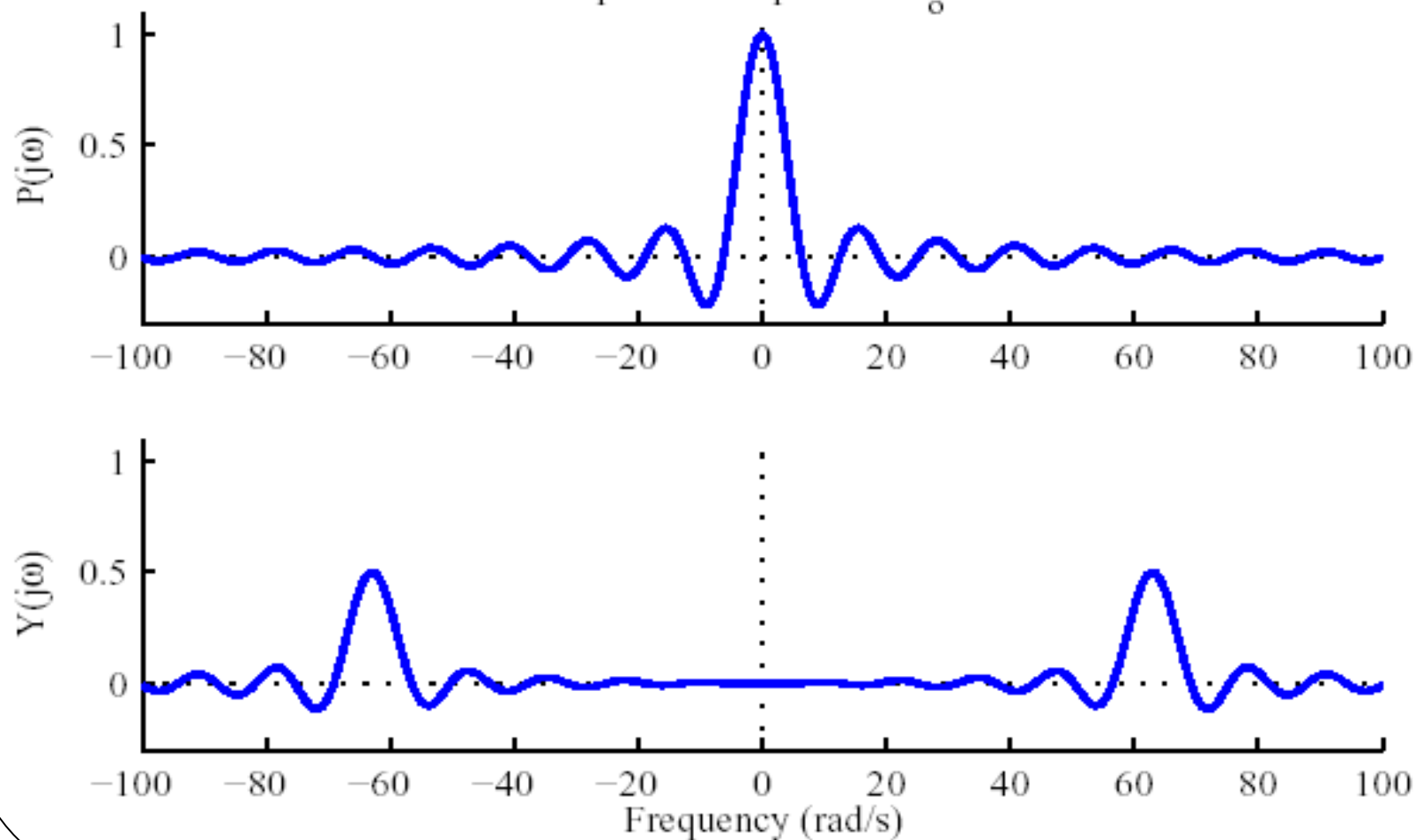
Plot the signal $y(t) = \cos(\omega_0 t) p(t)$ versus time and its Fourier transform for $f_0 = 10$ Hz and $T = 0.5$ s. Note that both signals are even so $\operatorname{Im}\{X(j\omega)\} = \operatorname{Im}\{P(j\omega)\} = 0$.

Example : Modulated Pulse Signal



Example : Modulated Pulse Spectrum

Modulated Pulse Spectral Amplitude $\omega_o=62.832$ $T=0.500$



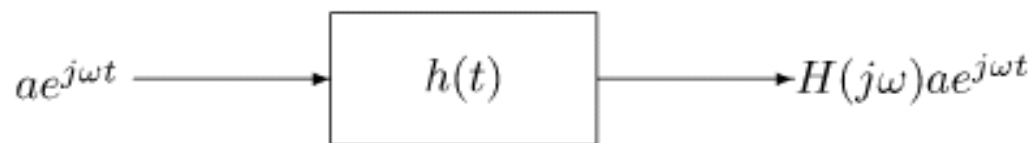
Convolution Property

$$y(t) = h(t) * x(t) \longleftrightarrow Y(j\omega) = H(j\omega)X(j\omega)$$

$$\text{where } h(t) \longleftrightarrow H(j\omega)$$

A consequence of the eigenfunction property:

$$x(t) = \int_{-\infty}^{\infty} \underbrace{\left(\frac{1}{2\pi} X(j\omega) d\omega \right)}_{\text{coefficient } a} e^{j\omega t}$$

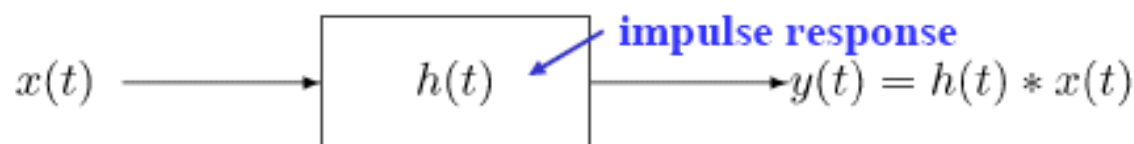


$$y(t) = \int_{-\infty}^{\infty} \underbrace{\left(H(j\omega) \frac{1}{2\pi} X(j\omega) d\omega \right)}_{H(j\omega) \cdot a} e^{j\omega t}$$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} \underbrace{H(j\omega) X(j\omega)}_{Y(j\omega)} e^{j\omega t} d\omega$$

Synthesis equation
for $y(t)$

The Frequency Response Revisited



$$Y(j\omega) = H(j\omega)X(j\omega)$$

← frequency response

The frequency response of a CT LTI system is simply the Fourier transform of its impulse response

Example : $x(t) = e^{j\omega_0 t}$ → $H(j\omega)$ → $y(t)$

Recall

$$e^{j\omega_0 t} \longleftrightarrow 2\pi\delta(\omega - \omega_0)$$

$$Y(j\omega) = H(j\omega)X(j\omega) = H(j\omega)2\pi\delta(\omega - \omega_0) = 2\pi H(j\omega_0)\delta(\omega - \omega_0)$$

⇓ inverse FT

$$y(t) = H(j\omega_0)e^{j\omega_0 t}$$

Example : A differentiator

$$y(t) = \frac{dx(t)}{dt} \quad \text{- an LTI system}$$

Differentiation property: $Y(j\omega) = j\omega X(j\omega)$

$\Downarrow$

$$H(j\omega) = j\omega$$

1) Amplifies high frequencies (enhances sharp edges)

2) $+\pi/2$ phase shift ($j = e^{j\pi/2}$)

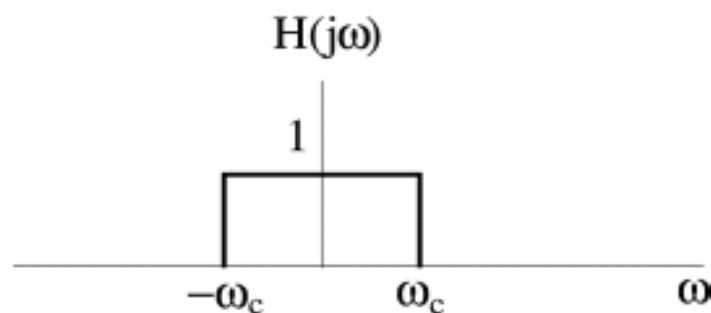
Larger at high ω_0

phase shift

$$\frac{d}{dt} \sin \omega_0 t = \omega_0 \cos \omega_0 t = \omega_0 \sin(\omega_0 t + \frac{\pi}{2})$$

$$\frac{d}{dt} \cos \omega_0 t = -\omega_0 \sin \omega_0 t = \omega_0 \cos(\omega_0 t + \frac{\pi}{2})$$

Example : Impulse Response of an Ideal Lowpass Filter



$$\begin{aligned} h(t) &= \frac{1}{2\pi} \int_{-\omega_c}^{\omega_c} e^{j\omega t} d\omega \\ &= \frac{\sin \omega_c t}{\pi t} \end{aligned}$$

$$= \frac{\omega_c}{\pi} \operatorname{sinc}\left(\frac{\omega_c t}{\pi}\right)$$

$$\text{Define: } \operatorname{sinc}(\theta) = \frac{\sin \pi \theta}{\pi \theta}$$

Questions:

1) Is this a causal system? **No.**

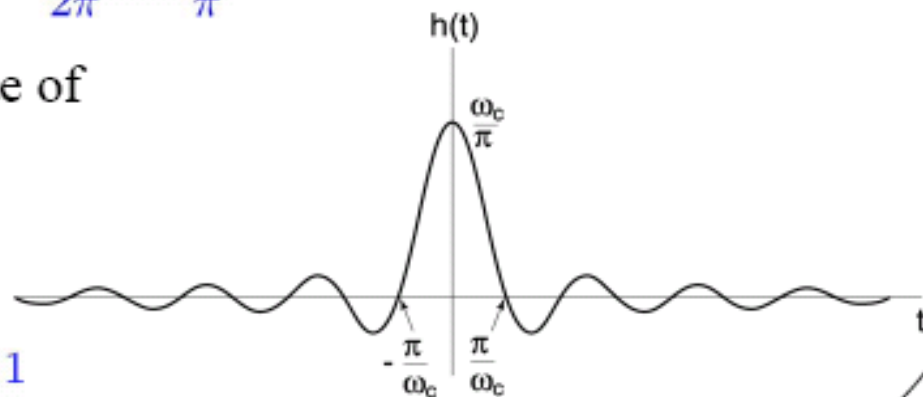
2) What is $h(0)$?

$$h(0) = \frac{1}{2\pi} \int_{-\infty}^{\infty} H(j\omega) d\omega = \frac{2\omega_c}{2\pi} = \frac{\omega_c}{\pi}$$

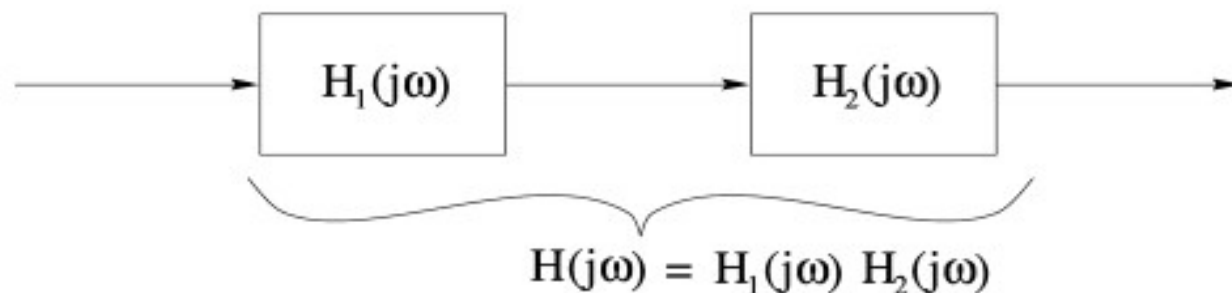
3) What is the steady-state value of the step response, i.e. $s(\infty)$?

$$s(t) = \int_{-\infty}^t h(t) dt$$

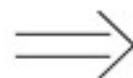
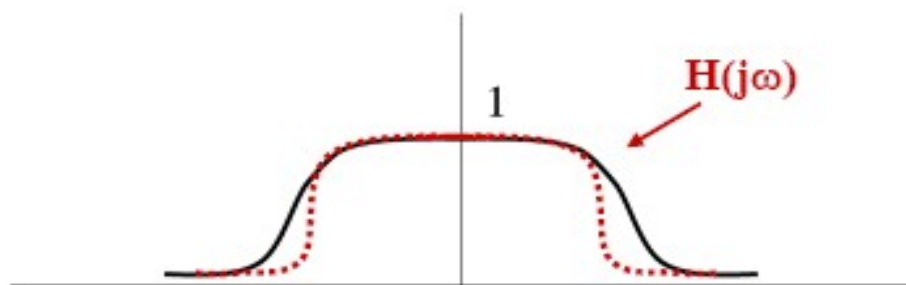
$$s(\infty) = \int_{-\infty}^{\infty} h(t) dt = H(j0) = 1$$



Example : Cascading filtering operations

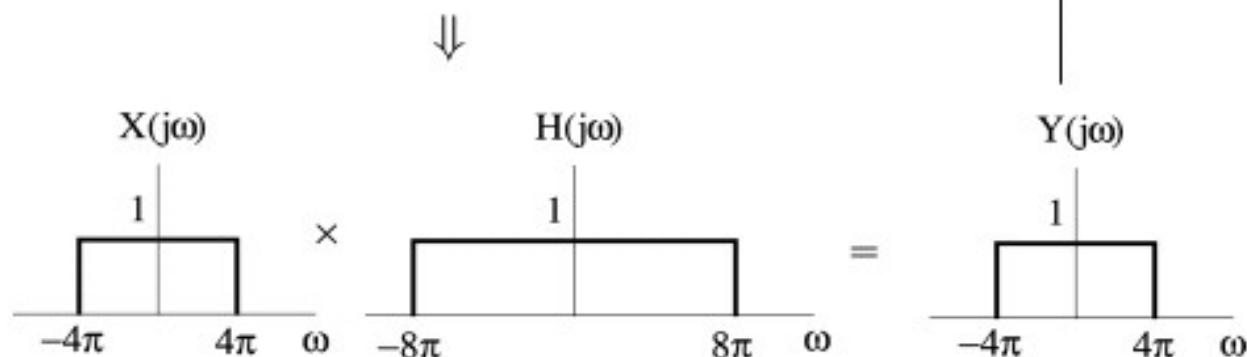


e.g. $H_1(j\omega) = H_2(j\omega)$



$H(j\omega) = H_1^2(j\omega)$ has a sharper frequency selectivity

Example : $\underbrace{\frac{\sin 4\pi t}{\pi t}}_{x(t)} * \underbrace{\frac{\sin 8\pi t}{\pi t}}_{h(t)} = ?$ $Y(j\omega) = X(j\omega)$
 $\Rightarrow y(t) = x(t)$



Example : $e^{-at^2} * e^{-bt^2} = ?$ $\sqrt{\frac{\pi}{a+b}} \cdot e^{-(\frac{ab}{a+b})t^2}$

$\Downarrow$

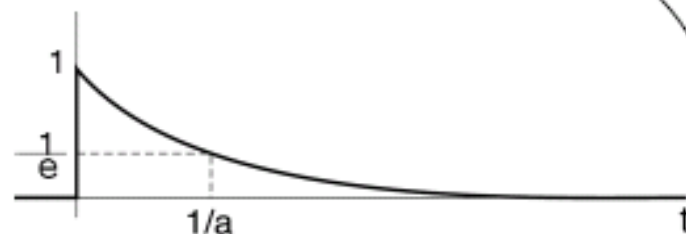
$$\sqrt{\frac{\pi}{a}} e^{-\frac{\omega^2}{4a}} \times \sqrt{\frac{\pi}{b}} e^{-\frac{\omega^2}{4b}} = \frac{\pi}{\sqrt{ab}} e^{-\frac{\omega^2}{4} \left(\frac{1}{a} + \frac{1}{b} \right)}$$

$\Uparrow$

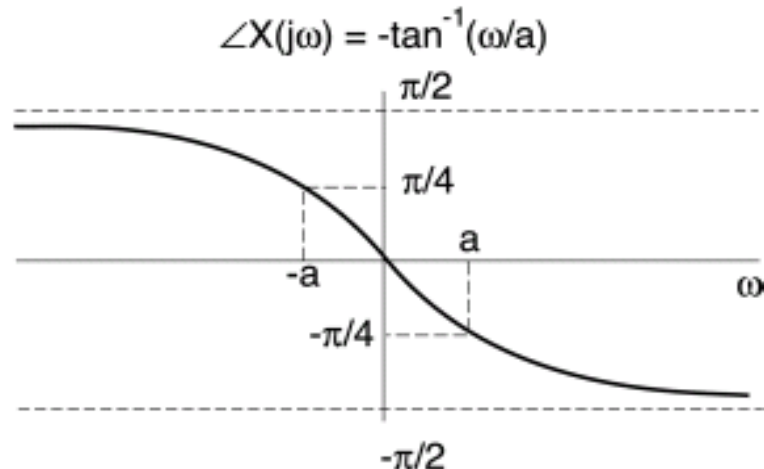
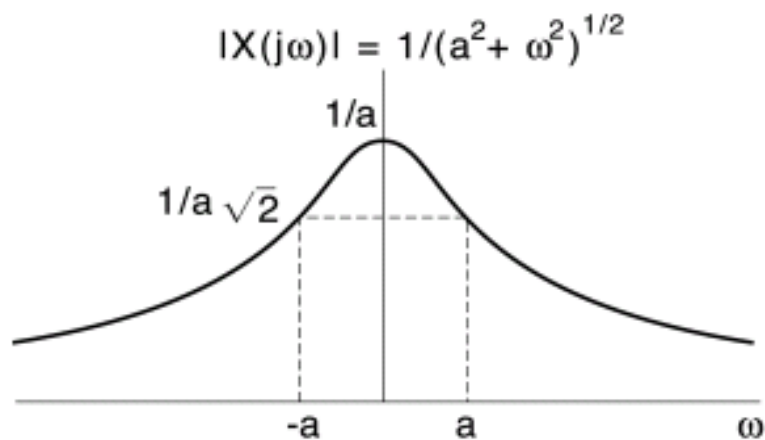
Gaussian $\times$ Gaussian = Gaussian $\Rightarrow$ Gaussian $*$ Gaussian = Gaussian

Example from last lecture

$$x(t) = e^{-at}u(t), \quad a > 0$$



$$\begin{aligned} X(j\omega) &= \int_{-\infty}^{\infty} x(t)e^{-j\omega t} dt = \int_0^{\infty} \underbrace{e^{-at}e^{-j\omega t}}_{e^{-(a+j\omega)t}} dt \\ &= -\left(\frac{1}{a+j\omega}\right) e^{-(a+j\omega)t} \Big|_0^{\infty} = \frac{1}{a+j\omega} \end{aligned}$$



Example :

$$h(t) = e^{-t}u(t), \quad x(t) = e^{-2t}u(t)$$

$$y(t) = h(t) * x(t)$$

$\Downarrow$

$$Y(j\omega) = H(j\omega)X(j\omega) = \frac{1}{(1+j\omega)} \cdot \frac{1}{(2+j\omega)}$$

- a rational function of $j\omega$, ratio of polynomials of $j\omega$

$\Downarrow$ Partial fraction expansion

$$Y(j\omega) = \frac{1}{1+j\omega} - \frac{1}{2+j\omega}$$

$\Downarrow$ inverse FT

$$y(t) = [e^{-t} - e^{-2t}]u(t)$$

Example : LTI Systems Described by LCCDE's
(Linear-constant-coefficient differential equations)

$$\sum_{k=0}^N a_k \frac{d^k y(t)}{dt^k} = \sum_{k=0}^M b_k \frac{d^k x(t)}{dt^k}$$

Using the Differentiation Property

$$\frac{d^k x(t)}{dt^k} \longleftrightarrow (j\omega)^k X(j\omega)$$

$\Downarrow$ Transform both sides of the equation

$$\sum_{k=0}^N a_k \cdot (j\omega)^k Y(j\omega) = \sum_{k=0}^M b_k \cdot (j\omega)^k X(j\omega)$$

$\Downarrow$

$$Y(j\omega) = \underbrace{\left[\frac{\sum_{k=0}^M b_k (j\omega)^k}{\sum_{k=0}^N a_k (j\omega)^k} \right]}_{H(j\omega)} X(j\omega)$$

- 1) Rational, can use PFE to get $h(t)$
- 2) If $X(j\omega)$ is rational
e.g. $x(t) = \sum c_l e^{-at} u(t)$
then $Y(j\omega)$ is also rational

Parseval's Relation

$$\underbrace{\int_{-\infty}^{\infty} |x(t)|^2 dt}_{\text{Total energy in the time-domain}} = \underbrace{\frac{1}{2\pi} \int_{-\infty}^{\infty} |X(j\omega)|^2 d\omega}_{\text{Total energy in the frequency-domain}} \quad \frac{1}{2\pi} |X(j\omega)|^2 \text{ - Spectral density}$$

Multiplication Property

FT is highly symmetric,

$$x(t) \stackrel{\mathcal{F}^{-1}}{=} \frac{1}{2\pi} \int_{-\infty}^{\infty} X(j\omega) e^{j\omega t} d\omega, \quad X(j\omega) \stackrel{\mathcal{F}}{=} \int_{-\infty}^{\infty} x(t) e^{-j\omega t} dt$$

We already know that: $x(t) * y(t) \longleftrightarrow X(j\omega) \cdot Y(j\omega)$

Then it isn't a surprise that:

$$x(t) \cdot y(t) \longleftrightarrow \frac{1}{2\pi} X(j\omega) * Y(j\omega)$$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} X(j\theta) Y(j(\omega - \theta)) d\theta$$

— A consequence of *Duality*

Convolution in ω

Examples of the Multiplication Property

$$r(t) = s(t) \cdot p(t) \quad \longleftrightarrow \quad R(j\omega) = \frac{1}{2\pi} [S(j\omega) * P(j\omega)]$$

$$\text{For } p(t) = \cos \omega_0 t \quad \longleftrightarrow \quad P(j\omega) = \pi \delta(\omega - \omega_0) + \pi \delta(\omega + \omega_0)$$

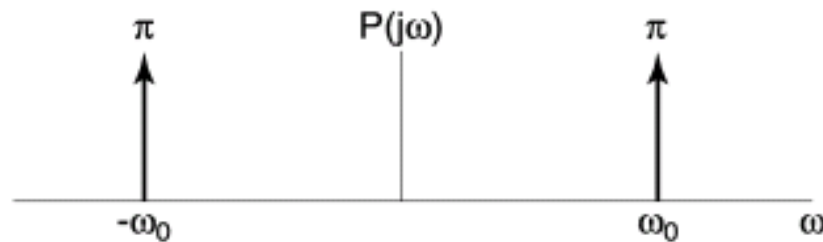
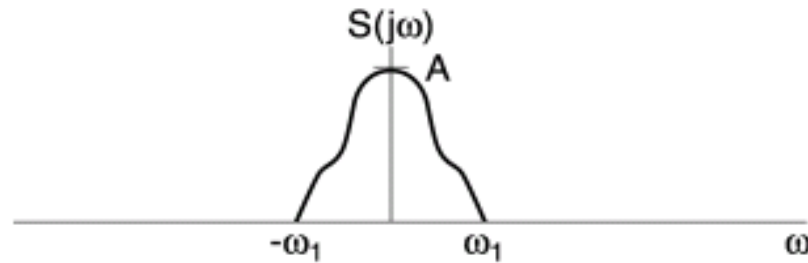


For any $s(t)$...

$$R(j\omega) = \frac{1}{2} S(j(\omega - \omega_0)) + \frac{1}{2} S(j(\omega + \omega_0))$$

Example (continued)

$r(t) = s(t) \cdot \cos(\omega_0 t)$
Amplitude modulation
(AM)



All of the info in $s(t)$ is preserved when multiplied by a sinusoidal

$$R(j\omega) = \frac{1}{2} [S(j(\omega - \omega_0)) + S(j(\omega + \omega_0))]$$

$$R(j\omega) = \frac{1}{2\pi} [S(j\omega) * P(j\omega)]$$



Drawn assuming:

$$\omega_0 - \omega_1 > 0$$

$$\text{i.e. } \omega_0 > \omega_1$$

Integration Transform Derived

First note that

$$\begin{aligned}x(t) * u(t) &= \int_{-\infty}^{+\infty} x(\tau) u(t - \tau) d\tau \\&= \int_{-\infty}^t x(\tau) d\tau\end{aligned}$$

Thus, by the convolution property

$$\begin{aligned}\mathcal{F}\{x(t) * u(t)\} &= \mathcal{F}\left\{\int_{-\infty}^t x(\tau) d\tau\right\} \\&= X(j\omega) \cdot U(j\omega) \\&= X(j\omega) \left(\pi\delta(\omega) + \frac{1}{j\omega}\right) \\&= \frac{1}{j\omega}X(j\omega) + \pi X(0)\delta(\omega)\end{aligned}$$

Integration

$$\int_{-\infty}^t x(\tau) d\tau \xleftrightarrow{\mathcal{F}} \frac{1}{j\omega} X(j\omega) + \pi X(0)\delta(\omega)$$

- The lower limit of the integral is different than the integral defined for the Laplace transform.

Duality

$$\mathcal{F}\{x(t)\} = X(j\omega) \triangleq \int_{-\infty}^{+\infty} x(t) e^{-j\omega t} dt$$

$$\mathcal{F}^{-1}\{X(j\omega)\} = x(t) \triangleq \frac{1}{2\pi} \int_{-\infty}^{+\infty} X(j\omega) e^{j\omega t} d\omega$$

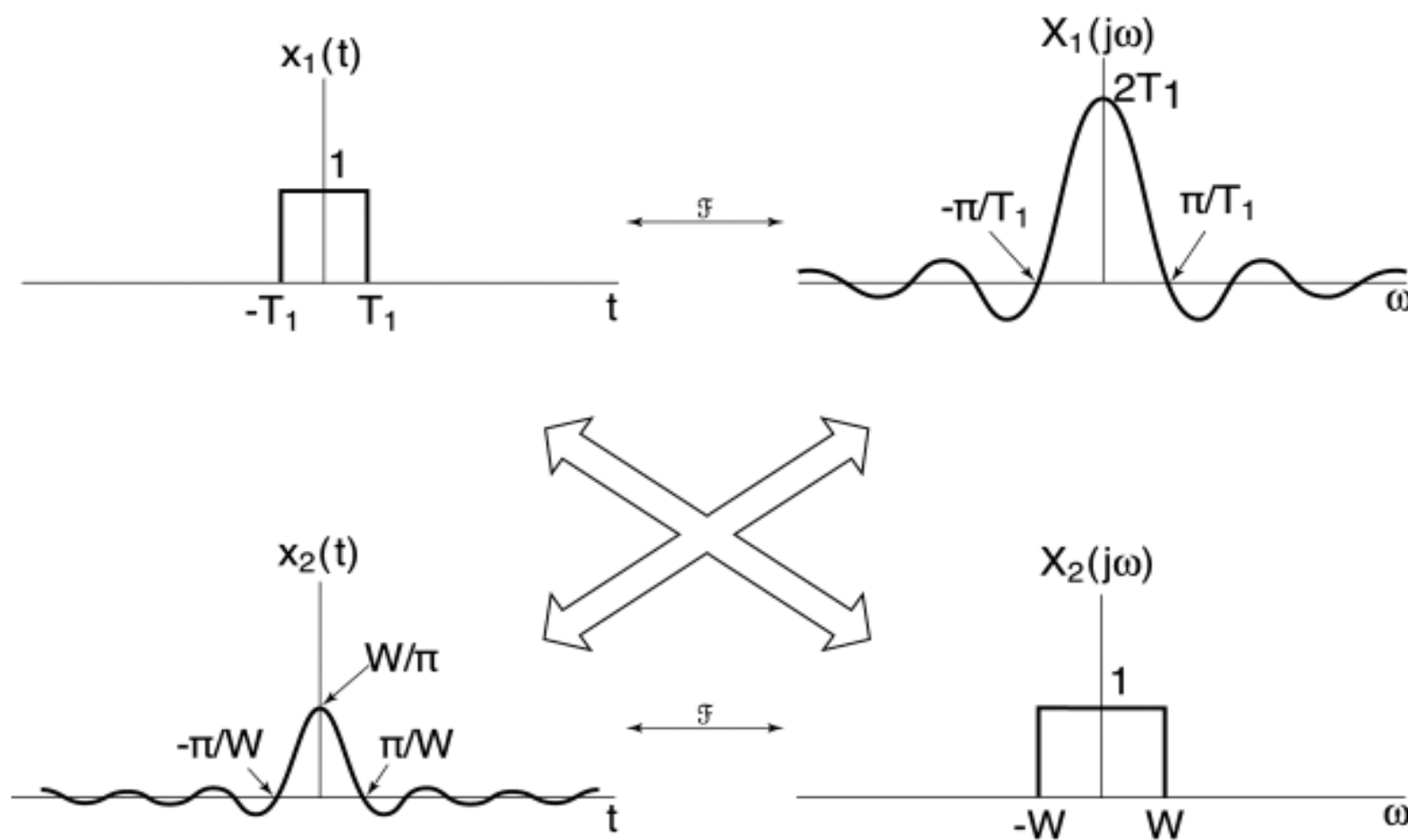
- We have already seen several examples where a Fourier transform pair had almost the exact inverse relationship
- For example

$$\begin{aligned} p_T(t) &\stackrel{\mathcal{F}}{\longleftrightarrow} 2T \operatorname{sinc}\left(\frac{\omega T}{\pi}\right) \\ 2W \operatorname{sinc}\left(\frac{tW}{\pi}\right) &\stackrel{\mathcal{F}}{\longleftrightarrow} 2\pi p_W(\omega) \end{aligned}$$

- This symmetry applies to Fourier transforms in general
- It is due to the similarity of the synthesis and analysis equations

Example of CTFT duality

Square pulse in either time or frequency domain



Example : Duality Derived

Suppose that $x(t) \xleftrightarrow{\mathcal{F}} X(j\omega)$. Solve for the Fourier transform of $X(jt)$ as a function of $x(\omega)$.

$$\mathcal{F}\{X(jt)\} = \Psi(j\omega) = \int_{-\infty}^{+\infty} X(jt) e^{-j\omega t} dt$$

$$\Psi(ju) = \int_{-\infty}^{+\infty} X(jt) e^{-jut} dt$$

$$= \int_{-\infty}^{+\infty} X(j\omega) e^{-ju\omega} d\omega$$

$$= 2\pi \left(\frac{1}{2\pi} \int_{-\infty}^{+\infty} X(j\omega) e^{j\omega(-u)} d\omega \right)$$

$$\Psi(ju) = 2\pi x(-u)$$

$$\Psi(j\omega) = 2\pi x(-\omega)$$

$$\mathcal{F}\{X(jt)\} = 2\pi x(-\omega)$$

Duality Continued

Thus

$$x(t) \xleftrightarrow{\mathcal{F}} X(j\omega) \quad X(jt) \xleftrightarrow{\mathcal{F}} 2\pi x(-\omega) \quad X(-jt) \xleftrightarrow{\mathcal{F}} 2\pi x(\omega)$$

- This is a handy property
- If we know $x(t) \xleftrightarrow{\mathcal{F}} X(j\omega)$ then we can easily find the Fourier transform of $X(jt)$

Signal Multiplication & Frequency Convolution Derived

Use duality to determine the Fourier transform of $x(t) \cdot z(t)$. Hint: use duality and the convolution property applied to $x(t) * h(t)$.

$$y(t) = x(t) * h(t) \quad \xleftrightarrow{\mathcal{F}} \quad Y(j\omega) = X(j\omega) \cdot H(j\omega)$$

$$Y(jt) \quad \xleftrightarrow{\mathcal{F}} \quad 2\pi y(-\omega)$$

$$X(jt) \cdot H(jt) \quad \xleftrightarrow{\mathcal{F}} \quad 2\pi (x(-\omega) * h(-\omega))$$

$$X(jt) \cdot H(jt) \quad \xleftrightarrow{\mathcal{F}} \quad \frac{1}{2\pi} (2\pi x(-\omega)) * (2\pi h(-\omega))$$

$$x(t) \cdot w(t) \quad \xleftrightarrow{\mathcal{F}} \quad \frac{1}{2\pi} X(j\omega) * W(j\omega)$$

This property is important for modulation & sampling to be discussed later

