

FOURIER SERIES

Complex Exponentials as Eigenfunctions of LTI Systems

Fourier Series of CT Periodic Signals

CT Fourier Series Properties

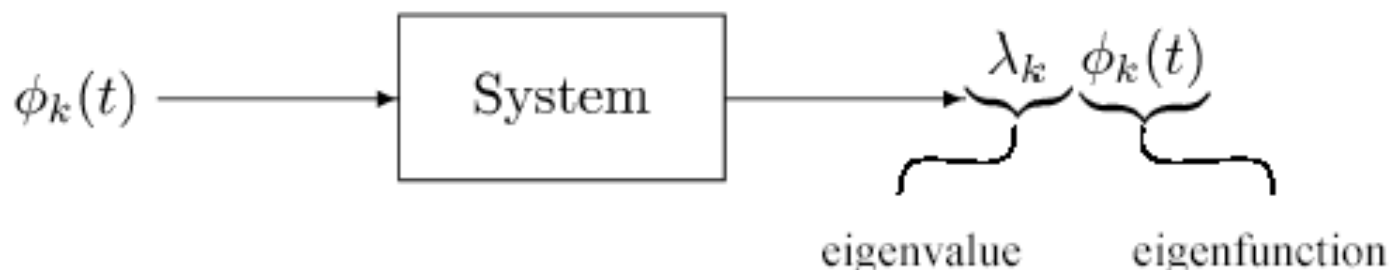
Periodic Convolution

Symmetry Advantage

Gibbs Phenomenon

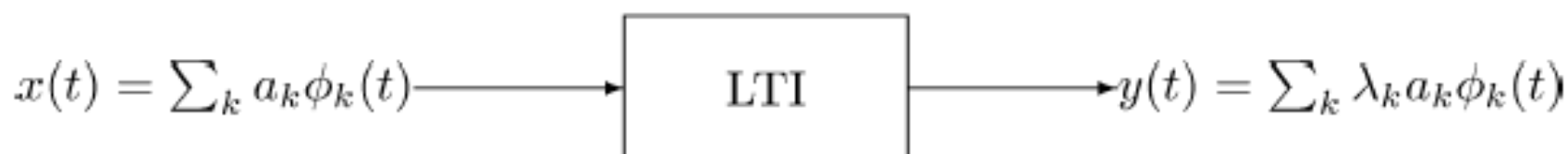
The eigenfunctions $\phi_k(t)$ and their properties

(Focus on CT systems now, but results apply to DT systems as well.)



Eigenfunction in $\rightarrow$ same function out with a “gain”

From the superposition property of LTI systems:



Now the task of finding response of LTI systems is to determine λ_k .

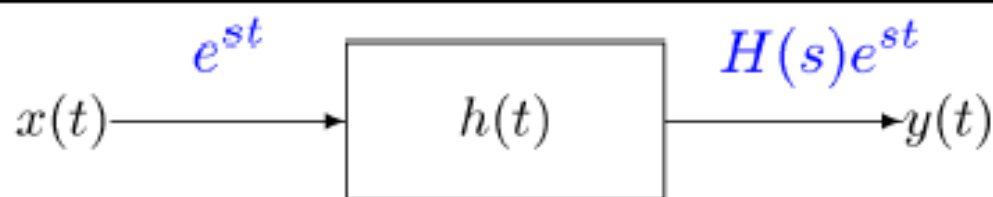
Complex Exponentials as the Eigenfunctions of any LTI Systems

$x(t) = e^{st}$ $\longrightarrow$ $\boxed{h(t)}$ $\longrightarrow$ $y(t)$

$$\begin{aligned} y(t) &= \int_{-\infty}^{\infty} h(\tau) e^{s(t-\tau)} d\tau \\ &= \left[\int_{-\infty}^{\infty} h(\tau) e^{-s\tau} d\tau \right] e^{st} \\ &= \underbrace{H(s)}_{\text{eigenvalue}} \underbrace{e^{st}}_{\text{eigenfunction}} \end{aligned}$$

$x[n] = z^n$ $\longrightarrow$ $\boxed{h[n]}$ $\longrightarrow$ $y[n]$

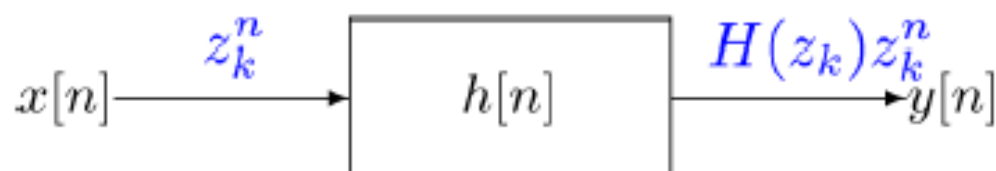
$$\begin{aligned} y[n] &= \sum_{m=-\infty}^{\infty} h[m] z^{n-m} \\ &= \left[\sum_{m=-\infty}^{\infty} h[m] z^{-m} \right] z^n \\ &= \underbrace{H(z)}_{\text{eigenvalue}} \underbrace{z^n}_{\text{eigenfunction}} \end{aligned}$$



$$H(s) = \int_{-\infty}^{\infty} h(t)e^{-st} dt$$

$$x(t) = \sum_k a_k e^{s_k t} \longrightarrow y(t) = \sum_k H(s_k) a_k e^{s_k t}$$

DT:



$$H(z) = \sum_{n=-\infty}^{\infty} h[n] z^{-n}$$

$$x[n] = \sum_k a_k z_k^n \longrightarrow y[n] = \sum_k H(z_k) a_k z_k^n$$

What kinds of signals can we represent as “sums” of complex exponentials?

For Now: Focus on restricted sets of complex exponentials

CT: $s = j\omega$ – purely imaginary,
i.e., signals of the form $e^{j\omega t}$

DT: $z = e^{j\omega}$,
i.e., signals of the form $e^{j\omega n}$

Magnitude 1



CT & DT Fourier Series and Transforms

Periodic Signals

CT complex exponential (sinusoid):

$$x(t) = e^{st}|_{s=j\omega} = e^{j\omega t} = \cos(\omega t) + j \sin(\omega t)$$

$$s = j\omega \Rightarrow \operatorname{Re}\{s\} = 0, \operatorname{Im}\{s\} = \omega$$

DT complex exponential (sinusoid):

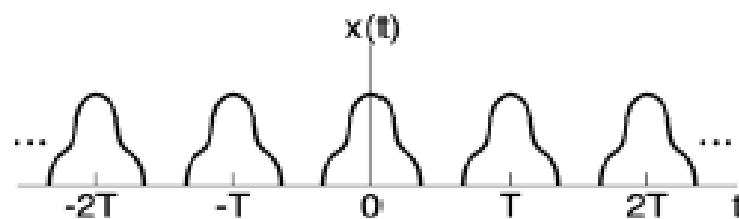
$$z^n = (|z| e^{j\angle z})^n = |z|^n e^{j\omega n}, \quad \omega = \angle z, \quad |z| = 1$$

$$x[n] = z^n|_{|z|=1} = e^{j\omega n} = \cos(\omega n) + j \sin(\omega n)$$

Fourier Series Representation of CT Periodic Signals

$$x(t) = x(t + T) \quad \text{for all } t$$

- smallest such T is the *fundamental period*
- $\omega_0 = \frac{2\pi}{T}$ is the *fundamental frequency*



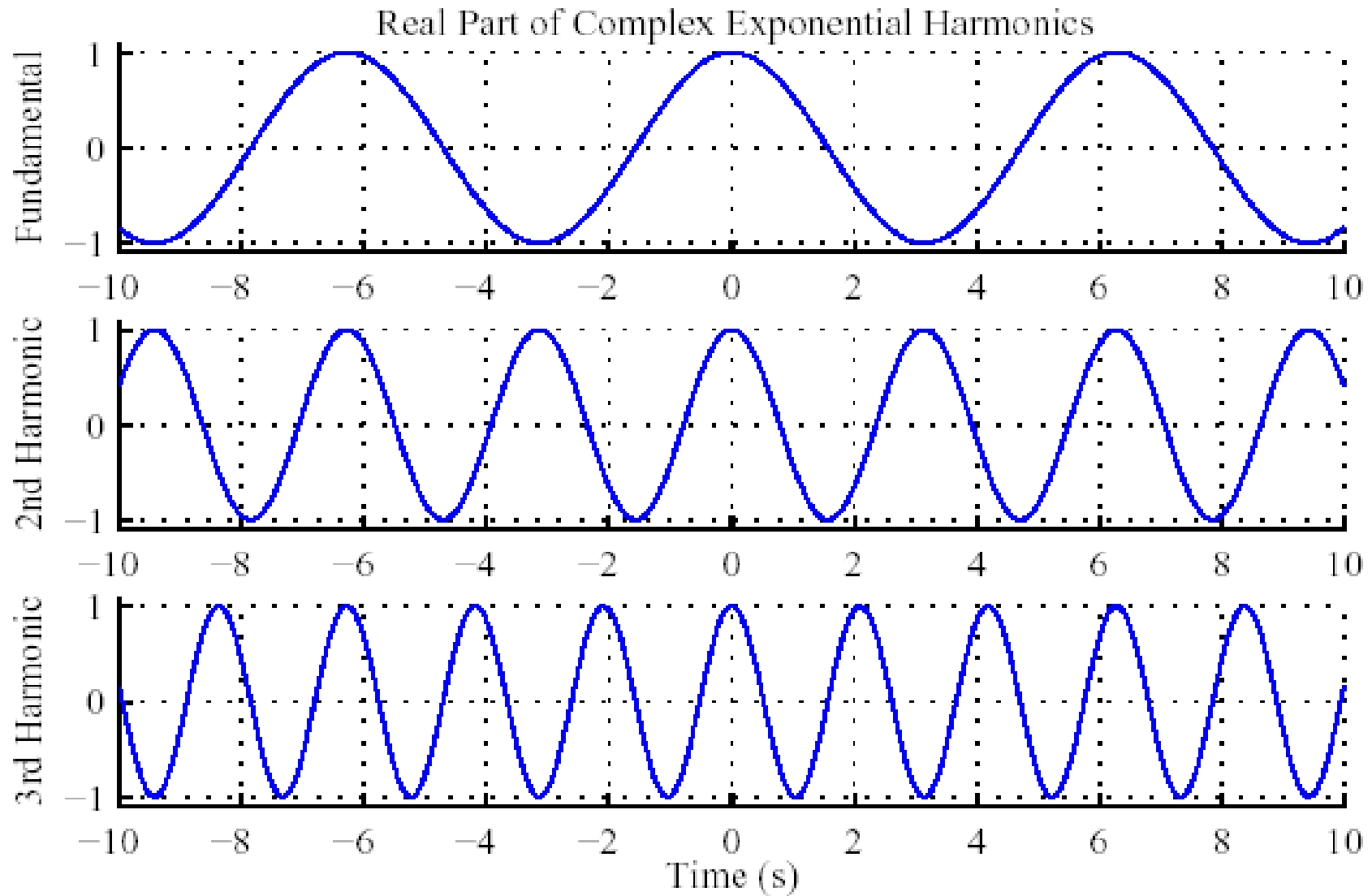
$$e^{j\omega t} \text{ periodic with period } T \Leftrightarrow \omega = k\omega_0$$



$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t} = \sum_{k=-\infty}^{\infty} a_k e^{jk2\pi t/T}$$

- periodic with period T
- $\{a_k\}$ are the *Fourier (series) coefficients* or *spectral coefficients*
- $k = 0$ DC
- $k = \pm 1$ first harmonic
- $k = \pm 2$ second harmonic

Example $x(t) = e^{jt}$



Question #1: How do we find the Fourier coefficients?

First, for simple periodic signals consisting of a few sinusoidal terms

$$\text{Ex: } x(t) = \cos 4\pi t + 2 \sin 8\pi t$$

$$\text{Euler's relation (memorize!)} = \frac{1}{2} [e^{j4\pi t} + e^{-j4\pi t}] + \frac{2}{2j} [e^{j8\pi t} - e^{-j8\pi t}]$$

$$\omega_0 = 4\pi \quad T = \frac{2\pi}{\omega_0} = \frac{2\pi}{4\pi} = \frac{1}{2}$$

$$a_0 = 0 - \text{no dc component}$$

$$a_1 = \frac{1}{2}$$

$$a_{-1} = \frac{1}{2}$$

$$a_2 = \frac{1}{j}$$

$$a_{-2} = -\frac{1}{j}$$

$$a_3 = 0$$

$$a_{-3} = 0$$

$$\vdots$$

- For *real* periodic signals, there are two other commonly used forms for CT Fourier series:

$$x(t) = a_0 + \sum_{k=1}^{\infty} [\alpha_k \cos k\omega_0 t + \beta_k \sin k\omega_0 t]$$

or

$$x(t) = a_0 + \sum_{k=1}^{\infty} [\gamma_k \cos(k\omega_0 t + \theta_k)]$$

- Because of the eigenfunction property of $e^{j\omega t}$, we will usually use the complex exponential form

- A consequence of this is that we need to include terms for *both* positive and negative frequencies:

$$e^{jk\omega_0 t} \quad , \quad e^{-jk\omega_0 t}$$

$$\text{Remember } \cos(k\omega_0 t) = \frac{1}{2}(e^{jk\omega_0 t} + e^{-jk\omega_0 t})$$

$$\text{and } \sin(k\omega_0 t) = \frac{1}{2j}(e^{jk\omega_0 t} - e^{-jk\omega_0 t})$$

Now, the complete answer to Question #1

(Given $x(t)$,
how find a_k ?)

Suppose

- 1) multiply by $e^{-jn\omega_0 t}$
- 2) integrate over one period

$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t}$$

$\Downarrow$

- 1) multiply by $e^{-jn\omega_0 t}$
- 2) integrate over one period

$$\begin{aligned} \int_T x(t) e^{-jn\omega_0 t} dt &= \int_T \left(\sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t} \right) e^{-jn\omega_0 t} dt \\ &= \sum_{k=-\infty}^{\infty} a_k \left(\int_T e^{j(k-n)\omega_0 t} dt \right) \end{aligned}$$

(Here $\int_T$ denotes integral over *any* interval of length T (one period).)

Next, note that

$$\begin{aligned} \int_T e^{j(k-n)\omega_0 t} dt &= \begin{cases} T, & k = n \\ 0, & k \neq n \end{cases} \\ &= T\delta[k - n] \quad \text{Orthogonality} \\ &\Downarrow \end{aligned}$$

$$\int_T x(t) e^{-jn\omega_0 t} dt = \sum_{k=-\infty}^{\infty} a_k \left(\int_T e^{j(k-n)\omega_0 t} dt \right) = \sum_{k=-\infty}^{\infty} a_k \cdot T \delta[k-n]$$

$$\int_T x(t) e^{-jn\omega_0 t} dt = a_n T$$



CT Fourier Series Pair ($\omega_0 = \frac{2\pi}{T}$)

$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t} \quad (\text{Synthesis equation})$$

$$a_k = \frac{1}{T} \int_T x(t) e^{-jk\omega_0 t} dt \quad (\text{Analysis equation})$$

Alternative Forms: Amplitude-Phase

There are alternative expressions for Fourier series. Since $a_{-k} = a_k^*$,

$$\begin{aligned}x(t) &= \sum_{k=-\infty}^{\infty} a_k e^{jk\omega t} = a_0 + \sum_{k=1}^{\infty} a_k e^{jk\omega t} + \sum_{k=1}^{\infty} a_{-k} e^{-jk\omega t} \\&= a_0 + \sum_{k=1}^{\infty} (a_k e^{jk\omega t}) + (a_k e^{jk\omega t})^* \\&= a_0 + 2 \sum_{k=1}^{\infty} \operatorname{Re} \{ a_k e^{jk\omega t} \} = a_0 + 2 \sum_{k=1}^{\infty} \operatorname{Re} \{ |a_k| e^{j\theta_k} e^{jk\omega t} \} \\&= a_0 + 2 \sum_{k=1}^{\infty} |a_k| \operatorname{Re} \left\{ e^{j(k\omega t + \theta_k)} \right\} \\&= a_0 + 2 \sum_{k=1}^{\infty} A_k \cos(k\omega t + \theta_k)\end{aligned}$$

Here $A_k \triangleq |a_k|$ and $\theta_k \triangleq \angle a_k$ is the complex phase angle of a_k .

Alternative Forms: Trigonometric

$$\begin{aligned}x(t) &= \sum_{k=-\infty}^{\infty} a_k e^{jk\omega t} \\&= a_0 + 2 \sum_{k=1}^{\infty} \operatorname{Re} \{ a_k e^{jk\omega t} \} \\&= a_0 + 2 \sum_{k=1}^{\infty} \operatorname{Re} \{ a_k \cos(k\omega t) + ja_k \sin(k\omega t) \} \\&= a_0 + 2 \sum_{k=1}^{\infty} \operatorname{Re} \{ a_k \} \cos(k\omega t) - \operatorname{Im} \{ a_k \} \sin(k\omega t) \\&= a_0 + 2 \sum_{k=1}^{\infty} B_k \cos(k\omega t) - C_k \sin(k\omega t)\end{aligned}$$

where $B_k \triangleq \operatorname{Re} \{ a_k \}$ and $C_k \triangleq \operatorname{Im} \{ a_k \}$.

Alternate Forms Summary

Exponential: $x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega t}$

Amplitude-Phase: $x(t) = a_0 + 2 \sum_{k=1}^{\infty} A_k \cos(k\omega t + \theta_k)$

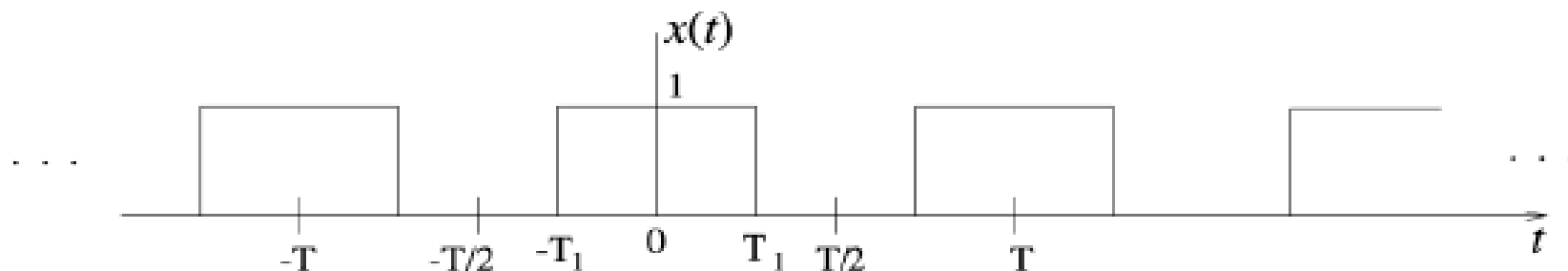
Trigonometric: $x(t) = a_0 + 2 \sum_{k=1}^{\infty} B_k \cos(k\omega t) - C_k \sin(k\omega t)$

$$a_k = A_k e^{j\theta_k} = B_k + jC_k$$

$$A_k = \sqrt{B_k^2 + C_k^2} = |a_k| \qquad \theta_k = \angle(B_k + jC_k) = \angle a_k$$

$$B_k = A_k \cos(\theta_k) = \operatorname{Re} \{a_k\} \qquad C_k = A_k \sin(\theta_k) = \operatorname{Im} \{a_k\}$$

Ex: Periodic Square Wave



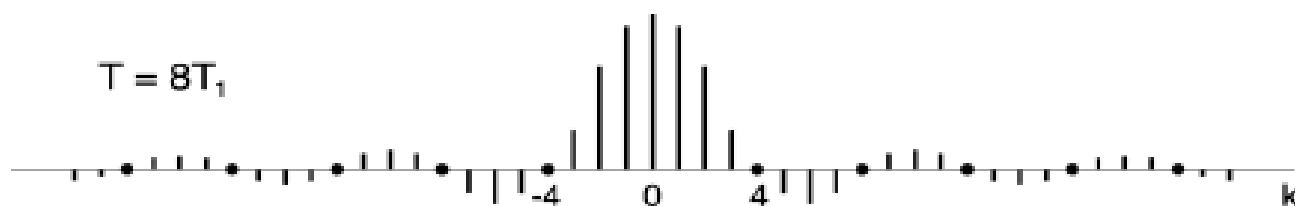
For $k = 0$

$$a_0 = \frac{1}{T} \int_{-T/2}^{T/2} x(t) dt = \frac{2T_1}{T}$$

DC component
is just the
average

For $k \neq 0$

$$\begin{aligned} a_k &= \frac{1}{T} \int_{-T/2}^{T/2} x(t) e^{-jk\omega_0 t} dt = \frac{1}{T} \int_{-T_1}^{T_1} e^{-jk\omega_0 t} dt \\ &= -\frac{1}{jk\omega_0 T} e^{-jk\omega_0 t} \Big|_{-T_1}^{T_1} = \frac{\sin k\omega_0 T_1}{k\pi} \quad \left(\omega_0 = \frac{2\pi}{T} \right) \end{aligned}$$



CT Fourier Series Pairs

$$\left(\omega_0 = \frac{2\pi}{T} \right)$$

Review:

$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t} = \sum_{k=-\infty}^{\infty} a_k e^{j2\pi kt/T}$$

$$a_k = \frac{1}{T} \int_T x(t) e^{-jk\omega_0 t} dt$$

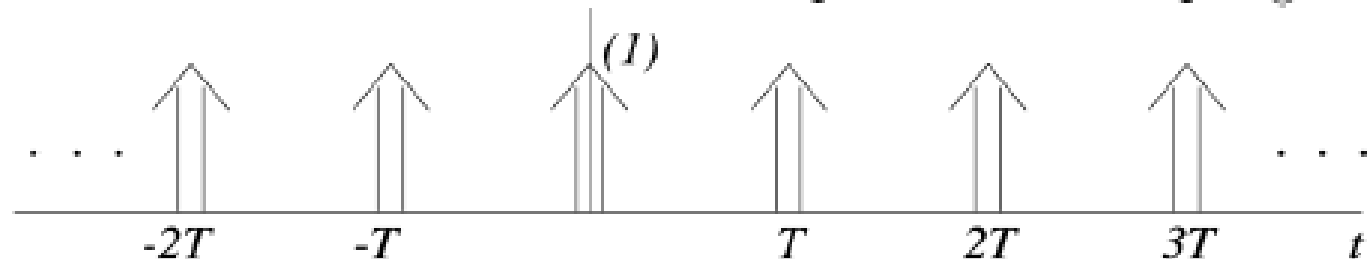
Skip it in future
for shorthand

$$x(t) \xleftrightarrow{FS} a_k$$

Another (important!) example: Periodic Impulse Train

$$x(t) = \sum_{n=-\infty}^{\infty} \delta(t - nT) \quad - \quad \text{Sampling function}$$

important for sampling



$$\begin{aligned} a_k &= \frac{1}{T} \int_{-T/2}^{T/2} x(t) e^{-jk\omega_0 t} dt = \frac{1}{T} \int_{-T/2}^{T/2} \delta(t) e^{-jk\omega_0 t} dt \\ &= \frac{1}{T} \quad \text{for all } k ! \end{aligned}$$

$\Downarrow$

$$x(t) = \frac{1}{T} \sum_{k=-\infty}^{\infty} e^{jk\omega_0 t}$$

— All components have:
(1) the same amplitude,
&
(2) the same phase.

Dirichlet Conditions for Convergence

The Fourier series representation of a periodic signal $x(t)$ converges if *all* of the following conditions are met.

1. $\int_{T_0} |x(t)| dt < \infty$
2. Finite number of discontinuities in a period T_0
3. Finite number of distinct maxima and minima in T_0
4. $x(t)$ is single valued

These are sufficient, but not necessary, conditions.

- These conditions are usually satisfied in practical context
- When these conditions satisfied, the truncated Fourier series, with finite (N) no. of terms, will be the best approximation to the original function;
- The total squared-error magnitude over one period is used for comparison:

$$J = \int_T [x(t) - x_N(t)]^2 dt$$

$$x_N(t) = \sum_{k=-N}^N a_k e^{jk\omega t}$$

Properties of CT Fourier Series

- Linearity $x(t) \leftrightarrow a_k, y(t) \leftrightarrow b_k \Rightarrow \alpha x(t) + \beta y(t) \leftrightarrow \alpha a_k + \beta b_k$
- Conjugate Symmetry

$$x(t) \text{ is real} \Rightarrow a_{-k} = a_k^*$$

$\Downarrow$

$$\begin{aligned} a_k &= \operatorname{Re}\{a_k\} + j\operatorname{Im}\{a_k\} \\ &= |a_k|e^{j\angle a_k} \end{aligned}$$

$\operatorname{Re}\{a_k\}$ is even, $\operatorname{Im}\{a_k\}$ is odd

or

$|a_k|$ is even, $\angle a_k$ is odd

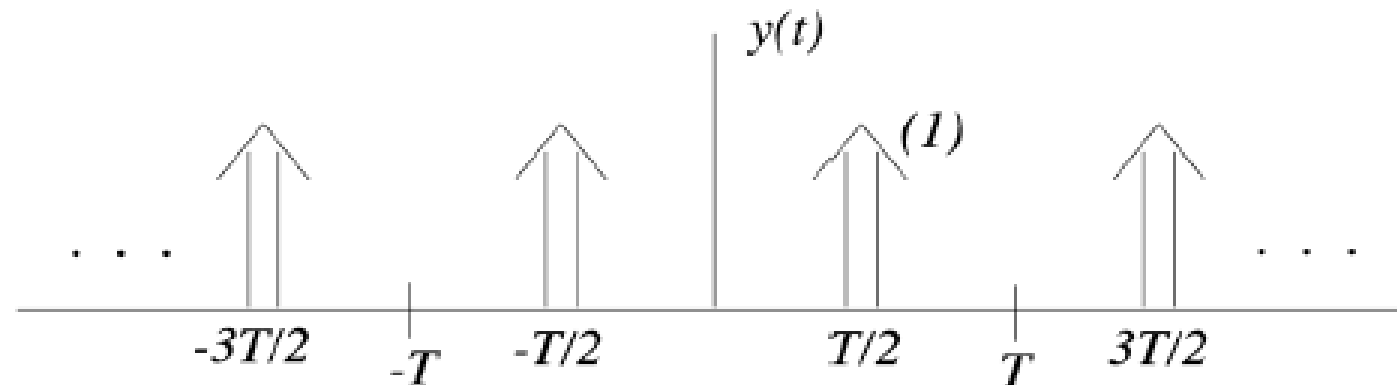
- Time shift
$$\begin{aligned} x(t) &\leftrightarrow a_k \\ x(t - t_0) &\leftrightarrow a_k e^{-jk\omega_0 t_0} = a_k e^{-jk2\pi t_0/T} \end{aligned}$$

Introduces a linear phase shift $\propto t_0$

Example: Shift by half period

$$y(t) = x(t - T/2) \leftrightarrow a_k e^{-jk\pi} = (-1)^k a_k$$

$$\text{using } e^{-jk\omega_0 T/2} = e^{-jk\pi}$$



$$y(t) \leftrightarrow (-1)^k a_k \quad \left(a_k = \frac{1}{T} = \text{F.C. of } \sum_{n=-\infty}^{\infty} \delta(t - nT) \right)$$

$$\parallel$$
$$\frac{(-1)^k}{T}$$

• Time Reversal

$y(t)=x(-t)$, the period T is unchanged,

$$y(t) = x(-t) = \sum_{k=-\infty}^{\infty} a_k e^{-jk\omega t} = \sum_{m=-\infty}^{\infty} a_{-m} e^{jm\omega t}$$

where $k = -m$ substitution is made.

$$x(t) \leftrightarrow a_k \quad \text{then} \quad x(-t) \leftrightarrow a_{-k}$$

Note that if $x(t)$ is even, $x(-t) = x(t)$ then $a_{-k} = a_k$;

And if $x(t)$ is odd; $x(-t) = -x(t)$ then $a_{-k} = -a_k$;

- **Time Scaling**

For a positive real number α , $x(\alpha t)$ will be periodic with T/α and fundamental frequency $\alpha\omega$;

$$x(\alpha t) = \sum_{k=-\infty}^{\infty} a_k e^{jk(\alpha\omega)t}$$

Note that the Fourier coefficients have not changed but the Fourier series representation *has changed*.

- **Parseval's Relation** (applies to power signals)

$$P = \underbrace{\frac{1}{T} \int_T |x(t)|^2 dt}_{\text{Average signal power}} = \sum_{k=-\infty}^{\infty} \underbrace{|a_k|^2}_{\text{Power in the } k_{th} \text{ harmonic}}$$

Energy is the same whether measured in the time-domain or the frequency-domain

- **Multiplication Property**

$$x(t) \leftrightarrow a_k, y(t) \leftrightarrow b_k \quad (\text{Both } x(t) \text{ and } y(t) \text{ are periodic with the same period } T)$$

$$\Downarrow$$

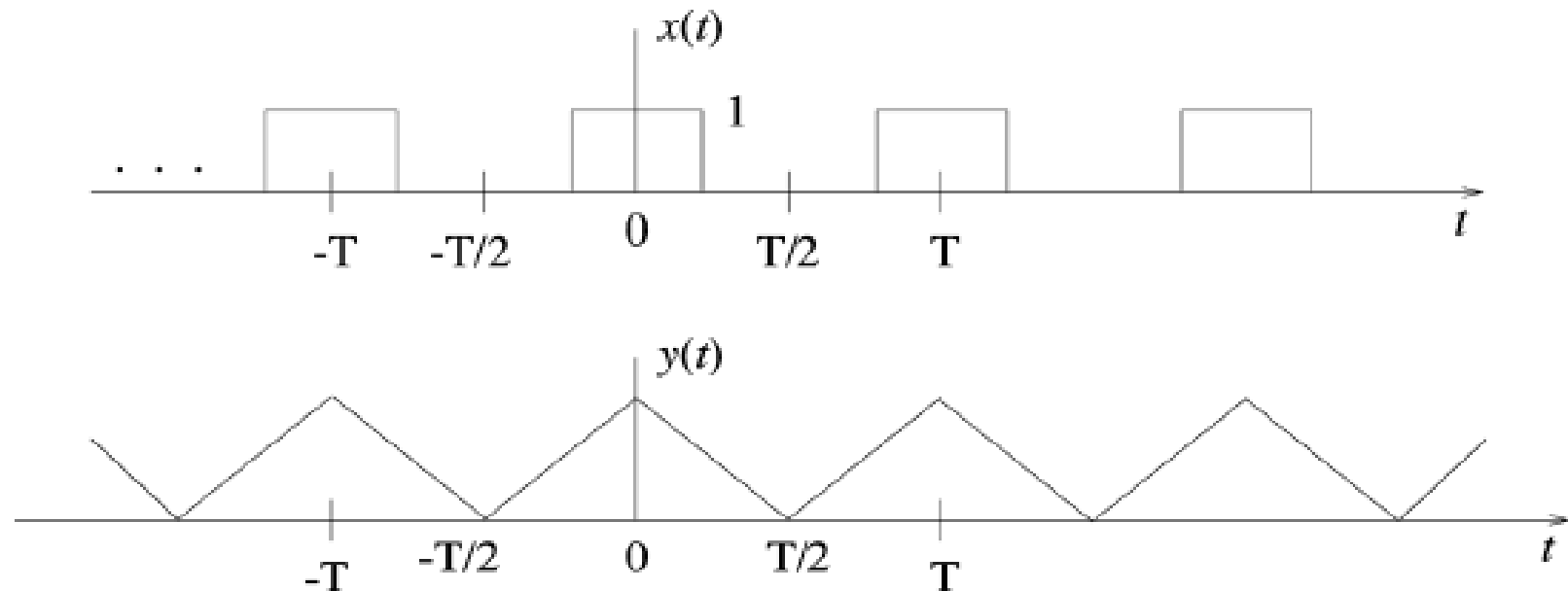
$$x(t) \cdot y(t) \leftrightarrow c_k = \sum_{l=-\infty}^{\infty} a_l b_{k-l} = a_k * b_k$$

Proof:

$$\underbrace{\sum_l a_l e^{jl\omega_0 t}}_{x(t)} \cdot \underbrace{\sum_m b_m e^{jm\omega_0 t}}_{y(t)} = \sum_{l,m} a_l b_m e^{j(l+m)\omega_0 t} \xrightarrow{l+m=k} \sum_k \left[\underbrace{\sum_l a_l b_{k-l}}_{c_k} \right] e^{jk\omega_0 t}$$

Periodic Convolution

$x(t), y(t)$ periodic with period T



$$x(t) * y(t) = \int_{-\infty}^{\infty} x(\tau) y(t - \tau) d\tau$$

E.g. If both $x(t)$ and $y(t)$ are positive, then

$$x(t) * y(t) = \infty$$

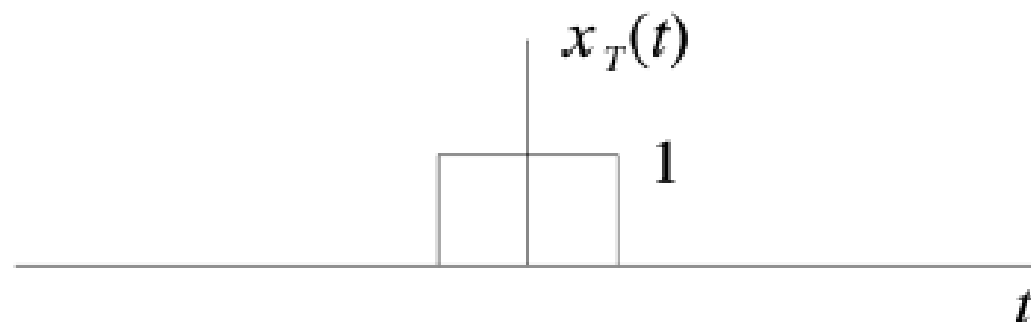
Periodic Convolution (continued)

Periodic convolution: Integrate over *any* one period (e.g. $-T/2$ to $T/2$)

$$z(t) = \int_{-T/2}^{T/2} x(\tau)y(t - \tau)d\tau = \int_{-\infty}^{\infty} x_T(\tau)y(t - \tau)d\tau$$

where

$$x_T(t) = \begin{cases} x(t) & -T/2 < t < T/2 \\ 0 & \text{otherwise} \end{cases}$$



Periodic Convolution (continued) Facts

- 1) $z(t)$ is periodic with period T

From Lecture #2, $x(t) = x(t + T) \rightarrow y(t) = y(t + T)$ for LTI systems.

In the convolution, treat $y(t)$ as the input and $x_T(t)$ as $h(t)$

- 2) Doesn't matter what period over which we choose to integrate:

$$z(t) = \int_T x(\tau)y(t - \tau)d\tau = x(t) \otimes y(t)$$

- 3) **Periodic
convolution
in time**

$$x(t) \leftrightarrow a_k, y(t) \leftrightarrow b_k, z(t) \leftrightarrow c_k$$

$$\begin{aligned} c_k &= \frac{1}{T} \int_T z(t) e^{-jk\omega_0 t} dt = \frac{1}{T} \int_T \left(\int_T x(\tau) y(t - \tau) d\tau \right) e^{-jk\omega_0 t} dt \\ &= \int_T \underbrace{\left(\frac{1}{T} \int_T y(t - \tau) e^{-jk\omega_0(t - \tau)} dt \right)}_{b_k} x(\tau) e^{-jk\omega_0 \tau} d\tau \end{aligned}$$

$$= \int_T b_k x(\tau) e^{-jk\omega_0 \tau} d\tau = T a_k b_k$$

**Multiplication
in frequency!**

Products of Even & Odd Signals

The product of even and odd signal is either even or odd

$x(t)$ **even**, $y(t)$ **even** $\Rightarrow$ **even**

$$p(t) \triangleq x(t)y(t) = x(-t)y(-t) = p(-t)$$

$x(t)$ **even**, $y(t)$ **odd** $\Rightarrow$ **odd**

$$p(t) \triangleq x(t)y(t) = x(-t)[-y(-t)] = -p(-t)$$

$x(t)$ **odd**, $y(t)$ **even** $\Rightarrow$ **odd**

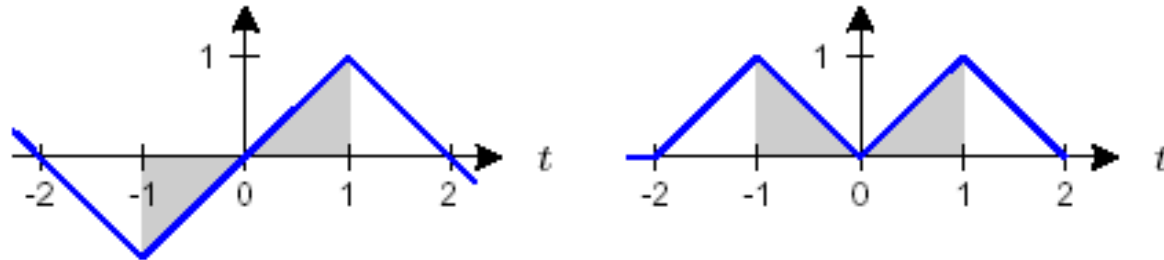
$$p(t) \triangleq x(t)y(t) = [-x(-t)]y(-t) = -p(-t)$$

$x(t)$ **odd**, $y(t)$ **odd** $\Rightarrow$ **even**

$$p(t) \triangleq x(t)y(t) = [-x(-t)][-y(-t)] = x(-t)y(-t) = p(-t)$$

The same relations hold for discrete-time signals

Symmetry Advantage



- Conceptually, the FS coefficients are the integrals of products
- The integral is nothing more than the area under a curve of this product
- If the product is odd, the total area is zero
- If the product is even, the total area is twice that of the left/right half
- Consider the trigonometric FS coefficients

Symmetry & FS Coefficients

$$x(t) = a_0 + 2 \sum_{k=1}^{\infty} B_k \cos(k\omega t) - C_k \sin(k\omega t)$$

$$\begin{aligned} B_k &= \frac{1}{T_0} \int_{T_0} x(t) \cos(k\omega t) dt \\ &= \frac{1}{T_0} \int_{-T_0/2}^{+T_0/2} x_e(t) \cos(k\omega t) dt + \frac{1}{T_0} \int_{-T_0/2}^{+T_0/2} x_o(t) \cos(k\omega t) dt \\ &= \frac{1}{T_0} \int_{-T_0/2}^{+T_0/2} x_e(t) \cos(k\omega t) dt \end{aligned}$$

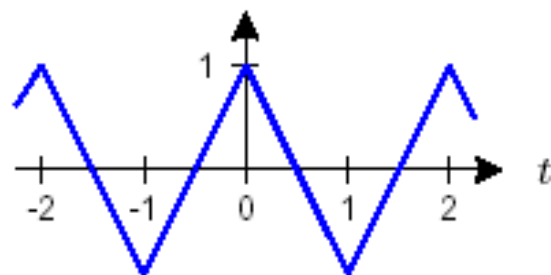
$$\begin{aligned} C_k &= -\frac{1}{T_0} \int_{T_0} x(t) \sin(k\omega t) dt \\ &= -\frac{1}{T_0} \int_{-T_0/2}^{+T_0/2} x_e(t) \sin(k\omega t) dt - \frac{1}{T_0} \int_{-T_0/2}^{+T_0/2} x_o(t) \sin(k\omega t) dt \\ &= -\frac{1}{T_0} \int_{-T_0/2}^{+T_0/2} x_o(t) \sin(k\omega t) dt \end{aligned}$$

Symmetry & FS Coefficients Continued 2

$$\begin{aligned} B_k &= \frac{1}{T_0} \int_{-T_0/2}^{+T_0/2} x_e(t) \cos(k\omega t) dt \\ &= \frac{2}{T_0} \int_0^{+T_0/2} x_e(t) \cos(k\omega t) dt \\ C_k &= -\frac{1}{T_0} \int_{-T_0/2}^{+T_0/2} x_o(t) \sin(k\omega t) dt \\ &= -\frac{2}{T_0} \int_0^{+T_0/2} x_o(t) \sin(k\omega t) dt \end{aligned}$$

- Symmetry can simplify calculation of the FS coefficients
- If $x(t)$ is even, $C_k = 0$
- If $x(t)$ is odd, $B_k = 0$
- Each product within the non-zero integrals is even

Example : Triangle Waveform

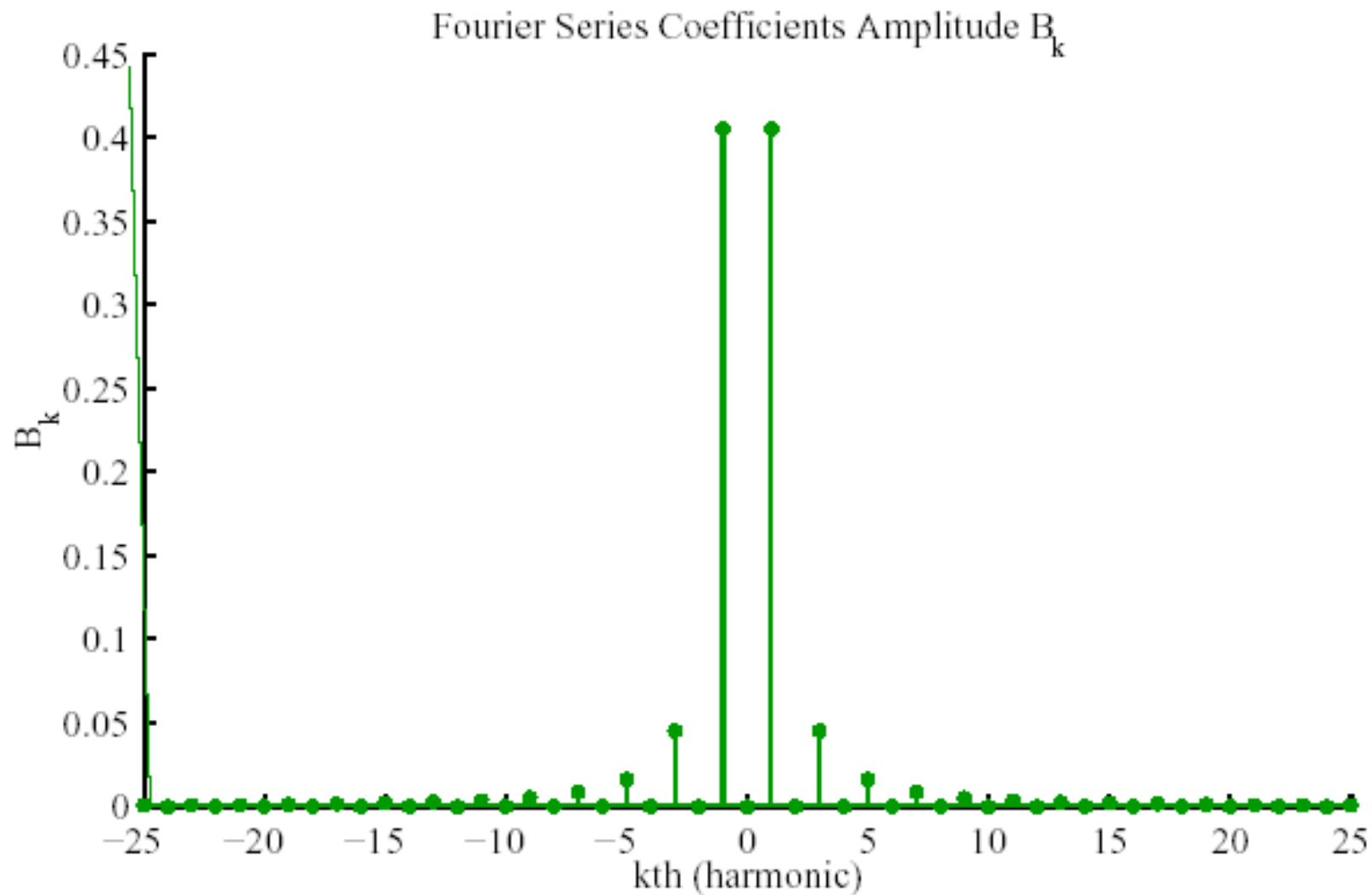


What is the fundamental period of $x(t)$? What type of symmetry does the signal have? Find the trigonometric Fourier series coefficients.

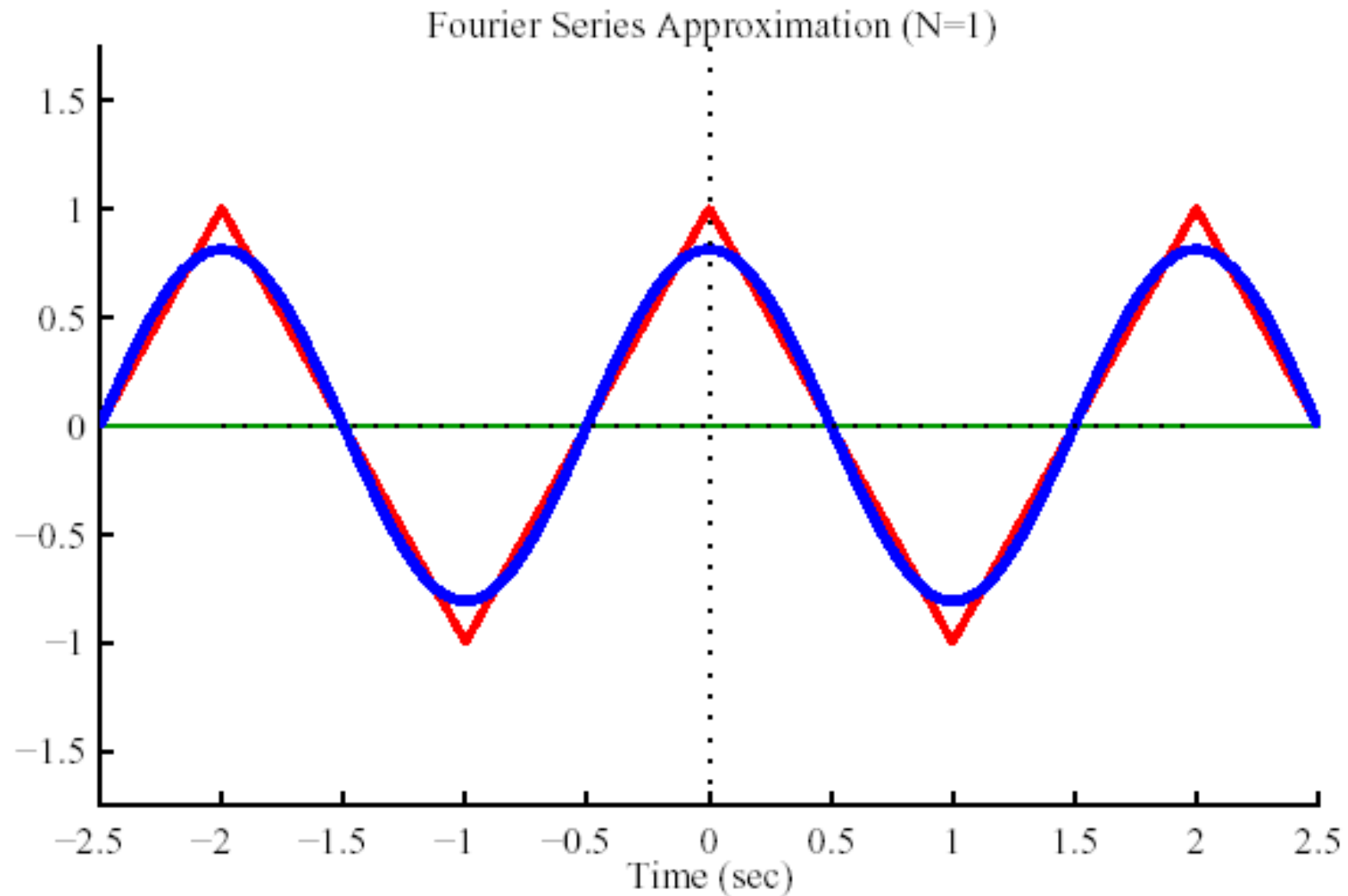
Plot the partial Fourier series sum for $N = 1, 5, 10, \& 50$. Hint:

$$\int t \cos(\omega t) dt = \frac{1}{\omega^2} \cos(\omega t) + \frac{t}{\omega} \sin(\omega t) + C$$

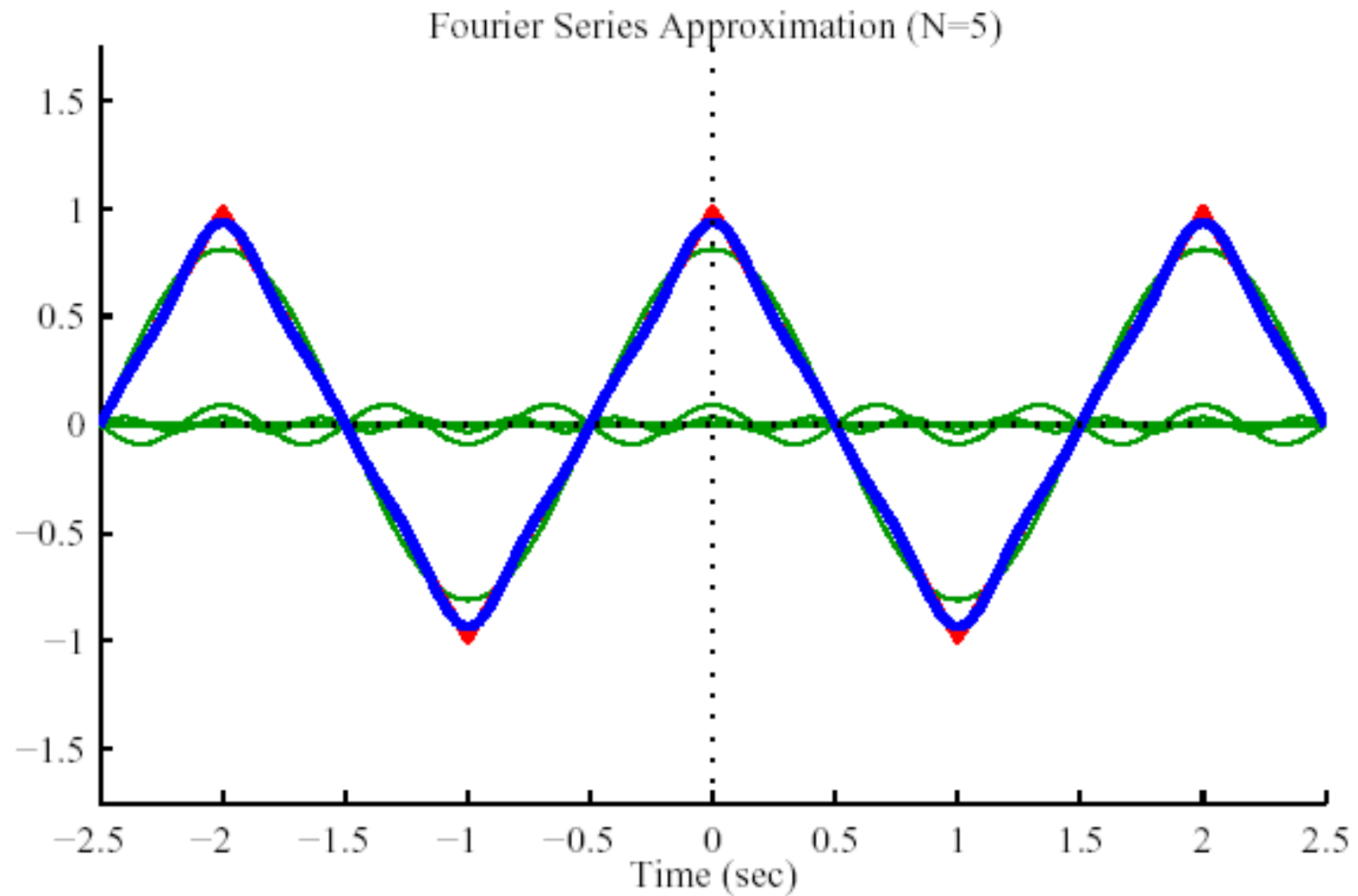
Coefficient Spectrum



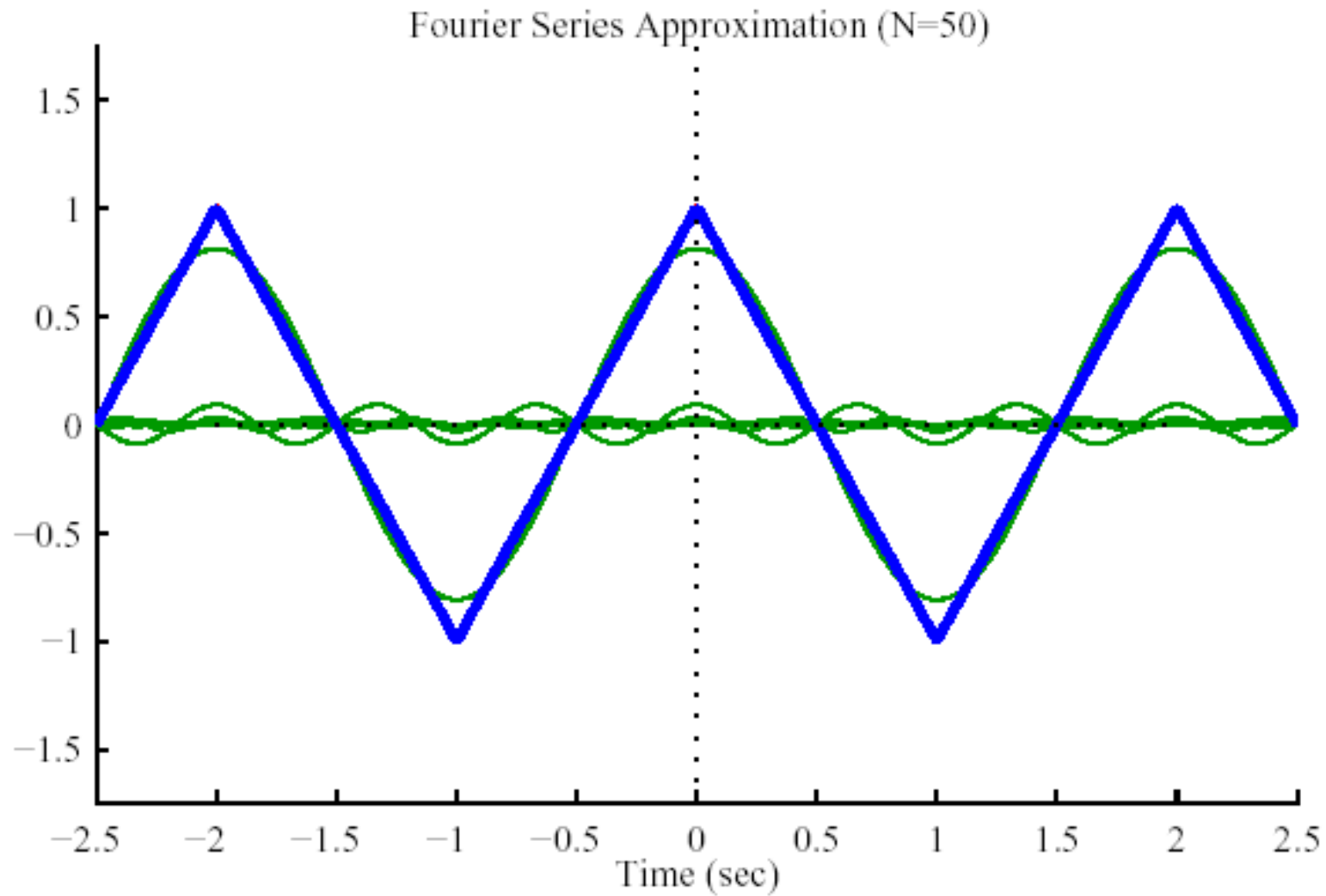
Partial Fourier Series $N=1$



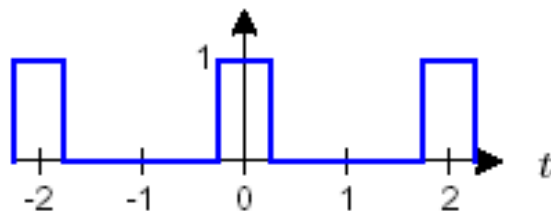
Partial Fourier Series $N=5$



Partial Fourier Series $N=50$

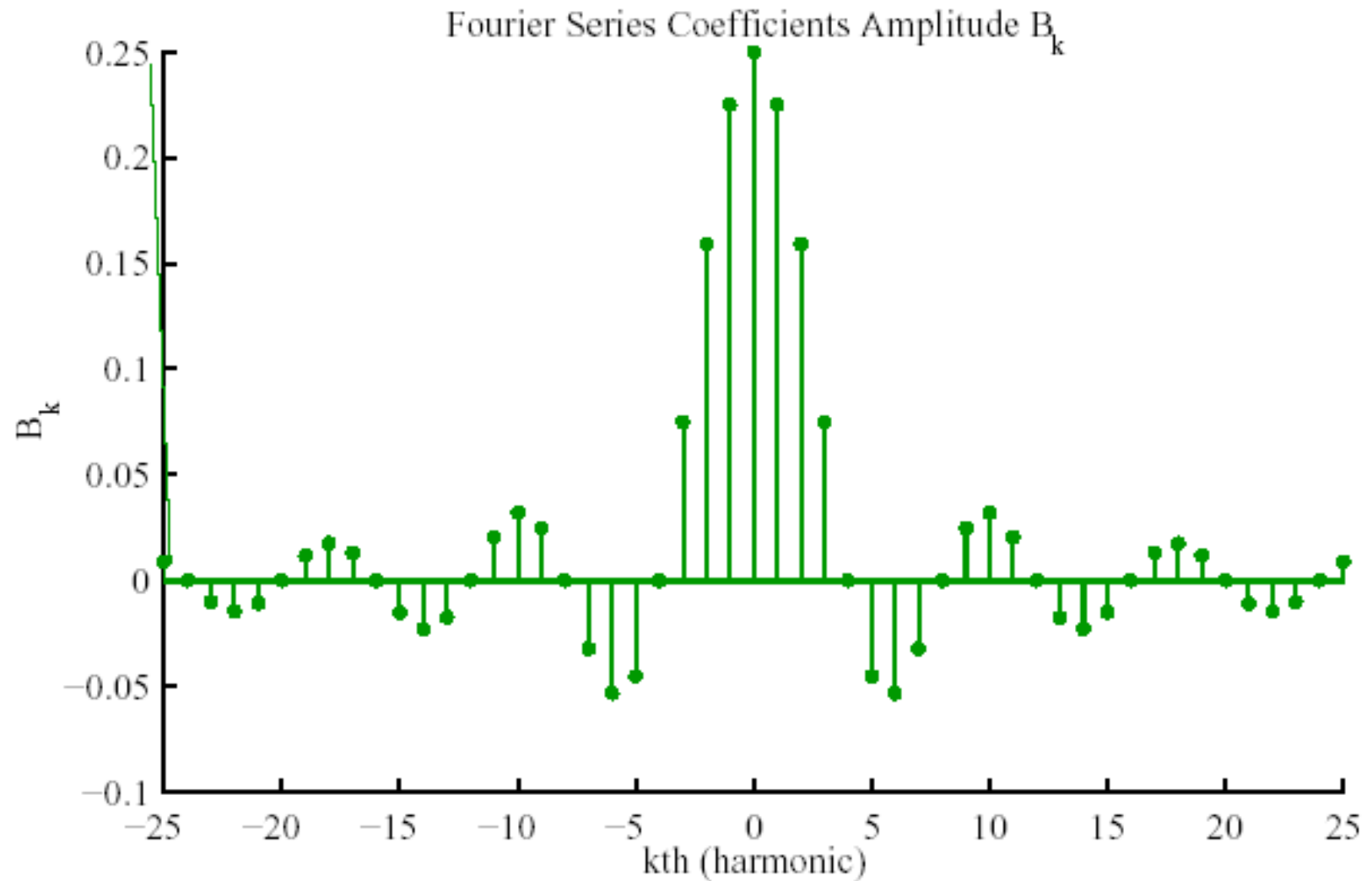


Example : Pulse Waveform

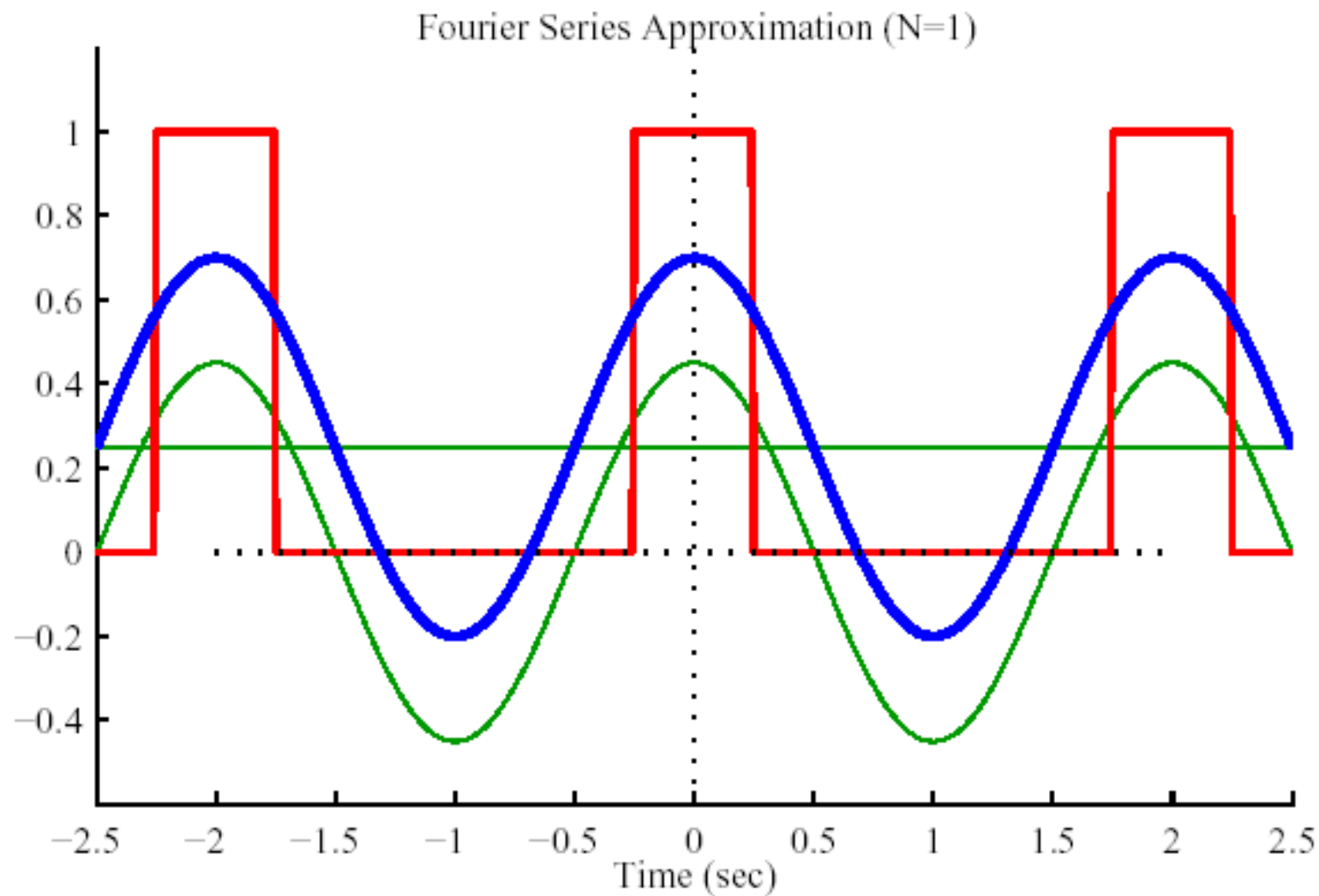


What is the fundamental period of $x(t)$? What type of symmetry does the signal have? Find the trigonometric Fourier series coefficients. Plot the partial Fourier series sum for $N = 1, 5, 10, 25, 50$ & 100.

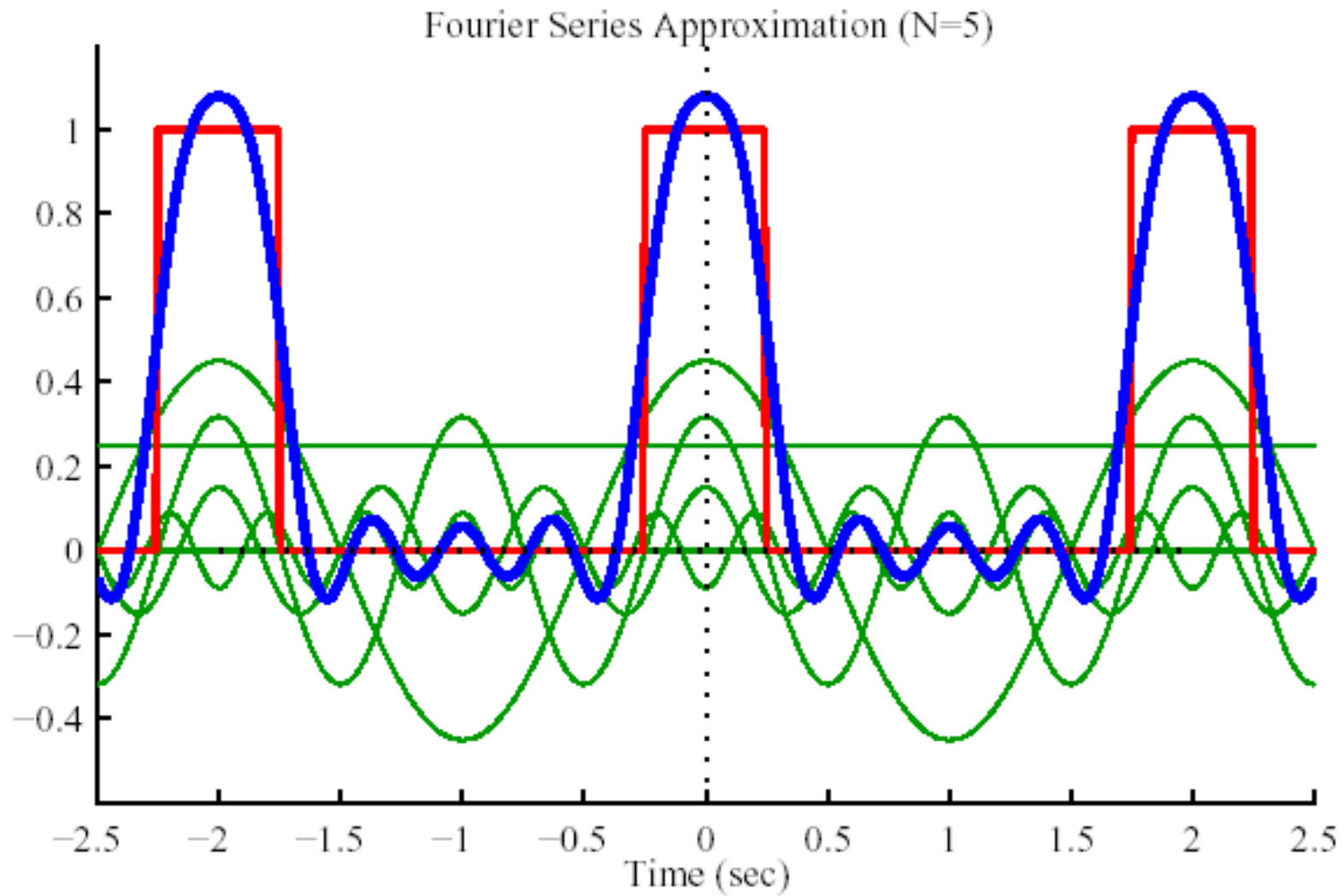
Coefficient Spectrum



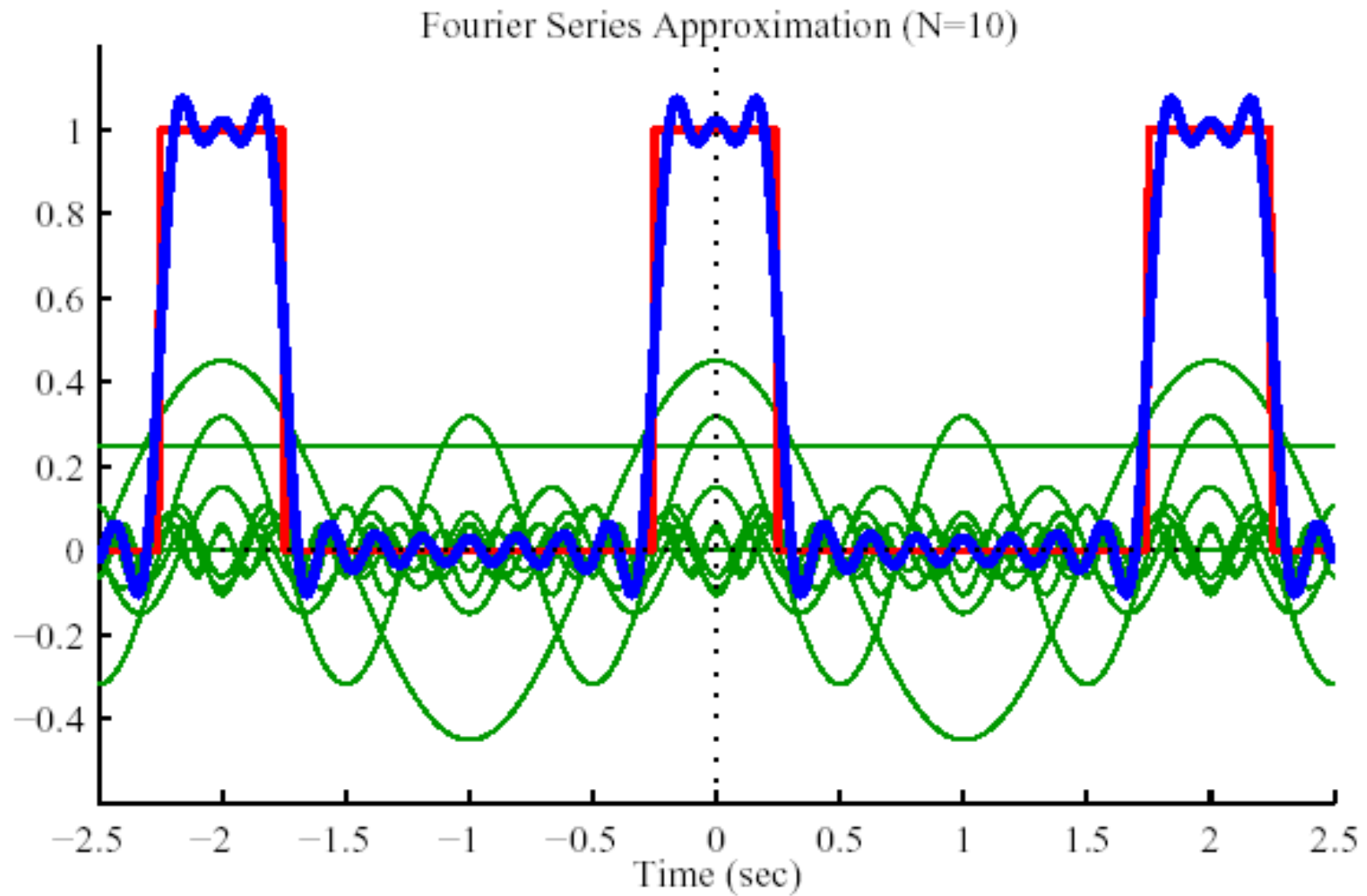
Partial Fourier Series $N=1$



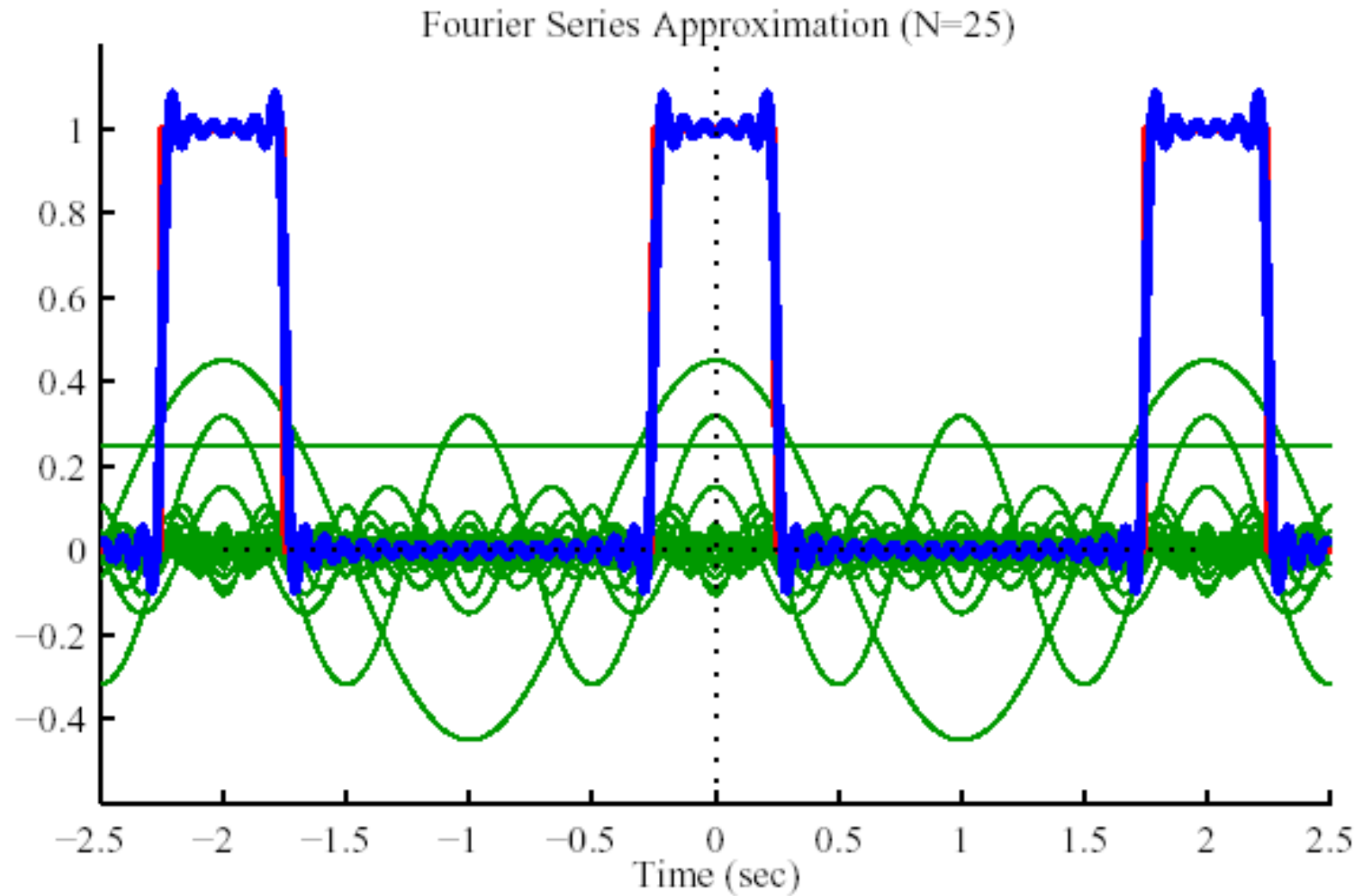
Partial Fourier Series $N=5$



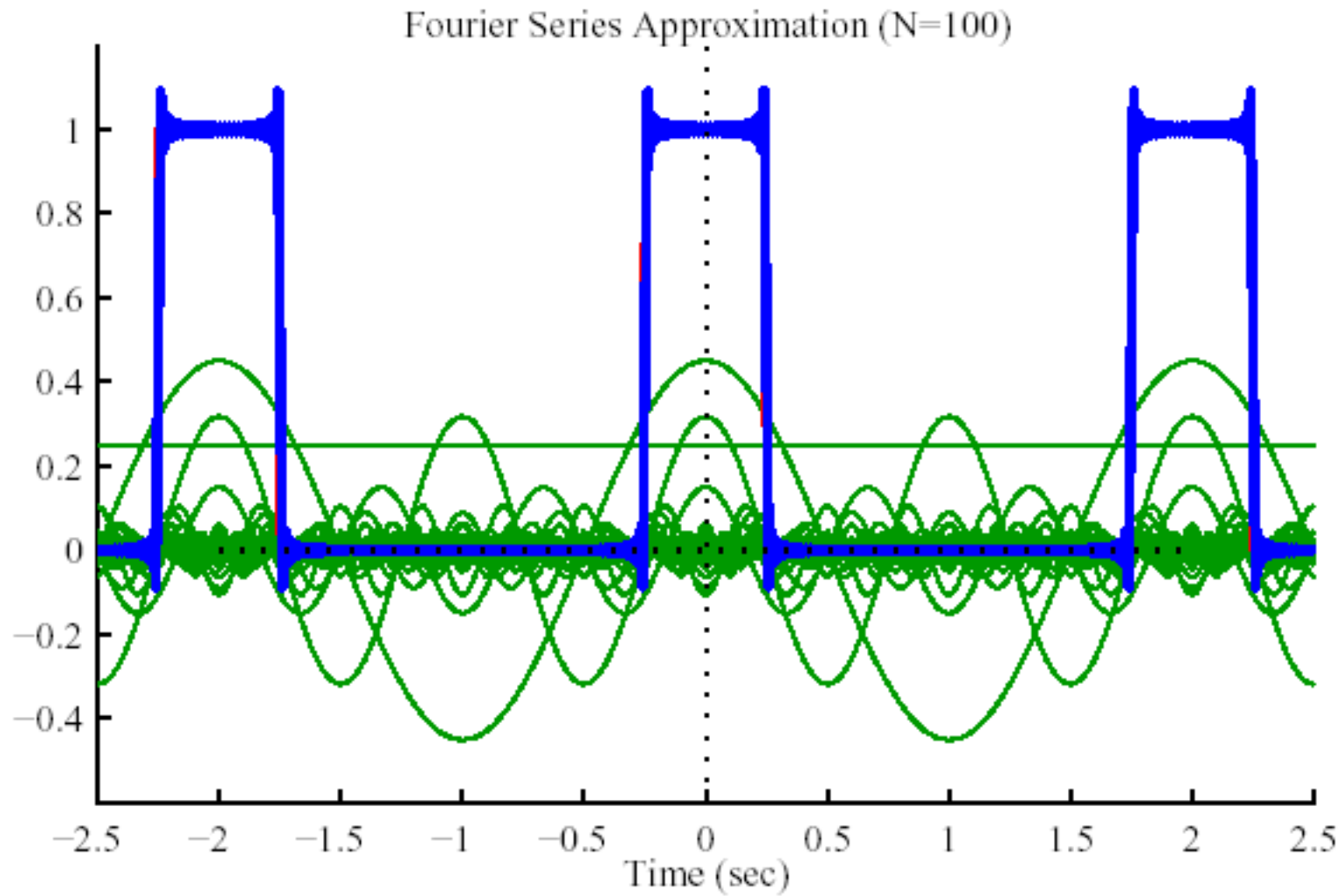
Partial Fourier Series $N=10$



Partial Fourier Series $N=25$



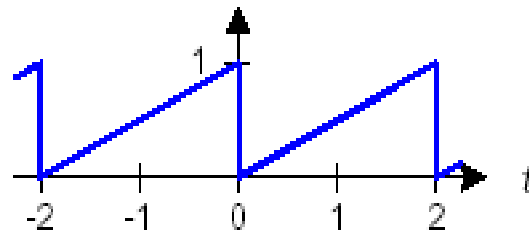
Partial Fourier Series $N=100$



Gibb's Phenomenon

- For finite Fourier series estimates there is apparent overshoot
- As N increases, the edges become sharper and more accurate
- The maximum overshoot (error) does not decrease as $N \rightarrow \infty$
- This is called **Gibb's phenomenon**

Example : Sawtooth Waveform



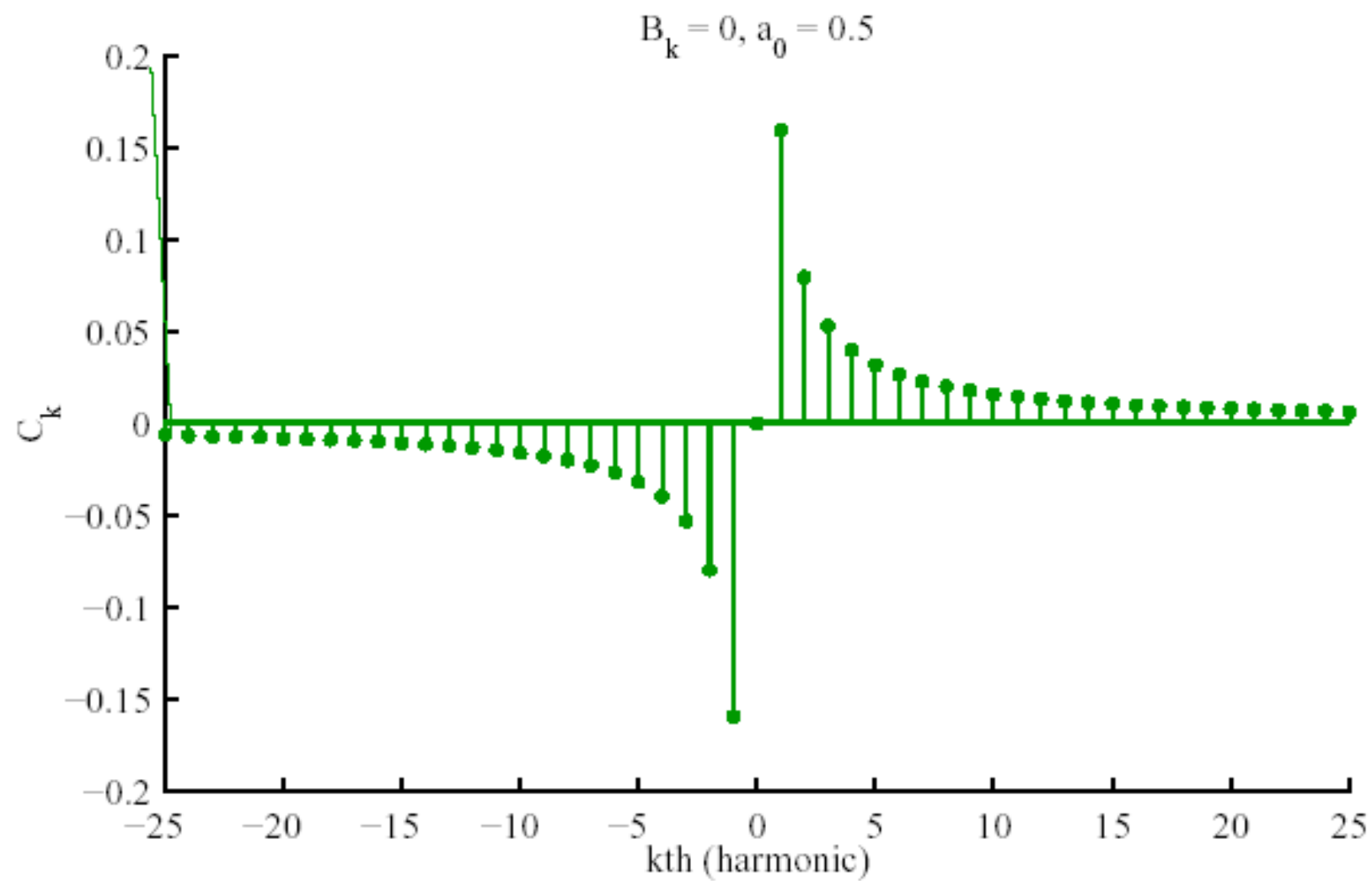
What is the fundamental period of $x(t)$? What type of symmetry does the signal have? Find the trigonometric Fourier series coefficients.

Plot partial Fourier series sum for $N = 1, 5, 10$, & 50 . Hint:

$$\int t \cos(\omega t) dt = \frac{1}{\omega^2} \cos(\omega t) + \frac{t}{\omega} \sin(\omega t) + C \text{ and}$$

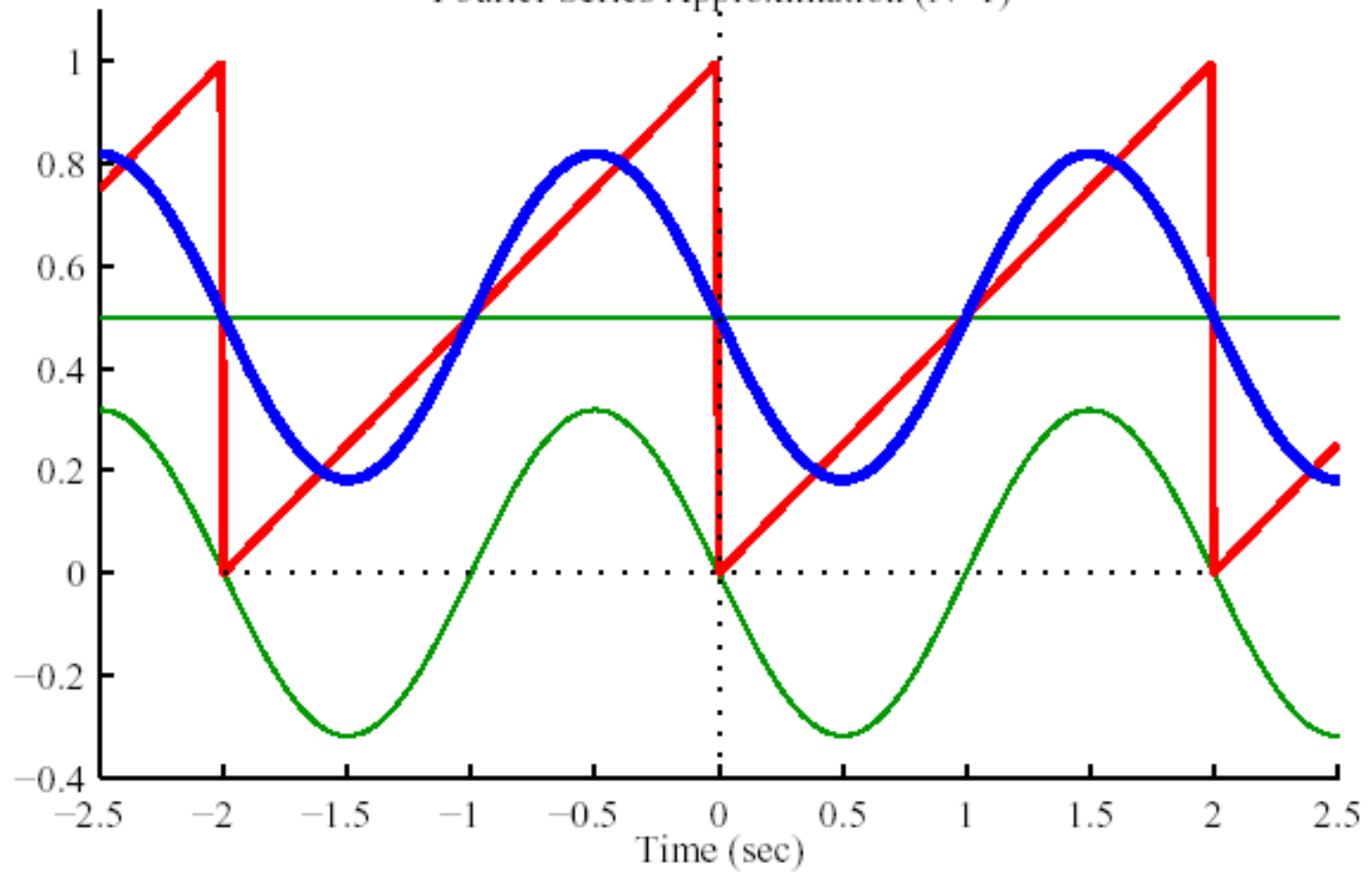
$$\int t \sin(\omega t) dt = \frac{1}{\omega^2} \sin(\omega t) - \frac{t}{\omega} \cos(\omega t) + C$$

Coefficient Spectrum

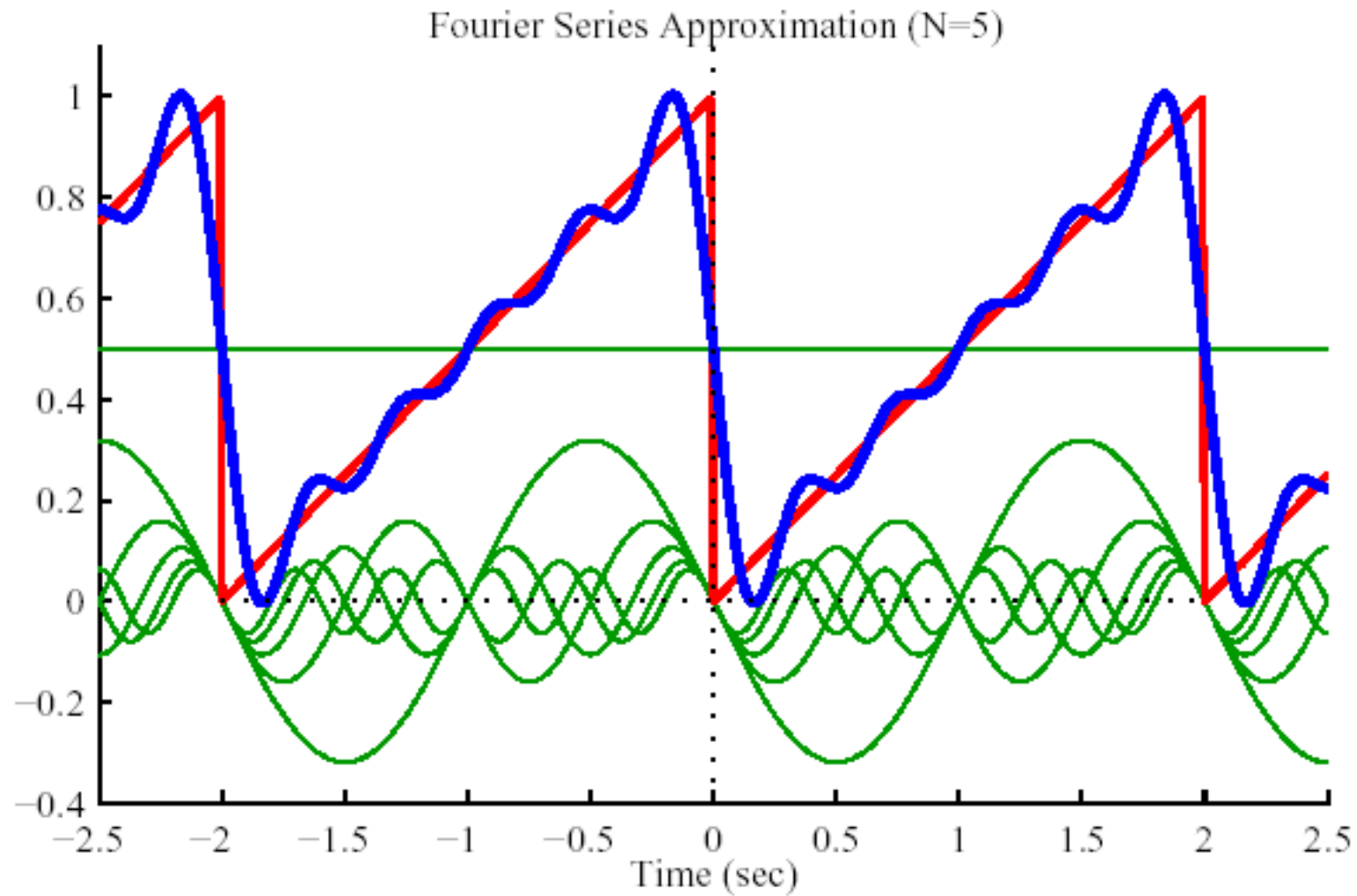


Partial Fourier Series $N=1$

Fourier Series Approximation ($N=1$)



Partial Fourier Series $N=5$



Partial Fourier Series $N=100$

