

CONVOLUTION

Impulse Response Sequence of DT LTI Systems

Convolution Sum for DT LTI Systems

Properties of Convolution Sum

Impulse Response of LTI Systems

Convolution Integral for CT LTI Systems

Properties of Convolution Integral

Operational Definition of the Unit Impulse

Exploiting Superposition and Time-Invariance

$$x[n] = \sum_k a_k x_k[n] \xrightarrow{\text{Linear System}} y[n] = \sum_k a_k y_k[n]$$

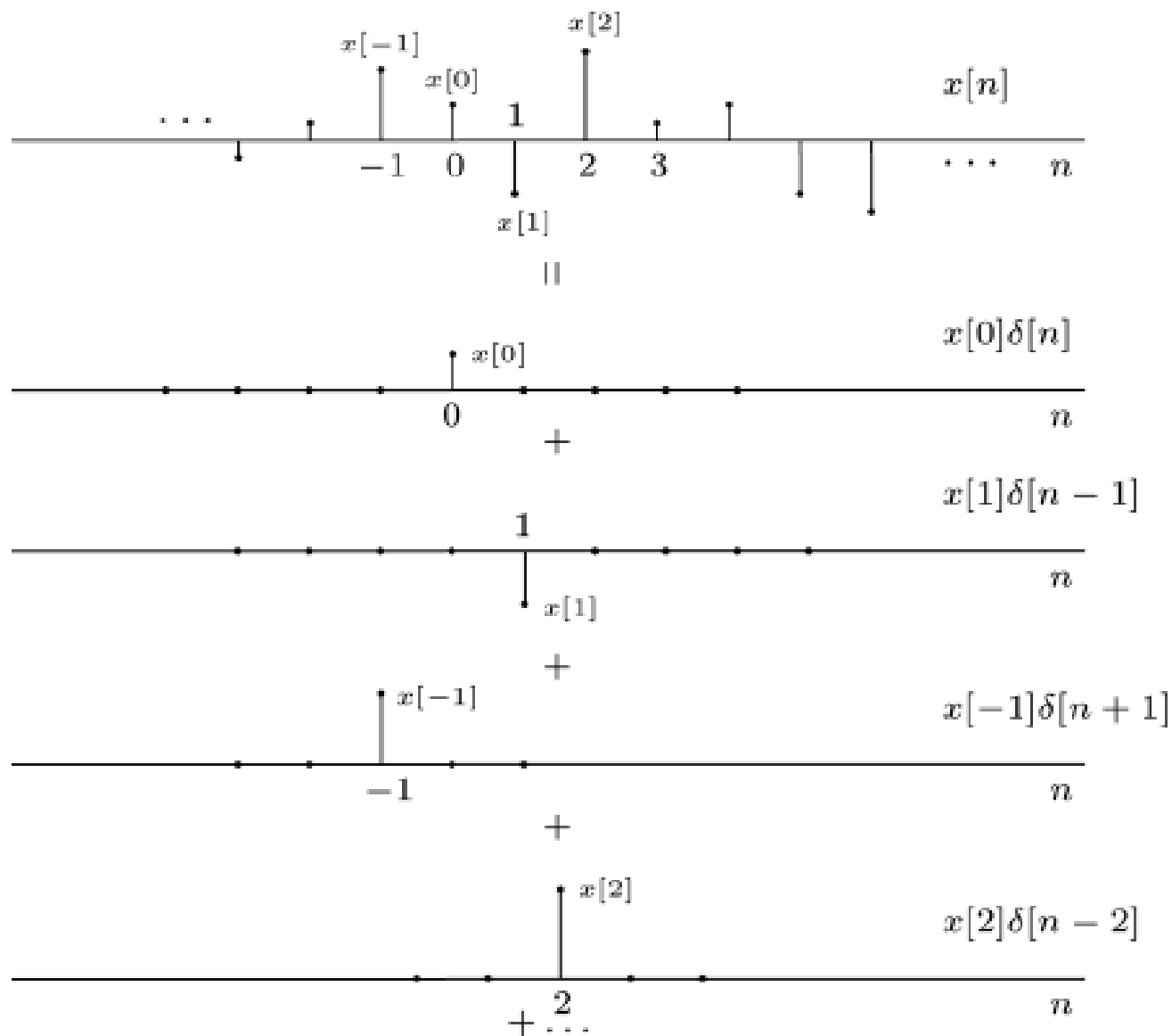
Question: Are there sets of “basic” signals so that:

- a) We can represent rich classes of signals as linear combinations of these building block signals.
- b) The response of LTI Systems to these basic signals are both *simple* and *insightful*.

Fact: For LTI Systems (CT or DT) there are two natural choices for these building blocks

Focus for now:	DT	Shifted unit samples
	CT	Shifted unit impulses

Representation of DT Signals Using Unit Samples



That is ...

$$x[n] = \cdots + x[-2]\delta[n+2] + x[-1]\delta[n+1] + x[0]\delta[n] + x[1]\delta[n-1] + \cdots$$

$\Downarrow$

$$x[n] = \sum_{k=-\infty}^{\infty} \underbrace{x[k]} \underbrace{\delta[n-k]}$$

Coefficients

Basic Signals

The Sifting Property of the Unit Sample

Impulse response of DT LTI systems:

Consider,

$$y(n) + a_1 y(n-1) + \dots + a_n y(n-k) = x(n) \quad (1)$$

where $x(n)$ is the impulse function $\delta(n)$, with $\delta(n) = 0$ for $n \neq 0$ and $\delta(n) = 1$ for $n = 0$. Let $h(n)$ be the impulse response sequence. Since $h(n)$ must satisfy the above equation for all n , we can write,

$$h(n) + a_1 h(n-1) + \dots + a_n h(n-k) = \delta(n)$$

As all the terms on the left, except the first, will be zero for $n=0$, this equation simplifies to

$$h(0) = 1$$

Similarly, for $n = 1$, all the terms except the first two are zero, and we have

$$h(1) + a_1 h(0) = 0$$

Continuing with these steps yields k linear equations through which initial conditions $h(0), h(1), \dots, h(k-1)$ can be obtained.

Similar to the continuous case, we obtain the complementary solution for the difference equation with $\delta(n) = 0$ for $n \neq 0$ and calculate arbitrary constants to satisfy initial conditions.

Example: Consider

$$y(n) - 2 y(n-1) + 1.31 y(n-2) - 0.28 y(n-3) = x(n)$$

The characteristic equation (CE) is

$$\Delta(q) = q^3 - 2q^2 + 1.31q - 0.28 = 0 \quad \Rightarrow \quad q_1 = 0.5, \quad q_2 = 0.7, \quad q_3 = 0.8$$

Hence, the complementary solution is

$$y_c(n) = A (0.5)^n + B (0.7)^n + C (0.8)^n$$

To obtain i.c., we start with $h(0)=1$,

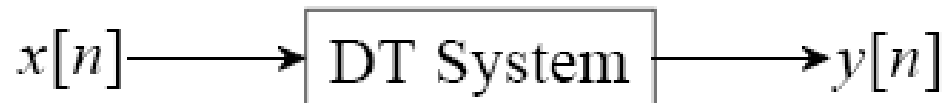
$$\begin{array}{lll} h(0) = 1 & \Rightarrow & h(0) = 1 \\ h(1) - 2 h(0) = 0 & \Rightarrow & h(1) = 2 \\ h(2) - 2 h(1) + 1.31 h(0) = 0 & \Rightarrow & h(2) = 2.69 \end{array}$$

Substituting these in the complementary solution gives,

$$\begin{array}{rcl} A + B + C & = & 1 \\ 0.5 A + 0.7 B + 0.8 C & = & 2 \\ 0.25 A + 0.49 B + 0.64 C & = & 2.69 \end{array} \quad \Rightarrow \quad \begin{array}{l} A=25/6 \\ B=-49/2 \\ C=64/3 \end{array}$$

Hence, the impulse response sequence $h(n)$ is obtained as

$$\begin{aligned} h(n) &= (25/6) (0.5)^n - (49/2) (0.7)^n + (64/3) (0.8)^n \\ &= 4.167 (0.5)^n - 24.5 (0.7)^n + 21.33 (0.8)^n \quad n \geq 0 \end{aligned}$$

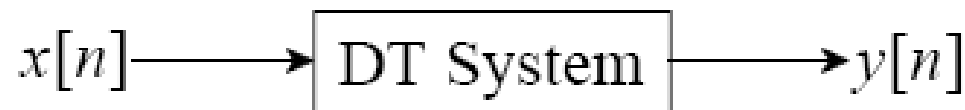


- Suppose the system is **linear**, and define $h_k[n]$ as the response to $\delta[n - k]$:

$$\delta[n - k] \rightarrow h_k[n]$$

From superposition: $\Downarrow$

$$x[n] = \sum_{k=-\infty}^{\infty} x[k] \delta[n - k] \rightarrow y[n] = \sum_{k=-\infty}^{\infty} x[k] h_k[n]$$



- Now suppose the system is **LTI**, and define the *unit sample response* $h[n]$:

$$\delta[n] \rightarrow h[n]$$

From TI: $\Downarrow$

$$\delta[n - k] \rightarrow h[n - k]$$

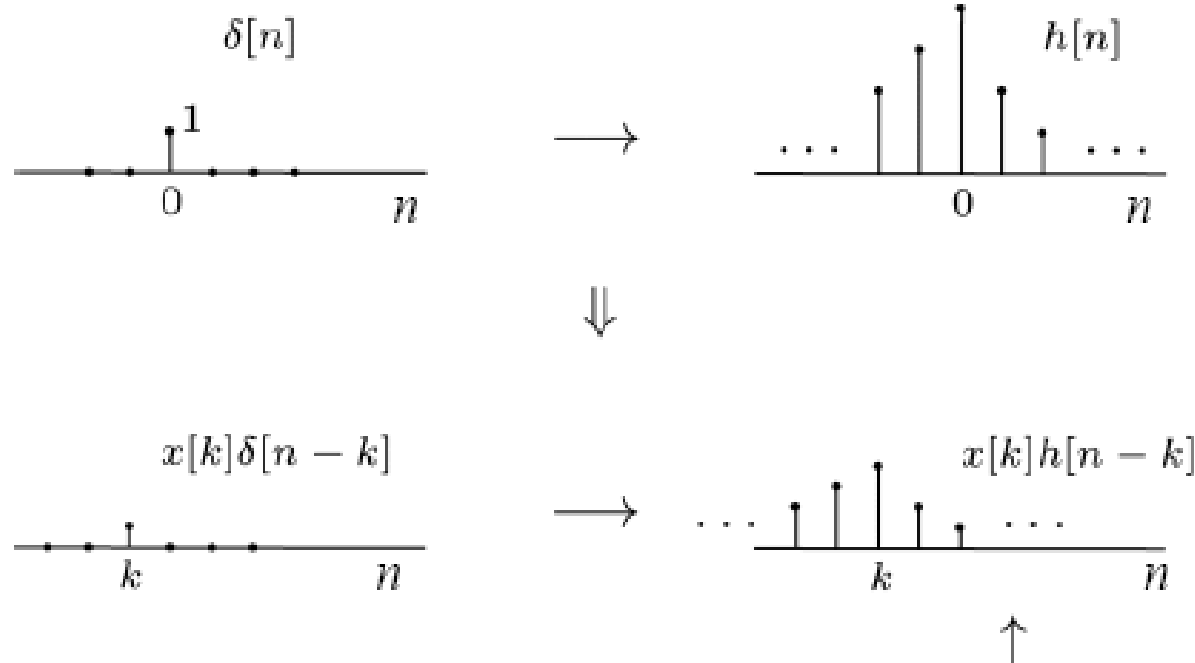
From LTI:

$$x[n] = \sum_{k=-\infty}^{\infty} x[k] \delta[n - k] \rightarrow y[n] = \underbrace{\sum_{k=-\infty}^{\infty} x[k] h[n - k]}_{\text{Convolution Sum}}$$

Convolution Sum Representation of Response of LTI Systems

$$y[n] = x[n] * h[n] = \sum_{k=-\infty}^{\infty} x[k]h[n-k]$$

Interpretation



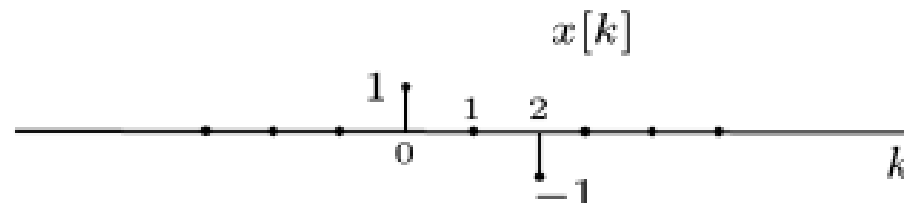
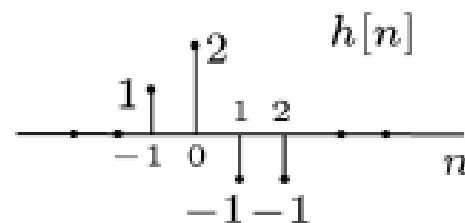
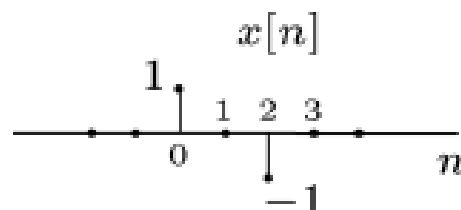
Sum up responses over all k

Visualizing the calculation of $y[n] = x[n] * h[n]$

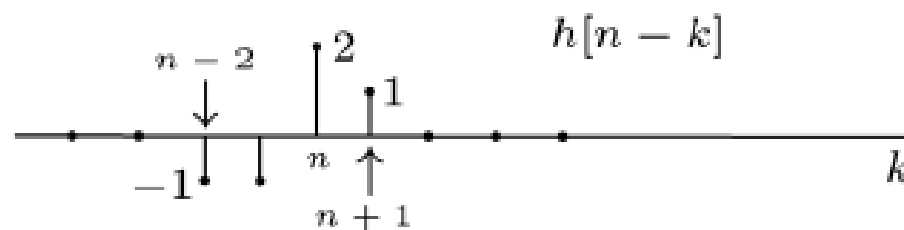
Choose value of n and consider it fixed

$$y[n] = \sum_{k=-\infty}^{\infty} x[k]h[n-k]$$

View as functions of k with n fixed

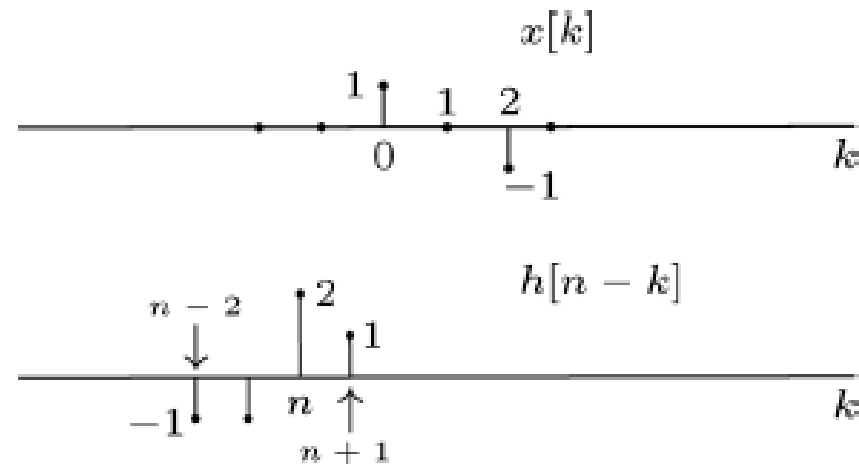


$y[0] = \sum$ prod of
overlap for
 $n = 0$



$y[1] = \sum$ prod of
overlap for
 $n = 1$

Calculating Successive Values: Shift, Multiply, Sum



$$y[n] = 0 \quad \text{for } n < -1$$

$$y[-1] = 1 \times 1 = 1$$

$$y[0] = 0 \times 1 + 1 \times 2 = 2$$

$$y[1] = (-1) \times 1 + 0 \times 2 + 1 \times (-1) = -2$$

$$y[2] = (-1) \times 2 + 0 \times (-1) + 1 \times (-1) = -3$$

$$y[3] = (-1) \times (-1) + 0 \times (-1) = 1$$

$$y[4] = (-1) \times (-1) = 1$$

$$y[n] = 0 \quad \text{for } n > 4$$

Properties of Convolution and DT LTI Systems

- 1) A DT LTI System is *completely characterized* by its unit sample response

Ex. #1: $h[n] = \delta[n - n_0]$

There are *many* systems with this response to $\delta[n]$

There is only *one* LTI System with this response to $\delta[n]$:

$$y[n] = x[n - n_0]$$



$$x[n] * \delta[n - n_0] = x[n - n_0]$$

Ex. #2:

$$y[n] = \sum_{k=-\infty}^n x[k] \quad - \text{An Accumulator}$$

Unit Sample response

$$h[n] = \sum_{k=-\infty}^n \delta[k] = u[n]$$

$\Downarrow$

$$x[n] * u[n] = \sum_{k=-\infty}^n x[k]$$

The Commutative Property

$$y[n] = x[n] * h[n] = h[n] * x[n]$$

Ex: Step response $s[n]$ of an LTI system

$$s[n] = u[n] * h[n] = h[n] * u[n]$$

↑
**step
input**

↑
“input”

↑
Unit Sample response
of accumulator

⇓

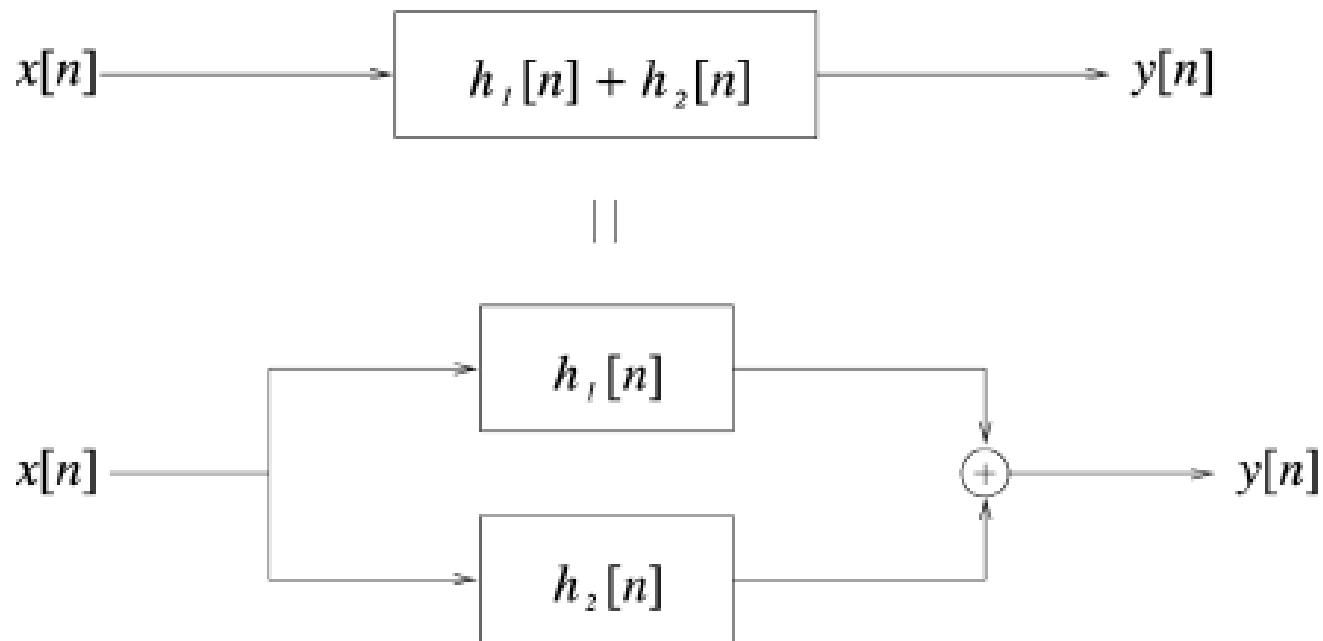
$$s[n] = \sum_{k=-\infty}^n h[k] \quad \text{running sum}$$

or $h[n] = s[n] - s[n-1]$

The Distributive Property

$$x[n] * (h_1[n] + h_2[n]) = x[n] * h_1[n] + x * h_2[n]$$

Interpretation



The Associative Property

$$x[n] * (h_1[n] * h_2[n]) = (x[n] * h_1[n]) * h_2[n]$$

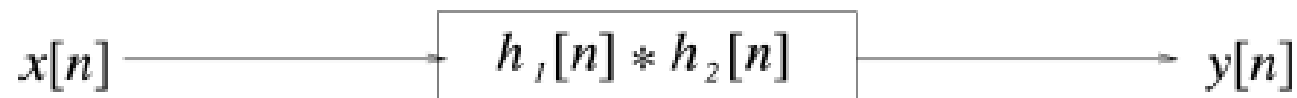
(Commutativity) ||

$$x[n] * (h_2[n] * h_1[n]) = (x[n] * h_2[n]) * h_1[n]$$

Implication (Very special to LTI Systems)



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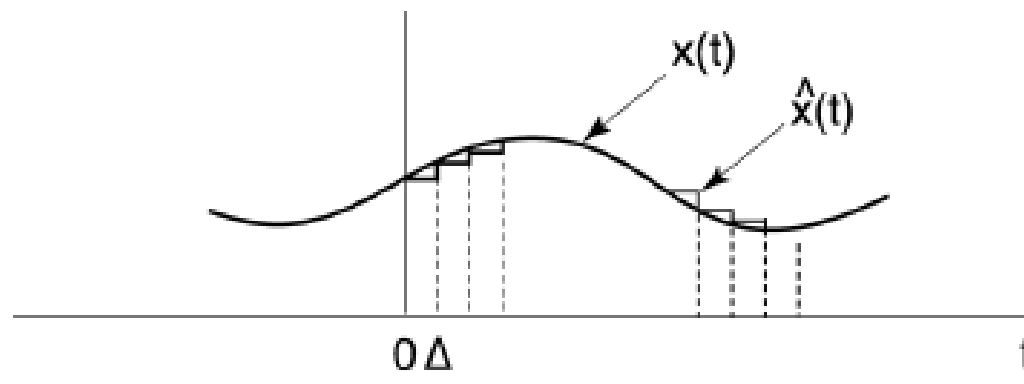
Properties of LTI Systems

1) Causality $\Leftrightarrow h[n] = 0$ for all $n < 0$

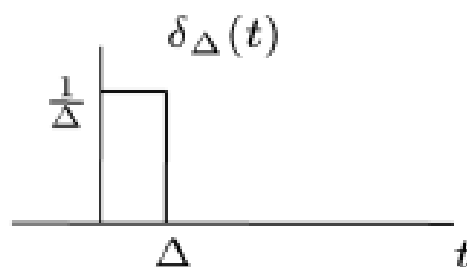
2) Stability $\Leftrightarrow \sum_{k=-\infty}^{+\infty} |h[k]| < \infty$

Representation of CT Signals

- Approximate any input $x(t)$ as a sum of shifted, scaled pulses

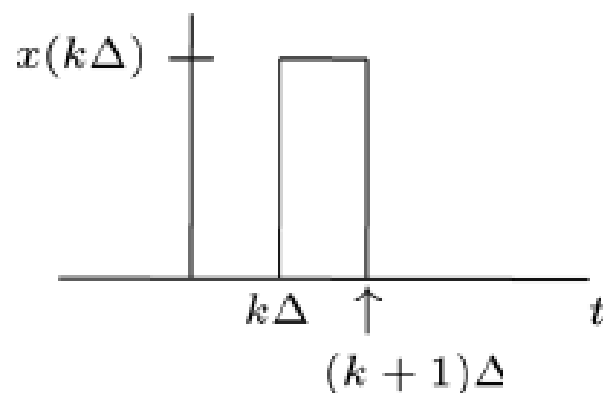


$$\hat{x}(t) = x(k\Delta) , k\Delta < t < (k+1)\Delta$$



$\delta_{\Delta}(t)$ has unit area

$\Delta\delta_{\Delta}(t)$ has unit amplitude



$$= x(k\Delta)\delta_{\Delta}(t - k\Delta)\Delta$$

$\Downarrow$

$$\hat{x}(t) = \sum_{k=-\infty}^{\infty} x(k\Delta)\delta_{\Delta}(t - k\Delta)\Delta$$

$\downarrow$ limit as $\Delta \rightarrow 0$

$$x(t) = \int_{-\infty}^{\infty} x(\tau)\delta(t - \tau)d\tau$$

The Sifting Property of the Unit Impulse

For the case of unit step signal, the above integral yields,

$$u(t) = \int_{-\infty}^{\infty} u(\tau) \delta(t - \tau) d\tau = \int_0^{\infty} \delta(t - \tau) d\tau$$

since $u(\tau)=0$ for $\tau<0$ and $u(\tau)=1$ for $\tau>0$.

Impulse response of CT LTI systems:

Consider a system described by the first-order differential equation,

$$y'(t) + a y(t) = x(t) \quad (1)$$

where $x(t)=\delta(t)$ and assume that $y=0$ for $t<0$.

$\delta(t)$ is applied at $t=0$, so outside the interval $0_- < t < 0_+$ we have $y=0$ at $t=0_-$ and

$$y'(t) + a y(t) = 0 \quad \text{for} \quad t > 0_+$$

This homogeneous solution can be obtained easily, if $y(0_+)$ is known. Integrating (1) between $t=0_-$ and $t=0_+$ after multiplying both sides by dt gives,

$$\int_{0_-}^{0_+} dy + \int_{0_-}^{0_+} a y dt = \int_0^{0_+} \delta(t) dt \quad \Rightarrow \quad [y(0_+) - y(0_-)] + 0 = 1$$

where the second term is zero since $y(t)$ is known to be finite, i.e. no discontinuities and $(0_-, 0_+)$ is infinitesimal;

$$y(0_-)=0 \quad \Rightarrow \quad y(0_+)=1.$$

Therefore, the impulse response is given by

$$h(t) = e^{-at} u(t)$$

where $u(t)$ is the step function. The response of a first order linear system to a unit impulse is identical to the response of the unforced system with the initial condition set to 1.

Similarly, the response of a unit impulse for the n -th order system is identical to the response of the unforced system with

$$y^{(n-1)}(0_+) = 1$$

and all the other initial conditions being zero.

Example: Consider,

$$y''(t) + 4 y'(t) + 3 y(t) = x(t)$$

where $y(0) = 0$ and $y'(0) = 1$. The roots of the characteristic polynomial are -1, -3, so we have the complementary function as

$$y_c(t) = A e^{-t} + B e^{-3t}$$

With i.c. we get

$$y_c(0) = A + B = 0$$

$$y'(0) = -A - 3B = 1 \quad \Rightarrow \quad A = 0.5, \quad B = -0.5$$

Consequently, the impulse response $h(t)$ is found as

$$h(t) = (0.5 e^{-t} - 0.5 e^{-3t}) u(t)$$

Now consider the more general case with derivatives of $x(t)$ included,

$$D(p) y(t) = N(p) x(t)$$

where it is assumed that $\deg D(p) = n$, $\deg N(p) \leq (n-1)$, so that the system function $L(p)$ is strictly proper.

The system equation is considered in two parts as

$$\begin{aligned} D(p) z(t) &= x(t) \\ y(t) &= N(p) z(t) \end{aligned}$$

such that the impulse response of the first equation is obtained and afterwards used in the second equation.

Response of a CT LTI System



$$\delta_{\Delta}(t) \longrightarrow h_{\Delta}(t)$$

$$\hat{x}(t) = \sum_{k=-\infty}^{\infty} x(k\Delta)\delta_{\Delta}(t - k\Delta)\Delta \longrightarrow \hat{y}(t) = \sum_{k=-\infty}^{\infty} x(k\Delta)h_{\Delta}(t - k\Delta)\Delta$$

$\Downarrow$

Impulse response:

$$\delta(t) \longrightarrow h(t)$$

Taking limits $\Delta \rightarrow 0$

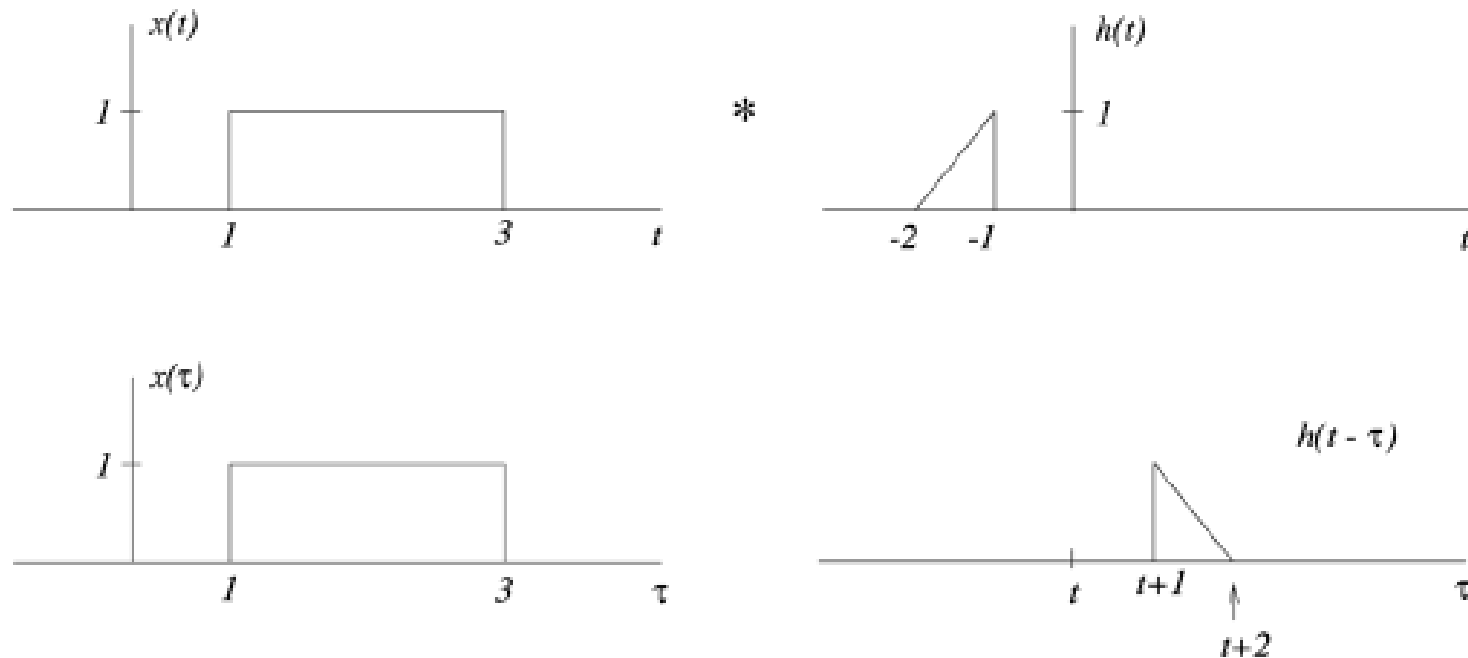
$$x(t) = \int_{-\infty}^{\infty} x(\tau)\delta(t - \tau)d\tau \longrightarrow y(t) = \underbrace{\int_{-\infty}^{\infty} x(\tau)h(t - \tau)d\tau}_{\text{Convolution Integral}}$$

Operation of CT Convolution

$$y(t) = x(t) * h(t) \equiv \int_{-\infty}^{\infty} x(\tau)h(t - \tau)d\tau$$

$$h(\tau) \xrightarrow{\text{Flip}} h(-\tau) \xrightarrow{\text{Slide}} h(t - \tau) \xrightarrow{\text{Multiply}} x(\tau)h(t - \tau) \xrightarrow{\text{Integrate}} \int_{-\infty}^{\infty} x(\tau)h(t - \tau)d\tau$$

Example: CT convolution



Time Interval

$x(\tau) \cdot h(t-\tau)$

Output

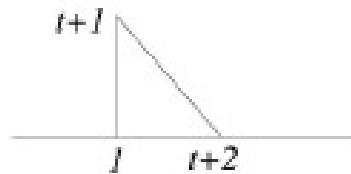
$$t < -1$$

$$0$$

$\Rightarrow$

$$y(t) = 0$$

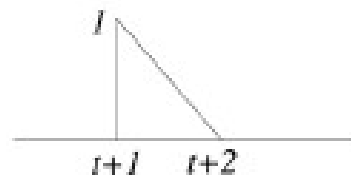
$$-1 < t < 0$$



$\Rightarrow$

$$y(t) = \frac{1}{2}(t+2)(t+2-1) \\ = \frac{1}{2}(t+1)^2$$

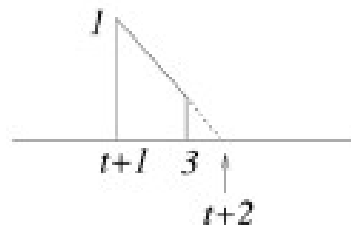
$$0 < t < 1$$



$\Rightarrow$

$$y(t) = \frac{1}{2} \cdot 1 \cdot 1 = \frac{1}{2}$$

$$1 < t < 2$$



$\Rightarrow$

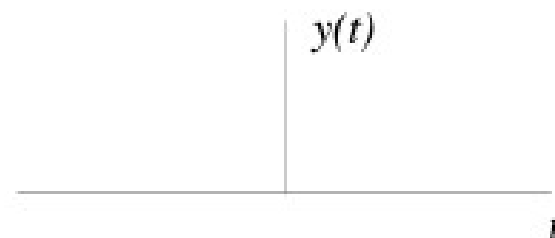
$$y(t) = \frac{1}{2} - \frac{1}{2}(t+2-3)(t-1) \\ = \frac{1}{2} - \frac{1}{2}(t-1)^2$$

$$t > 2$$

$$0$$

$\Rightarrow$

$$y(t) = 0$$



PROPERTIES AND EXAMPLES

1) Commutativity: $x(t) * h(t) = h(t) * x(t)$

2) $x(t) * \delta(t - t_0) = x(t - t_0)$ Sifting property: $x(t) * \delta(t) = x(t)$

3) An integrator: $y(t) = \int_{-\infty}^t x(\tau) d\tau$

$\Downarrow$

So if input $x(t) = \delta(t)$
output $y(t) = h(t)$

$$h(t) = \int_{-\infty}^t \delta(\tau) d\tau = u(t)$$

That is

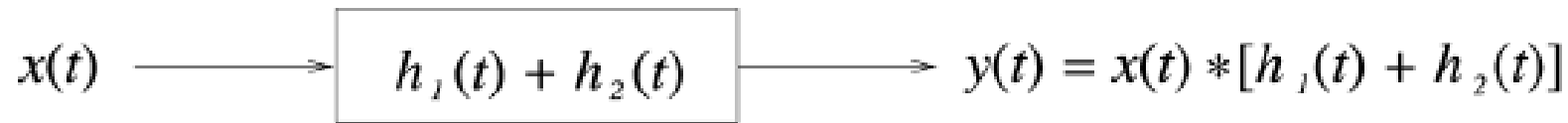
$$y(t) = x(t) * h(t) = \boxed{x(t) * u(t) = \int_{-\infty}^t x(\tau) d\tau}$$

running
integral

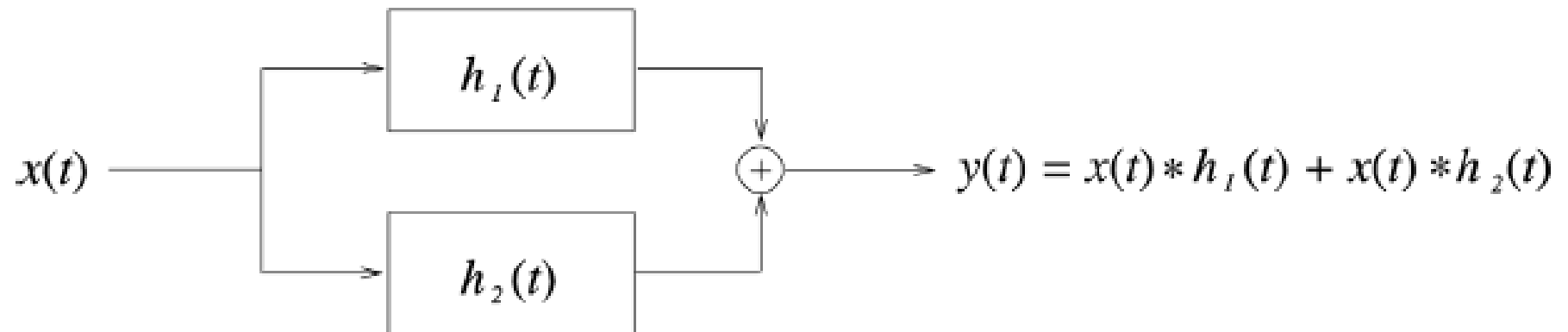
4) Step response:

$$s(t) = u(t) * h(t) = h(t) * u(t) = \int_{-\infty}^t h(\tau) d\tau \quad \text{or} \quad h(t) = ds/dt$$

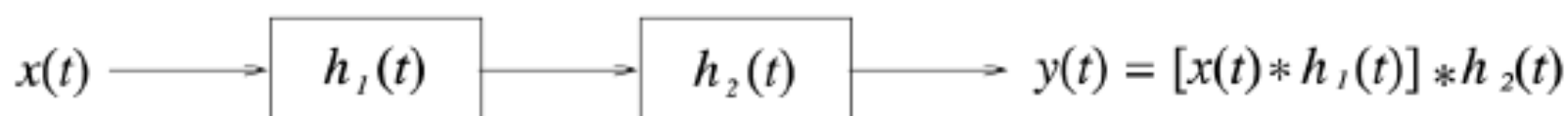
DISTRIBUTIVITY



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ASSOCIATIVITY



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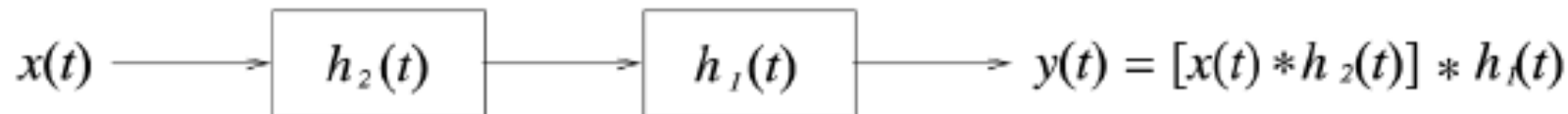


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← Commutativity



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Causality: CT LTI system is causal $\Leftrightarrow h(t) = 0, t < 0$

Stability: CT LTI system is stable $\Leftrightarrow \int_{-\infty}^{+\infty} |h(\tau)| d\tau < \infty$

A DT LTI system is

memoryless if $h(n) = 0$ for $n \neq 0$
and

with memory if $h(n) \neq 0$ for $n \neq 0$

A CT LTI system is

memoryless if $h(t) = 0$ for $t \neq 0$
and

with memory if $h(t) \neq 0$ for $t \neq 0$

Examples for memoryless systems;

(a) $h(n) = K \delta(n)$, $y(n) = K x(n)$ where $K = h(0)$

(b) $h(t) = K \delta(t)$, $y(t) = K x(t)$

The Operational Definition of the Unit Impulse $\delta(t)$

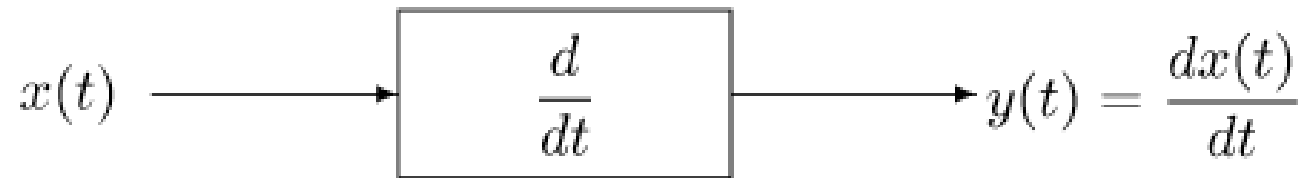
$\delta(t)$ — idealization of a unit-area pulse that is so short that, for any physical systems of interest to us, the system responds only to the area of the pulse and is insensitive to its duration

Operationally: The unit impulse is the signal which when applied to any LTI system results in an output equal to the impulse response of the system. That is,

$$\delta(t) * h(t) = h(t) \quad \text{for all } h(t)$$

— $\delta(t)$ is defined by what it does under convolution.

The Unit Doublet — Differentiator



Impulse response = unit doublet

$$u_1(t) = \frac{d\delta(t)}{dt}$$

The operational definition of the unit doublet:

$$x(t) * u_1(t) = \frac{dx(t)}{dt}$$

Triplets and beyond!

$$n > 0$$

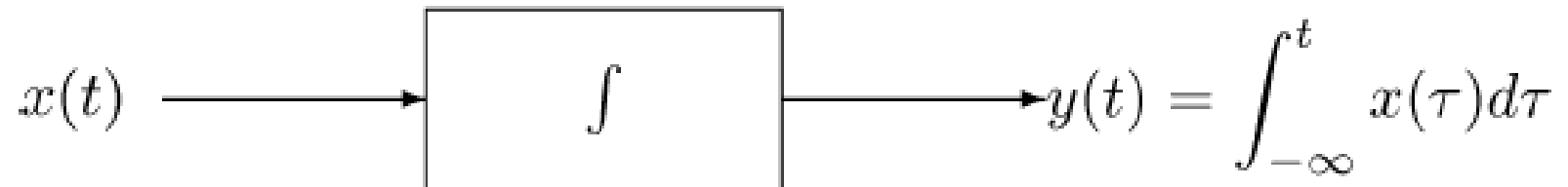
$$u_n(t) = \underbrace{u_1(t) * \cdots * u_1(t)}_{n \text{ times}}$$

 *n* is number of
differentiations

Operational definitions

$$x(t) * u_n(t) = \frac{d^n x(t)}{dt^n} \quad (n > 0)$$

Integrators



Impulse response: $u_{-1}(t) \equiv u(t)$

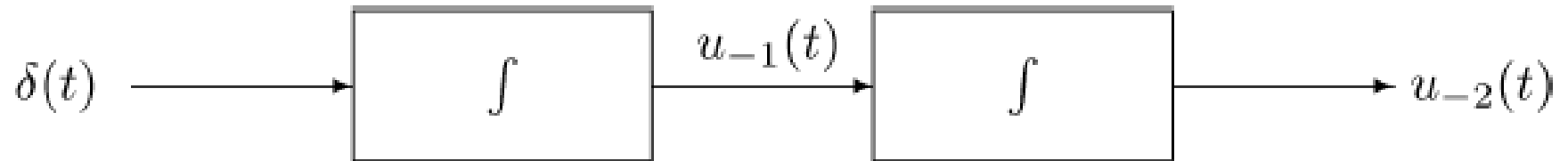
$\uparrow$
“-1 derivatives” = integral $\Rightarrow$ I.R. = unit step

Operational definition: $x(t) * u_{-1}(t) = \int_{-\infty}^t x(\tau) d\tau$

Cascade of n integrators:

$$u_{-n}(t) = \underbrace{u_{-1}(t) * \cdots * u_{-1}(t)}_{n \text{ times}} \quad (n > 0)$$

Integrators (continued)



$$\begin{aligned} u_{-2}(t) &= \int_{-\infty}^t u_{-1}(\tau) d\tau = \int_{-\infty}^t u(\tau) d\tau \\ &= t \cdot u(t) \quad \text{the unit ramp} \end{aligned}$$

More generally, for $n > 0$

$$u_{-n}(t) = \frac{t^{n-1}}{(n-1)!} u(t)$$

Notation

Define $u_0(t) = \delta(t)$

Then $u_n(t) * u_m(t) = u_{n+m}(t)$
 n and m can be $\pm$.

E.g. $u_1(t) * u_{-1}(t) = u_0(t)$

$\Downarrow$

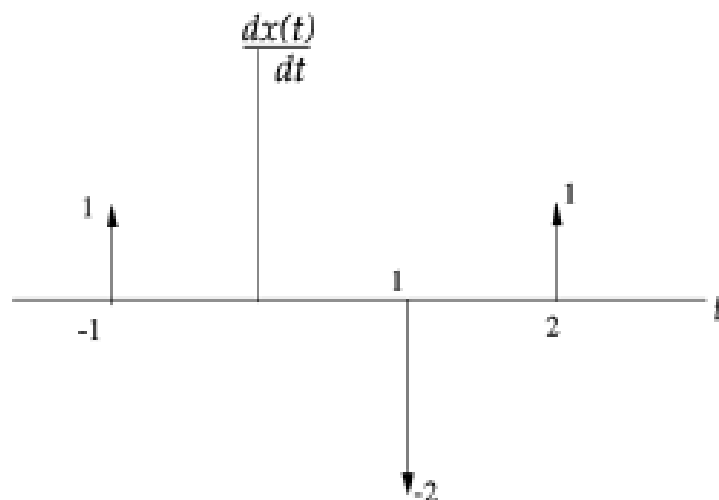
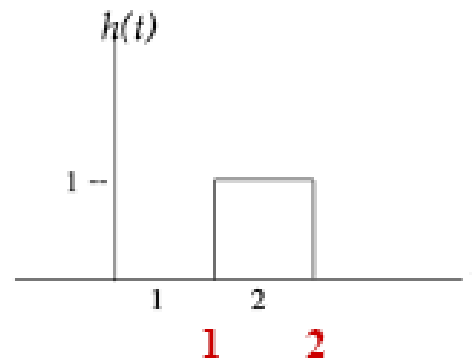
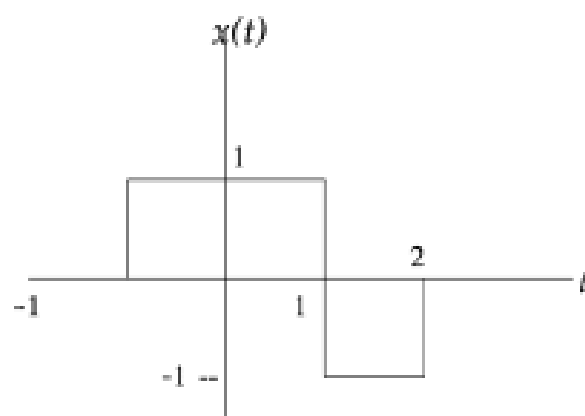
$$\left(\frac{d}{dt} u(t) \right) = \delta(t)$$

Sometimes Useful Tricks

$$\begin{aligned}x(t) * h(t) &= x(t) * \delta(t) * h(t) \\&= x(t) * u_1(t) * u_{-1}(t) * h(t) \\&= \{[x(t) * u_1(t)] * h(t)\} * u_{-1}(t)\end{aligned}$$

Differentiate first, then convolve, then integrate

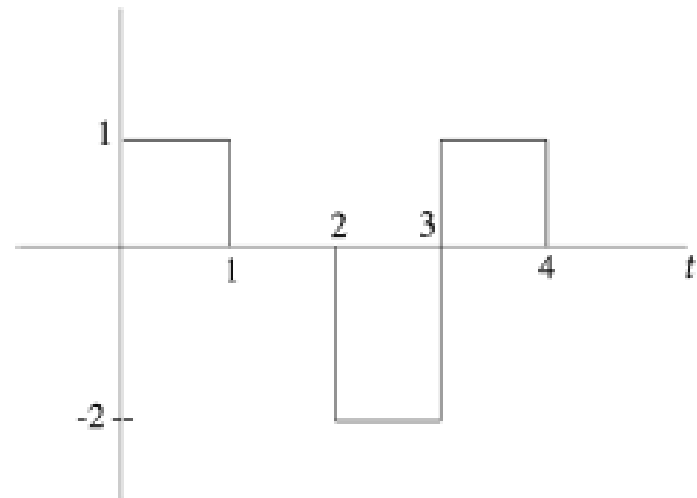
Example



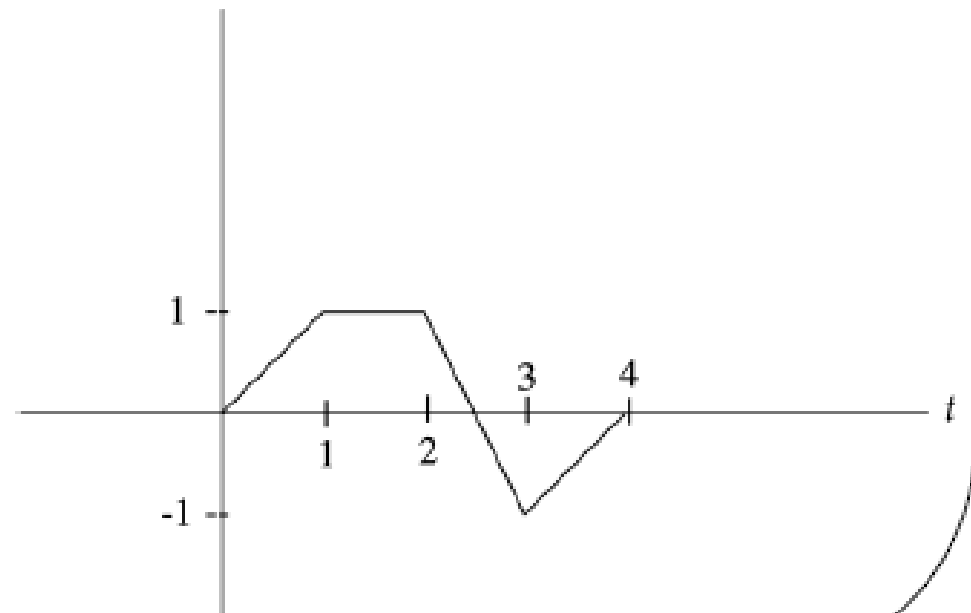
$$\frac{dx(t)}{dt} = \delta(t+1) - 2\delta(t-1) + \delta(t-2)$$

Example (continued)

$$\frac{dx(t)}{dt} * h(t) = h(t+1) - 2h(t-1) + h(t-2)$$

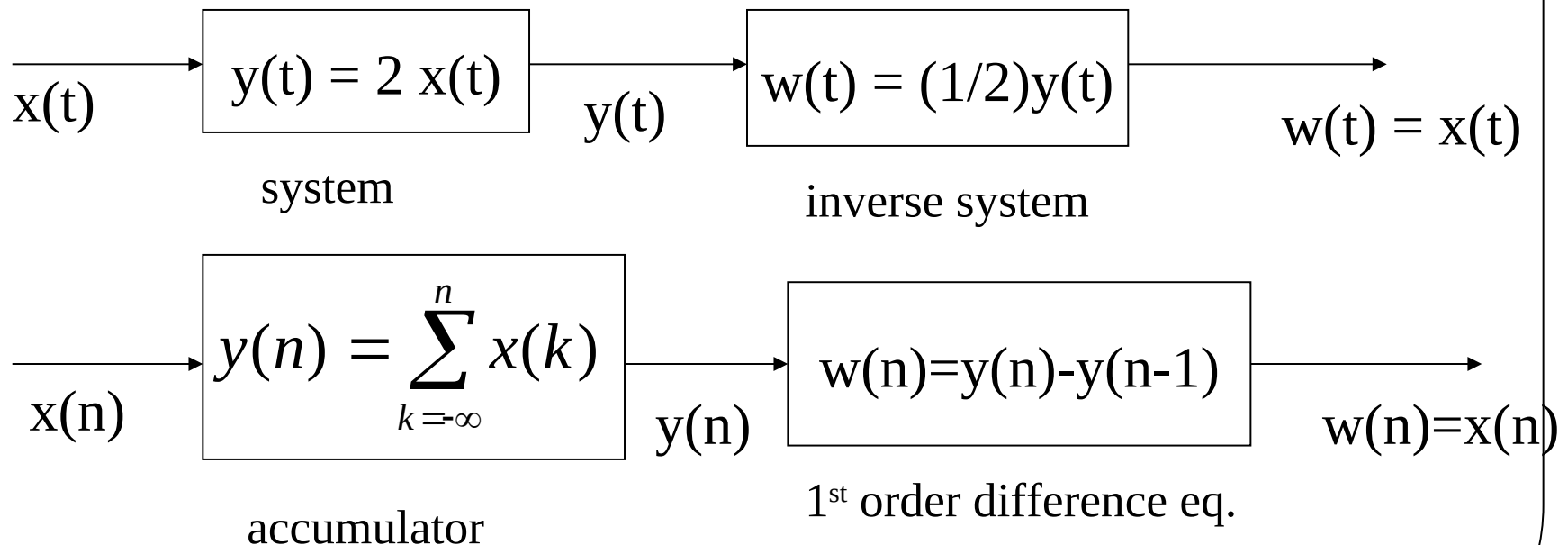


$$x(t) * h(t) = \int_{-\infty}^t \left[\frac{dx(\tau)}{d\tau} * h(\tau) \right] d\tau$$

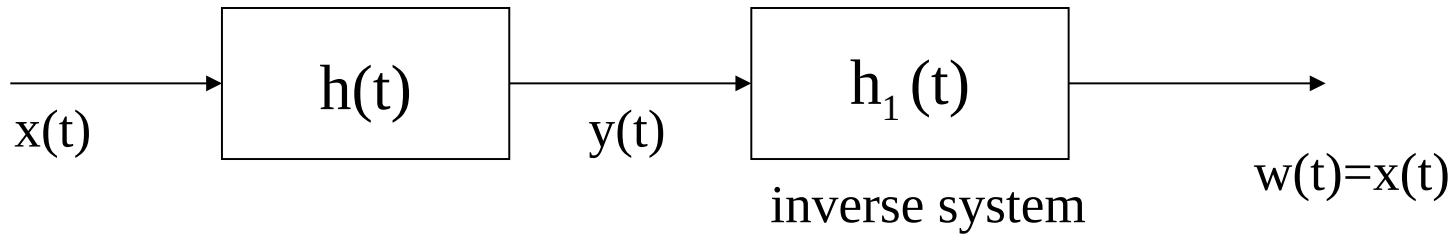


Inverse Systems

A system is said to be **invertible** if distinct inputs lead to distinct outputs. If a system is invertible, then an **inverse** system exists such that when cascaded with the original system yields an output equal to the input to the first system.

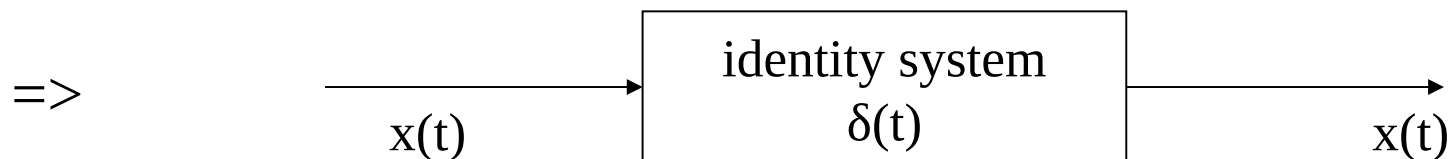


In terms of impulse responses;



The overall impulse response is $h(t) * h_1(t)$ where the condition that $h_1(t)$ must satisfy for it to be the impulse response of the inverse system is,

$$h(t) * h_1(t) = \delta(t)$$



Similar results for DT case: $h(n) * h_1(n) = \delta(n)$