

EEM 301 SIGNALS & SYSTEMS

Chapters

- 1 – Systems
- 2 – Signals
- 3 – Convolution
- 4 – Fourier Series
- 5 – Discrete-time Fourier Series
- 6 – Fourier Transform
- 7 – Discrete-time Fourier Transform

SYSTEMS

Classification of Systems

Differential Equation Models and the p -Operator

Difference Equation Models and the q -Operator

Solution of Linear Differential Equation

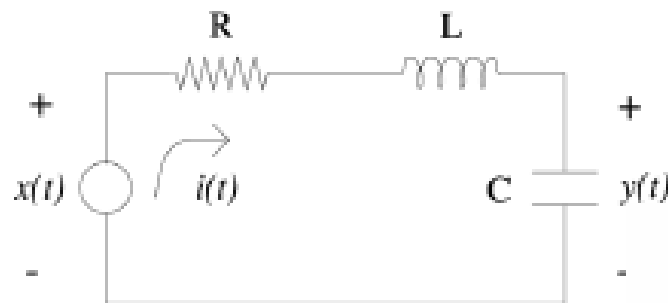
Solution of Linear Difference Equation

Stability

SYSTEM EXAMPLES



Ex. #1 RLC circuit



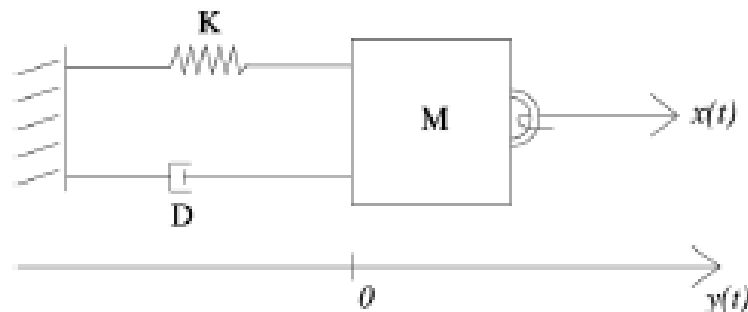
$$R i(t) + L \frac{di(t)}{dt} + y(t) = x(t)$$

$$i(t) = C \frac{dy(t)}{dt}$$

$\Downarrow$

$$LC \frac{d^2 y(t)}{dt^2} + RC \frac{dy(t)}{dt} + y(t) = x(t)$$

Ex. #2 Mechanical system



$x(t)$ - applied force

K - spring constant

D - damping constant

$y(t)$ - displacement from rest

Force Balance:

$$M \frac{d^2 y(t)}{dt^2} = x(t) - K y(t) - D \frac{dy(t)}{dt}$$

$\Downarrow$

$$M \frac{d^2 y(t)}{dt^2} + D \frac{dy(t)}{dt} + K y(t) = x(t)$$

Observation: Very different physical systems may be modeled mathematically in very similar ways.

Observations

- 1) A very rich class of systems (but by no means all systems of interest to us) are described by differential and difference equations.
- 2) Such an equation, by itself, does not completely describe the input-output behavior of a system: we need auxiliary conditions (initial conditions, boundary conditions).
- 3) In some cases the system of interest has time as the natural independent variable and is causal. However, that is not always the case.
- 4) Very different physical systems may have very similar mathematical descriptions.

Classification of Systems

Static and dynamic systems: A system is said to be **static** if its outputs at any instant depend only on the present value of the inputs. A circuit that has only resistors is static system. A system is said to be **dynamic** if its outputs depend on the present as well as the past values of the inputs. Such a system may be regarded as one with **memory**. A circuit that has one or more energy storing elements such as C or L is a dynamic system.

All dynamic systems can mathematically be represented by differential/difference equations.

Causal (nonanticipatory) and noncausal (anticipatory) systems

CAUSALITY

- A system is causal if the output does not anticipate future values of the input, i.e., if the output at any time depends only on values of the input up to that time.
- All real-time physical systems are causal, because time only moves forward. Effect occurs after cause. (Imagine if you own a noncausal system whose output depends on tomorrow's stock price.)
- Causality does not apply to spatially varying signals. (We can move both left and right, up and down.)
- Causality does not apply to systems processing recorded signals, e.g. taped sports games vs. live broadcast.

CAUSALITY (continued)

- Mathematically (in CT): A system $x(t) \rightarrow y(t)$ is causal if

$$\begin{array}{ll} \text{when} & x_1(t) \rightarrow y_1(t) \quad x_2(t) \rightarrow y_2(t) \\ \text{and} & x_1(t) = x_2(t) \quad \text{for all } t \leq t_0 \end{array}$$

$$\text{Then} \quad y_1(t) = y_2(t) \quad \text{for all } t \leq t_0$$

CAUSAL OR NONCAUSAL

$$y(t) = x^2(t - 1)$$

E.g. $y(5)$ depends on $x(4)$... causal

$$y(t) = x(t + 1)$$

E.g. $y(5) = x(6)$, y depends on future $\Rightarrow$ noncausal

$$y[n] = x[-n]$$

E.g. $y[5] = x[-5]$ ok, but

$y[-5] = x[5]$, y depends on future $\Rightarrow$ noncausal

$$y[n] = \left(\frac{1}{2}\right)^{n+1} x^3[n - 1]$$

E.g. $y[5]$ depends on $x[4]$... causal

Time-invariant and time-varying systems

TIME-INVARIANCE (TI)

Informally, a system is time-invariant (TI) if its behavior does not depend on what time it is.

- Mathematically (in DT): A system $x[n] \rightarrow y[n]$ is TI if for any input $x[n]$ and any time shift n_0 ,

$$\begin{array}{ll} \text{If} & x[n] \rightarrow y[n] \\ \text{then} & x[n - n_0] \rightarrow y[n - n_0] . \end{array}$$

- Similarly for a CT time-invariant system,

$$\begin{array}{ll} \text{If} & x(t) \rightarrow y(t) \\ \text{then} & x(t - t_0) \rightarrow y(t - t_0) . \end{array}$$

TIME-INVARIANT OR TIME-VARYING ?

$$y(t) = x^2(t + 1) \qquad x(t-t_0) \rightarrow y(t-t_0)=x^2((t-t_0)+1)$$

TI

$$y[n] = \left(\frac{1}{2}\right)^{n+1} x^3[n - 1]$$

Time-varying (NOT time-invariant)

$$x[n-n_0] \rightarrow y_1[n]=(1/2)^{n+1}x^3[n-n_0-1] \neq y[n-n_0]$$

NOW WE CAN DEDUCE SOMETHING!

Fact: If the input to a TI System is periodic, then the output is periodic with the same period.

“Proof”: Suppose $x(t + T) = x(t)$
 and $x(t) \rightarrow y(t)$

Then by TI

$$\begin{array}{ccc} x(t + T) & \rightarrow & y(t + T). \\ \uparrow & & \uparrow \end{array}$$

These are the
same input!

So these must be
the same output,
i.e., $y(t) = y(t + T)$

Linear and nonlinear systems

LINEARITY

A (CT) system is linear if it has the superposition property:

If $x_1(t) \rightarrow y_1(t)$ and $x_2(t) \rightarrow y_2(t)$

then $ax_1(t) + bx_2(t) \rightarrow ay_1(t) + by_2(t)$

$y[n] = x^2[n]$ Nonlinear, TI, Causal

$y(t) = x(2t)$ Linear, not TI, Noncausal

Can you find systems with other combinations ?

- e.g. Linear, TI, Noncausal

Linear, not TI, Causal

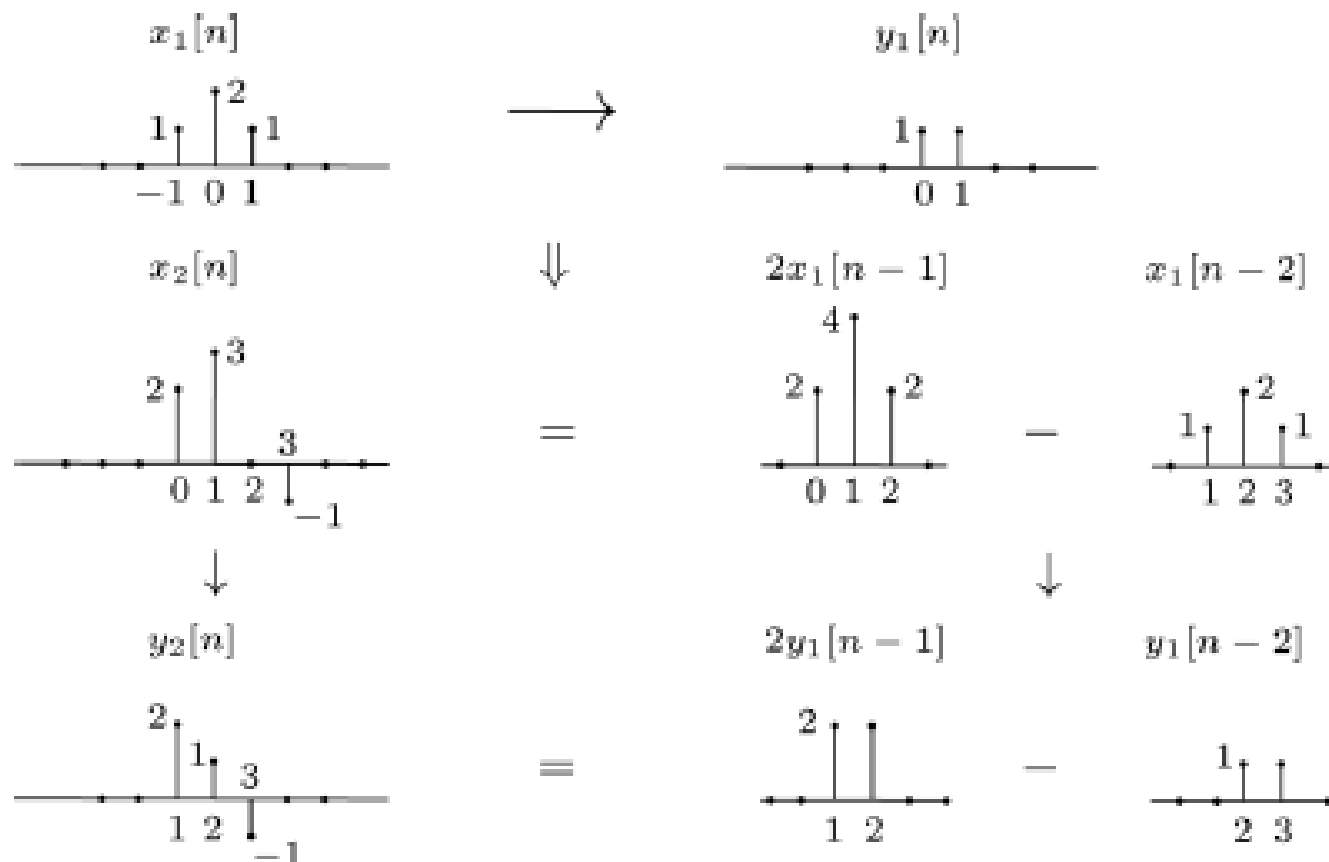
PROPERTIES OF LINEAR SYSTEMS

Superposition

If $x_k[n] \rightarrow y_k[n]$

Then $\sum_k a_k x_k[n] \rightarrow \sum_k a_k y_k[n]$

Example: DT LTI System



In general, a linear CT system is described by n-th order ordinary differential equation,

$$a_n \frac{d^n y}{dt^n} + a_{n-1} \frac{d^{n-1} y}{dt^{n-1}} + \dots + a_1 \frac{dy}{dt} + a_0 y = b_m \frac{d^m x}{dt^m} + \dots + b_0 x$$

where x is the input, y is the output, a_i, b_j are real constants; m, n are integers. The order “n” of the differential equation gives the order of the system and usually $n \geq m$.

We introduce the **operator** “p” to replace “d / dt” in the above equation,

$$(a_n p^n + a_{n-1} p^{n-1} + \dots + a_1 p + a_0) y(t) = (b_m p^m + b_{m-1} p^{m-1} + \dots + b_1 p + b_0) x(t)$$

or

$$D(p) y(t) = N(p) x(t)$$

where $D(p)$, $N(p)$ are polynomials in p ;

$$D(p) = a_n p^n + a_{n-1} p^{n-1} + \dots + a_1 p + a_0$$

$$N(p) = b_m p^m + b_{m-1} p^{m-1} + \dots + b_1 p + b_0$$

Note that $py(t)$ is simply a short-hand notation for the first order derivative w.r.t. (with respect to) time. However $y(t)p$ does not have any meaning, i.e. $py(t) \neq y(t)p$; it does not satisfy the commutativity property.

$$\Rightarrow \quad y(t) = L(p) x(t) \quad \text{where} \quad L(p) = \frac{N(p)}{D(p)}$$

is called the **system operator** or the **operational transfer function** or the **system function** of the system. $L(p)$ is a rational function and for most physical systems it is a proper rational function, i.e. $n \geq m$.

Example

Consider a series RC circuit. Kirchhoff's voltage law yields,

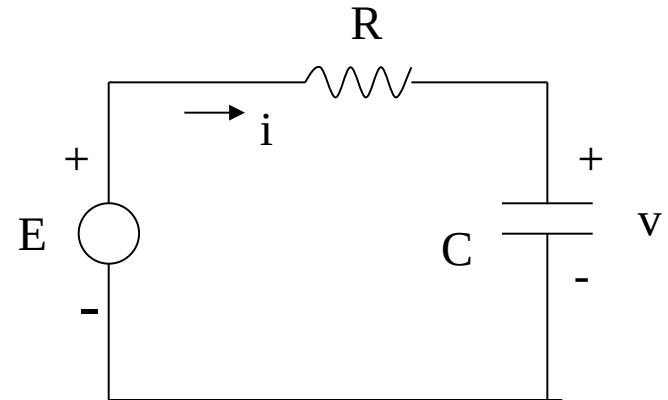
$$RC \frac{dv}{dt} + v = E$$

Using the operator notation, we get

$$(RCp+1) v = E$$

The system operator for this circuit is

$$L(p) = \frac{1}{RCp + 1}$$



Deterministic and stochastic systems

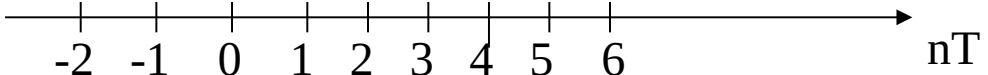
A system is said to be **deterministic** if the system operator $L(p)$, as well as the input(s) are known exactly. For such a system the output(s) for any given input can be determined for all future time if all the initial conditions are known.

If the parameters of the system operator or the input are not known precisely but can be described only in a statistical sense, such systems are called **stochastic**. Most real systems are subject to random disturbances and therefore most systems in the real world are stochastic in nature. However, we will consider only **linear time-invariant (LTI) deterministic** systems.

Continuous-time and discrete-time systems

If time takes any value in the continuous set of real numbers, such system descriptions are called **continuous-time (CT)** systems. For example, RLC circuits or mechanical systems involving motions of particles fall into this category.

With the advent of digital components and devices, the essential features of the input and output signals are of interest only at specific instants of time, given by

$$t = nT$$


where n is an integer and T is called the sampling interval. Such systems are called **discrete-time (DT)** systems.

DT LTI systems are described by, not differential, but **difference equations**. A general form of order n is

$$y(n) + a_1 y(n-1) + \dots + a_{N-1} y(n-N+1) + a_N y(n-N) \\ = b_0 x(n) + b_1 x(n-1) + \dots + b_N x(n-N)$$

or

$$y(n) = \sum_{k=0}^N b_k x(n-k) - \sum_{k=1}^N a_k y(n-k)$$

Note that the input and output are defined only at discrete values of time, i.e. nT and $x(n)$, $n=0,1, \dots$ are the **input sequences**, $y(n)$, $n=0,1,\dots$ are the **output sequences**.

Let “q” denote the **shift operator** that advances the sequence by one time interval; $q x(n) = x(n+1)$

Similarly, the inverse of q is the **unit delay operator**, i.e.

$$q^{-1} x(n) = x(n-1)$$

With this operator, the general difference equation becomes,

$$(1 + a_1 q^{-1} + \dots + a_N q^{-N}) y(n) = (b_0 + b_1 q^{-1} + \dots + b_N q^{-N}) x(n)$$

Multiplying both sides by q^N , we get

$$(q^N + a_1 q^{N-1} + \dots + a_N) y(n) = (b_0 q^N + b_1 q^{N-1} + \dots + b_N) x(n)$$

Hence, the linear **DT system function** is defined as

$$L(q) = \frac{N(q)}{D(q)} = \frac{b_0 q^N + b_1 q^{N-1} + \dots + b_N}{q^N + a_1 q^{N-1} + \dots + a_N}$$

Example: We can develop a model of a savings account that pays compound interest daily. Let the original amount be denoted by p_0 and the rate of interest by r percent per annum. At the end of the 1st day, amount of one day interest is

$$\frac{r}{365 \times 100} \times p(0)$$

where $p(0)=p_0$ (i.e. initial day amount). So the following days we have,

$$p(1) = p(0) + \frac{r}{365 \times 100} \times p(0)$$

$$p(2) = p(1) + \frac{r}{365 \times 100} \times p(1)$$

$$p(3) = p(2) + \frac{r}{365 \times 100} \times p(2) \quad \dots \quad p(n+1) = \left(1 + \frac{r}{36500}\right) p(n)$$

The last general expression gives us the 1st order linear time invariant (LTI) discrete-time (DT) model where the period T is equal to a day.

Example: Consider,

$$3y(n)+4y(n-1)+7y(n-2) = 2x(n)+5x(n-1)$$

We introduce the q-operator as

$$(3+4q^{-1} +7q^{-2}) y(n) = (2+5q^{-1}) x(n)$$

Multiply both sides by q^2 , we obtain

$$(3q^2 +4q+7) y(n) = (2q^2 +5q) x(n)$$

or

$$D(q) y(n) = N(q) x(n)$$

Hence, the system function is

$$L(q) = \frac{N(q)}{D(q)} = \frac{2q^2 + 5q}{3q^2 + 4q + 7}$$

Example

System Description

S/D L/NL TI/TV CT/DT Order

$$y(t) = x^2(t) + 4$$

$$y(t) = tx(t)$$

$$\frac{dy(t)}{dt} + ty(t) = x(t)$$

$$\frac{dy(t)}{dt} + y^2(t) = x(t)$$

$$\frac{d^2y}{dt^2} + \frac{dy}{dt} + 3y = x$$

$$\frac{d^2y}{dt^2} + \frac{dy}{dt} + 3y = x + 2x^2$$

$$\frac{d^2y}{dt^2} + y \frac{dy}{dt} + 3y = x$$

$$y(k) + 2y(k-1) - 3y(k-2) = 2x(k) + x(k-1)$$

$$y(k) + 2y(k-1) - 3y(k-2) = 2x(k) + x(k+1)$$

$$y(k) = kx(k) + k^2x(k-1)$$

$$y(k) = 2x(k) + 3[x(k-1)]^2$$

Solution of Linear Differential Equation

With operator notation,

$$D(p) y(t) = N(p) x(t)$$

where $D(p)$ is a polynomial of degree n with coefficient of p^n equal to 1. Such polynomials are called **monic** polynomial. For a unique solution we need n initial conditions (ic), i.e. the values of y and its first $n-1$ derivatives at $t=0$.

The complete solution $y(t)$ consists of two parts; **complementary function** and **particular integral**:

- (i) Complementary function; is the solution to the homogeneous eq. (HE) i.e. no input, $x(t)=0$. This solution is also known as the **natural (free) or zero-input response** of the system, denoted by $y_c(t)$.
- (ii) Particular integral; is the component of the solution due to the forcing function $x(t)$ and is independent of the initial conditions. It is also called **forced or zero-state response** of the system, denoted by $y_p(t)$.

$$y(t) = y_c(t) + y_p(t)$$

Determining the complementary function

Homogeneous differential eq.: $D(p) y(t) = 0$

The characteristic eq (CE): $D(p) = a_n p^n + a_{n-1} p^{n-1} + \dots + a_1 p + a_0 = 0$

where the n roots r_i are called natural modes.

The complementary function will have the form,

$$y_c(t) = A_1 y_1(t) + A_2 y_2(t) + \dots + A_n y_n(t)$$

where A_i are constants depending on the ic as well as the particular integral and $y_i(t)$ are exponential functions of the form:

- (i) For each distinct real root, r_i , the function $y_i(t)$ is $e^{r_i t}$
- (ii) For each pair of complex conjugate roots $a \pm j b$, the functions are of the form $e^{at} \cos bt$ and $e^{at} \sin bt$.
- (iii) For each real root with multiplicity m , the functions are $e^{r_i t}, t e^{r_i t}, \dots, t^{m-1} e^{r_i t}$.
- (iv) For $a \pm j b$ with multiplicity m , the functions are $e^{at} \cos bt, e^{at} \sin bt, t e^{at} \cos bt, t e^{at} \sin bt, \dots, t^{m-1} e^{at} \cos bt, t^{m-1} e^{at} \sin bt$.

Examples

$$(a) \quad \frac{d^3 y}{dt^3} + 8 \frac{d^2 y}{dt^2} + 37 \frac{dy}{dt} + 50y = 0$$

$$\Rightarrow D(p) = p^3 + 8p^2 + 37p + 50 = 0 \quad \Rightarrow \quad \text{roots: } -2, -3 \pm j4$$

$$\Rightarrow y_c(t) = A_1 e^{-2t} + A_2 e^{-3t} \cos 4t + A_3 e^{-3t} \sin 4t$$

$$(b) \quad \frac{d^2 y}{dt^2} + 10 \frac{dy}{dt} + 25y = 0$$

$$\Rightarrow D(p) = p^2 + 10p + 25 = 0 \quad \Rightarrow \quad \text{roots: } -5, -5$$

$$\Rightarrow y_c(t) = A_1 e^{-5t} + A_2 t e^{-5t}$$

Determining the particular integral

Non-homogeneous eq. (NHE): $D(p) y_p(t) = N(p) x(t)$

where $y_p(t)$ is the particular integral satisfying NHE and often called the **forced response** of the system.

For the special type of input $x(t) = A e^{st}$, $y_p(t)$ is obtained as

$$y_p(t) = \left. \frac{N(p)}{D(p)} \right|_{p=s} \times A e^{st} = \frac{N(s)}{D(s)} \times A e^{st}$$

where s is real or complex quantity.

Note that this method does not apply to all input functions. If $x(t)$ is a sinusoid $\Rightarrow x(t) = A \cos \omega t = \text{Re}[A e^{j\omega t}]$, $x(t) = A \sin \omega t = \text{Im}[A e^{j\omega t}]$ where $s = j\omega$.

The more general type of input; $x(t) = A e^{at} \cos bt = \text{Re}[A e^{st}]$ where $s = a + jb$.

Examples

(a)
$$\frac{d^3 y}{dt^3} + 8 \frac{d^2 y}{dt^2} + 37 \frac{dy}{dt} + 50y = 4e^{-3t}$$

$$\Rightarrow (p^3 + 8p^2 + 37p + 50) y(t) = 4e^{-3t}$$

The particular integral is obtained as

$$y_p(t) = \left[\frac{1}{p^3 + 8p^2 + 37p + 50} \right] \Big|_{p=-3} \times 4e^{-3t} = -0.25e^{-3t}$$

(b) We repeat with $x(t) = 4 \cos 3t$

$$\Rightarrow (p^3 + 8p^2 + 37p + 50) y(t) = 4 \cos 3t = \operatorname{Re}[4e^{st}], \text{ where } s = j3$$

$$y_p(t) = \operatorname{Re} \left[\frac{4e^{st}}{p^3 + 8p^2 + 37p + 50} \right] \Big|_{s=j3} = \operatorname{Re} \left[\frac{4(\cos 3t + j \sin 3t)}{-22 + j84} \right] = -\frac{22}{1885} \cos 3t + \frac{84}{1885} \sin 3t$$

The complete solution: $y(t) = y_c(t) + y_p(t)$

Example

$$\frac{d^3 y}{dt^3} + 8 \frac{d^2 y}{dt^2} + 37 \frac{dy}{dt} + 50y = 4e^{-3t}$$

Hence, the complete solution is

$$y(t) = y_c(t) + y_p(t) = A_1 e^{-2t} + A_2 e^{-3t} \cos 4t + A_3 e^{-3t} \sin 4t - 0.25 e^{-3t}$$

where A_1, A_2, A_3 are to be evaluated to satisfy ic.

Let $y(0) = 1, y'(0) = 2, y''(0) = 1$

From the complete solution we have at $t=0$;

$$y(0) = A_1 + A_2 - 0.25 = 1$$

$$y'(0) = -2A_1 - 3A_2 + 4A_3 + 0.75 = 2$$

$$y''(0) = -4A_1 - 7A_2 - 24A_3 - 2.25 = 1$$

$$\Rightarrow A_1 = 42/17, \quad A_2 = -83/68, \quad A_3 = 43/68$$

Note that if

$$y(t) = Ae^{at} \cos bt \quad \Rightarrow \quad y'(t) = A ae^{at} \cos bt - A be^{at} \sin bt$$

and if

$$y(t) = Ae^{at} \sin bt \quad \Rightarrow \quad y'(t) = A ae^{at} \sin bt + A be^{at} \cos bt$$

A more direct approach for obtaining the complete solution is provided by the Laplace transforms.

Solution of Linear Difference Equations

A linear difference equation of order N is

$$y(n) + a_1 y(n-1) + \dots + a_{n-1} y(n-N+1) + a_N y(n-N) = b_0 x(n) + b_1 x(n-1) + \dots + b_N x(n-N)$$

$$\Rightarrow y(n) = \sum_{k=0}^N b_k x(n-k) - \sum_{k=1}^N a_k y(n-k)$$

where the n-th sample of the output is expressed as a linear combination of the present and N past values of the input as well as the N past values of the output. This approach gives a simple “brute force” solution.

A **closed-form** solution can be obtained using the similar steps to that of the continuous case. The operator notation yields,

$$(1 + a_1 q^{-1} + \dots + a_N q^{-N}) y(n) = (b_0 + b_1 q^{-1} + \dots + b_N q^{-N}) x(n)$$

where q is the **unit advance (shift) operator**.

The complete solution is made of two parts; one is the $y_c(n)$ complementary part, the other particular solution $y_p(n)$;

$$y(n) = y_c(n) + y_p(n)$$

The complementary solution

Consider the HE with the characteristic polynomial as

$$D(q) = q^N + a_1 q^{N-1} + \dots + a_{N-1} q + a_N = 0$$

where r is any root of $D(q)$. By direct substitution it can be verified that the sequence

$$y(n) = A r^n$$

is a solution of the HE where A is a constant.

For all distinct roots, the complementary solution is of the form,

$$y_c(n) = \sum_{k=1}^N A_k y_k(n) = \sum_{k=1}^N A_k r_k^n$$

Steps

- (i) Calculate the N roots of $D(q)$, r_k , $k=1, 2, \dots, N$
- (ii) For each distinct real root, the sequence $y_k(n)$ is r_k^n
- (iii) For each pair of complex conjugate roots, $a \pm jb$, the sequences are of the form $(a \pm jb)^n$.

Note that instead of rectangular notation, one may use polar notation as

$$(a + jb) = \sqrt{a^2 + b^2} \angle \tan^{-1} \frac{b}{a} = r \angle \phi = r e^{j\phi} = r \cos \phi + j r \sin \phi$$

Thus, the two sequences corresponding to the pair of complex conjugate roots may also be written as $r^n \cos n\phi$ and $r^n \sin n\phi$, obtained from

$$r^n e^{j(n\phi)} = r^n \cos n\phi + j r^n \sin n\phi$$

(iv) For each multiple real root, r , of multiplicity m , the sequences are $r^n, n r^n, n^2 r^n, \dots, n^{(m-1)} r^n$.

(v) For each pair of complex conjugate roots $a \pm jb$ of multiplicity m , the sequences are

$$(a \pm jb)^n, n (a \pm jb)^n, n^2 (a \pm jb)^n, \dots, n^{(m-1)} (a \pm jb)^n.$$

In the polar form, $r e^{\pm j\varphi}$, the sequences are

$$r^n \cos n\varphi, r^n \sin n\varphi; n r^n \cos n\varphi, n r^n \sin n\varphi; n^2 r^n \cos n\varphi, n^2 r^n \sin n\varphi; \dots;$$

$$n^{(m-1)} r^n \cos n\varphi, n^{(m-1)} r^n \sin n\varphi.$$

Examples: Consider HE,

(a) $y(n) - 8 y(n-1) + 37 y(n-2) - 50 y(n-3) = 0$

$\Rightarrow D(q) = q^3 - 8 q^2 + 37 q - 50 = 0$

$\Rightarrow$ roots: $r_1 = 2, r_{2,3} = 3 \pm j4 = 5 e^{\pm j0.93}$

The complementary solution is given by the following sequences

$$y_c(n) = A_1 2^n + A_2 5^n \cos 0.93n + A_3 5^n \sin 0.93n$$

(b) $y(n+3) - 7 y(n+2) + 16 y(n+1) - 12 y(n) = 0$

$\Rightarrow D(q) = q^3 - 7 q^2 + 16 q - 12 = 0$

$\Rightarrow$ roots: $r_1 = 2, r_2 = 2, (m=2), r_3 = 3$

The complementary solution is

$$y_c(n) = A_1 2^n + A_2 n 2^n + A_3 3^n$$

The particular solution

The particular solution $y_p(n)$ of the NHE (i.e. nonzero input) depends on the forcing function and is also called the **forced response**. Similar to the differential equations, the particular solution can be determined for the special case of complex exponential sequence.

Example

Consider NHE of Ex.(a) with the exponential sequence as,

$$y(n+3) - 8 y(n+2) + 37 y(n+1) - 50 y(n) = 8 (0.5)^n$$

Form $D(q) y(n) = N(q) x(n)$

$$\Rightarrow (q^3 - 8 q^2 + 37 q - 50) y(n) = 8 (0.5)^n$$

Similar to the continuous case, the particular solution is readily obtained for the given exponential sequence as

$$y_p(n) = \left[\frac{N(q)}{D(q)} \right] \bigg|_{q=0.5} \times 8(0.5)^n = \frac{8(0.5)^n}{0.125 - (8 \times 0.25) + (37 \times 0.5) - 50} = -\frac{64}{267} \times (0.5)^n$$

The complete solution

$$y(n) = y_c(n) + y_p(n)$$

The unknown constants in the complete solution are determined in such a way that all of the boundary conditions are satisfied.

In an **initial value** problem, N number of initial conditions for a N-th order difference equation must be given.

Example

Consider the previous example with the boundary conditions given as $y(0) = 2$, $y(1) = 3$, $y(2) = 5$.

The complete solution is found as

$$y(n) = A_1 2^n + A_2 5^n \cos 0.93n + A_3 5^n \sin 0.93n - (64/267) (0.5)^n$$

$$y(0) = A_1 + A_2 - 64/267 = 2$$

$$y(1) = 2A_1 + 5A_2 \cos 0.93 + 5A_3 \sin 0.93 - (32/267) = 3$$

$$y(2) = 4A_1 + 25A_2 \cos 1.85 + 25A_3 \sin 1.85 - (16/267) = 5$$

$$\Rightarrow A_1 = 2.1886, \quad A_2 = 0.05108, \quad A_3 = -0.35666$$

For the solution of difference equation, there is also a direct approach using the z-transforms which will be presented in the sequel.

Stability

Stability is directly related to the usefulness of the system response. A system is said to be **stable** if the output remains bounded for all bounded inputs.

CT: The **natural response** of a LTI CT system will remain bounded $\Leftrightarrow$ all the roots of $D(p)$ satisfy the following;

- (i) The real parts of all simple roots must be ≤ 0
- (ii) The real parts of all multiple roots must be < 0

DT: The **natural response** of a LTI DT system will remain bounded $\Leftrightarrow$ all the roots of $D(q)$ satisfy the following;

- (i) The magnitude of all simple roots must be ≤ 1
- (ii) The magnitude of all multiple roots must be < 1

For the stability of the **complete response** in the general bounded-input-bounded-output (**BIBO**) sense;

(a) A LTI CT system is

- (i) stable in the BIBO sense if all the roots of $D(p)$ have negative real parts,
- (ii) unstable if the real part of any root ≥ 0

(b) A LTI DT system is

- (i) stable in the BIBO sense if all the roots of $D(q)$ have magnitudes less than one,
- (ii) unstable if the magnitude of any root ≥ 1

Note that we have excluded simple roots of $D(p)$ with zero real parts in (a) and simple roots of $D(q)$ with magnitude equal to one in (b). The reason is that a bounded natural response does not necessarily imply stability. e.g. simple roots of $D(p)$ with zero real parts will cause a bounded sinusoid in the natural response, but the particular solution will grow out of bounds if the input function is a sinusoid of the same frequency.

Example

Consider

$$y''(t) + 4y(t) = x(t) \Rightarrow D(p) = p^2 + 4 = 0 \Rightarrow p_{1,2} = \pm j2$$

The complementary function is

$$y_c(t) = A \cos 2t + B \sin 2t \quad (\text{a bounded signal})$$

For the input $x(t)=2\cos 2t$, the particular integral is

$$y_p(t) = 0.5 t \sin 2t$$

which will be unbounded although the input itself is bounded.

