

SIGNALS

Continuous-time (CT) and Discrete-time (DT) Signals

Periodic Signals

Even and Odd Signals

Complex Exponential Signals

Some Common Signals

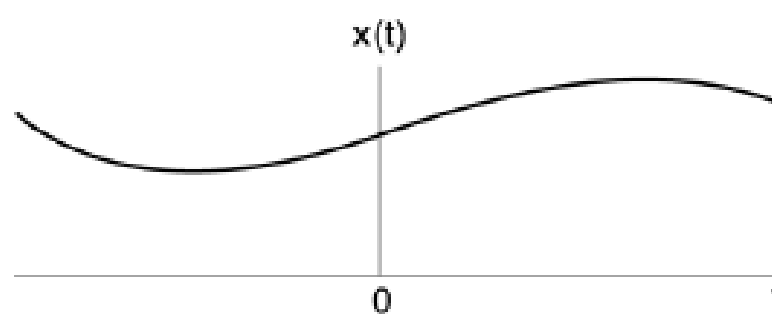
Power and Energy Signals

SIGNALS

Signals are functions of independent variables that carry information. For example:

- Electrical signals --- voltages and currents in a circuit
- Acoustic signals --- audio or speech signals (analog or digital)
- Video signals --- intensity variations in an image (e.g. a CAT scan)

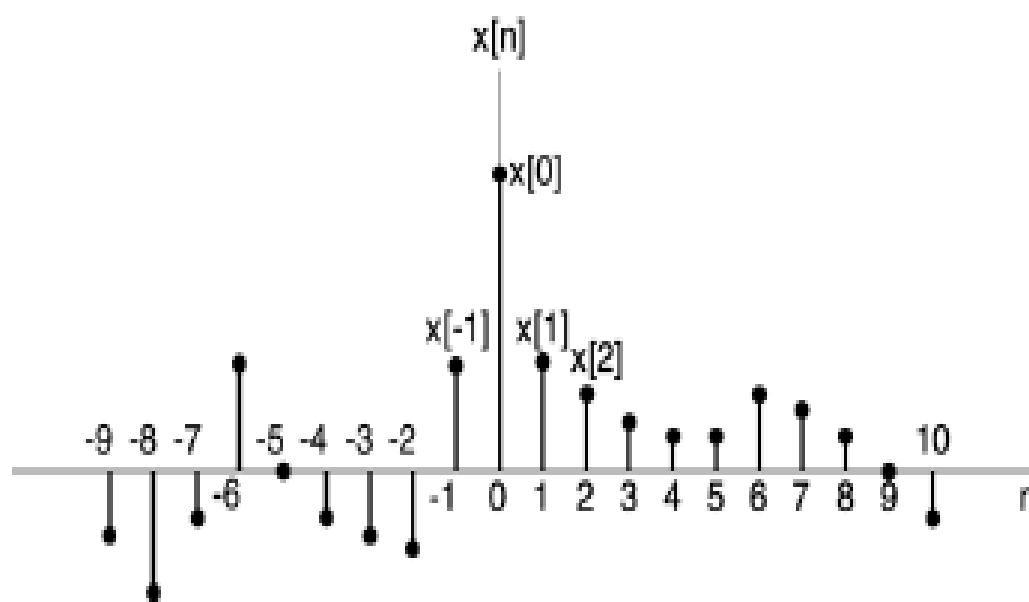
CT Signals



- Most of the signals in the physical world are CT signals—E.g. voltage & current, pressure, temperature, velocity, etc.

DT Signals

- $x[n]$, n — integer, time varies discretely



A **DT** signal has values for only discrete instants of time, that is, the domain of t is a discrete set of numbers.

Such a signal is usually denoted as $x(n)$ or $x[n]$, instead of $x(t)$, where n is an integer.

A DT signal can also be obtained by **sampling** a CT signal.

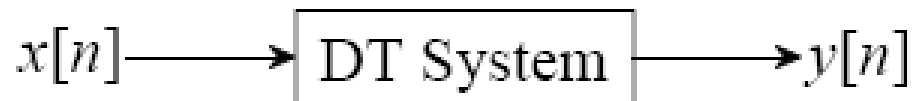
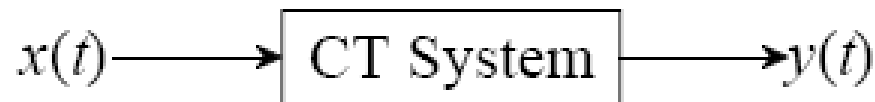
In such cases, n may be replaced by kT , where k is an integer and T is called the sampling interval. Hence, the sampled signal is denoted by $x(kT)$ or $x(k)$ for simplicity.

A DT signal $x(n)$ is often called a **sequence**.

SYSTEMS

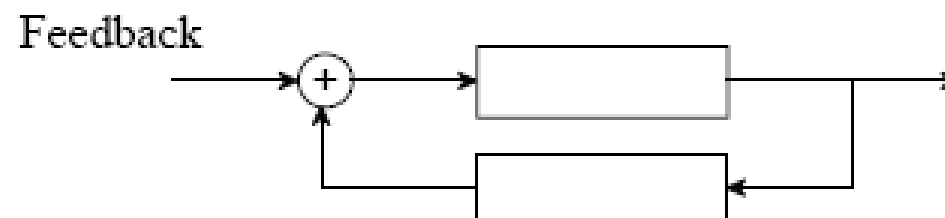
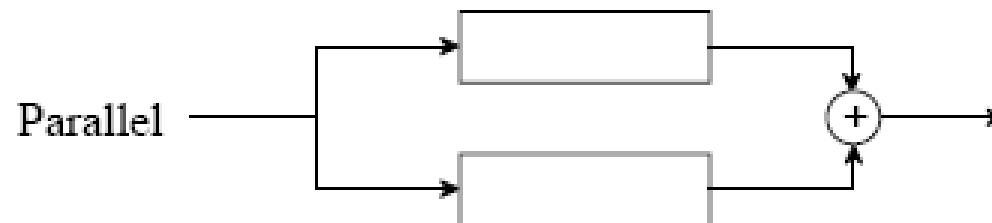
For the most part, our view of systems will be from an input-output perspective:

A system responds to applied input signals, and its response is described in terms of one or more output signals



SYSTEM INTERCONNECTIONS

- An important concept is that of interconnecting systems
 - To build more complex systems by interconnecting simpler subsystems
 - To modify response of a system
- Signal flow (Block) diagram



Periodic Signal: A CT signal $x(t)$ is said to be **periodic** iff

$$x(t+T) = x(t) \quad \text{for all } -\infty < t < \infty$$

The smallest +ve value of T for which this holds is called the **period** of the signal. For integer k , T may be replaced by kT .

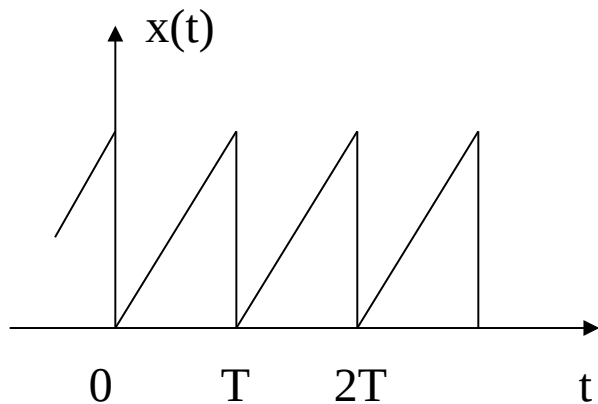
Similarly, DT signal $x[n]$ is said to be periodic if and only if

$$x[n+N] = x[n] \quad \text{for all } -\infty < n < \infty$$

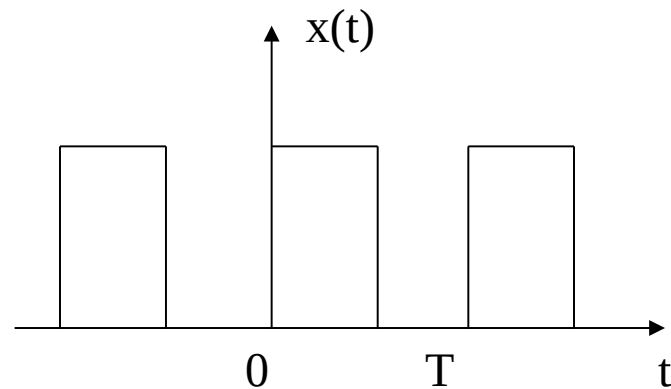
The smallest +ve integer value of N for which this holds is called the **period** of the signal.

Any signal that is **not** periodic is said to be **aperiodic**.

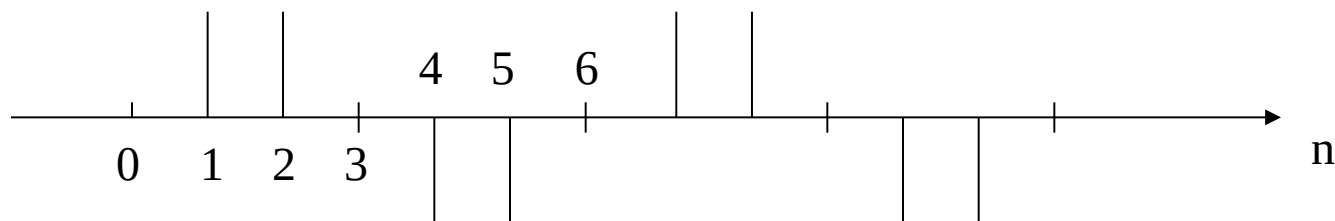
Examples



saw-tooth



train of pulses



A sinusoidal sequence, $x(n)=A \sin(2\pi n/N)$ where the period is $N=6$

The most common example of a CT periodic signal is the sinusoid described by

$$x(t) = A \cos(\omega t - \varphi)$$

where A is the amplitude, ω is the angular frequency in rad/sec, φ is the phase in radians. The angular frequency is given by $\omega = 2\pi f$, where f is the frequency in hertz (or cycles/sec).

The period T , in seconds, is given by

$$T = 1/f = 2\pi/\omega$$

Sum of $x_1(t) = A_1 \cos(\omega_1 t)$, $x_2(t) = A_2 \cos(\omega_2 t)$ is periodic, iff the ratio T_2/T_1 is a rational number.

Example

Consider the three sinusoidal (periodic) signals,

$$x_1(t) = 10 \sin 3\pi t, \quad x_2(t) = 20 \sin 7\pi t, \quad x_3(t) = 10 \sin 32t$$

where $\omega_1 = 2\pi/T_1 = 3\pi$ and the period for $x_1(t)$ is $T_1 = 2\pi/3\pi = 2/3$
and $\omega_2 = 2\pi/T_2 = 7\pi$ and the period for $x_2(t)$ is $T_2 = 2\pi/7\pi = 2/7$
and $\omega_3 = 2\pi/T_3 = 32$ and the period for $x_3(t)$ is $T_3 = 2\pi/32 = \pi/16$

The ratio of T_2/T_1 is $3/7$ which is a rational fraction and thus the sum of x_1 and x_2 is also a periodic signal.

The ratio of T_3/T_1 is $(\pi/16)/(2/3)$ is an irrational number and therefore the sum of x_1 and x_3 will not be periodic.

Even and Odd Signals

A signal is said to be **even** if

$$x(t) = x(-t), \quad x(n) = x(-n)$$

Note that an even function is symmetrical about the vertical axis.

A signal is said to be **odd** if

$$x(t) = -x(-t), \quad x(n) = -x(-n)$$

An odd function is antisymmetric about the vertical axis.

Remember that $\sin k\omega t$ is odd while $\cos k\omega t$ is even for all values of k .

Any $x(t)$ or $x(n)$ signal can be expressed as a sum of even and odd signals;

$$x(t) = x_e(t) + x_o(t),$$

$$x(n) = x_e(n) + x_o(n),$$

where

$$x_e(t) = (1/2) [x(t) + x(-t)], \quad \text{even};$$

$$x_o(t) = (1/2) [x(t) - x(-t)], \quad \text{odd};$$

$$x_e(n) = (1/2) [x(n) + x(-n)], \quad \text{even};$$

$$x_o(n) = (1/2) [x(n) - x(-n)], \quad \text{odd};$$

Complex Exponential Signals

Recall the Euler's relationship

$$\exp(j\theta) = e^{j\theta} = \cos \theta + j \sin \theta$$

and a sinusoid can also be expressed as

$$A \cos(\omega t + \varphi) = \text{Re}[A e^{j\omega t} e^{j\varphi}] \quad \text{or} \quad A \sin(\omega t + \varphi) = \text{Im}[A e^{j\omega t} e^{j\varphi}]$$

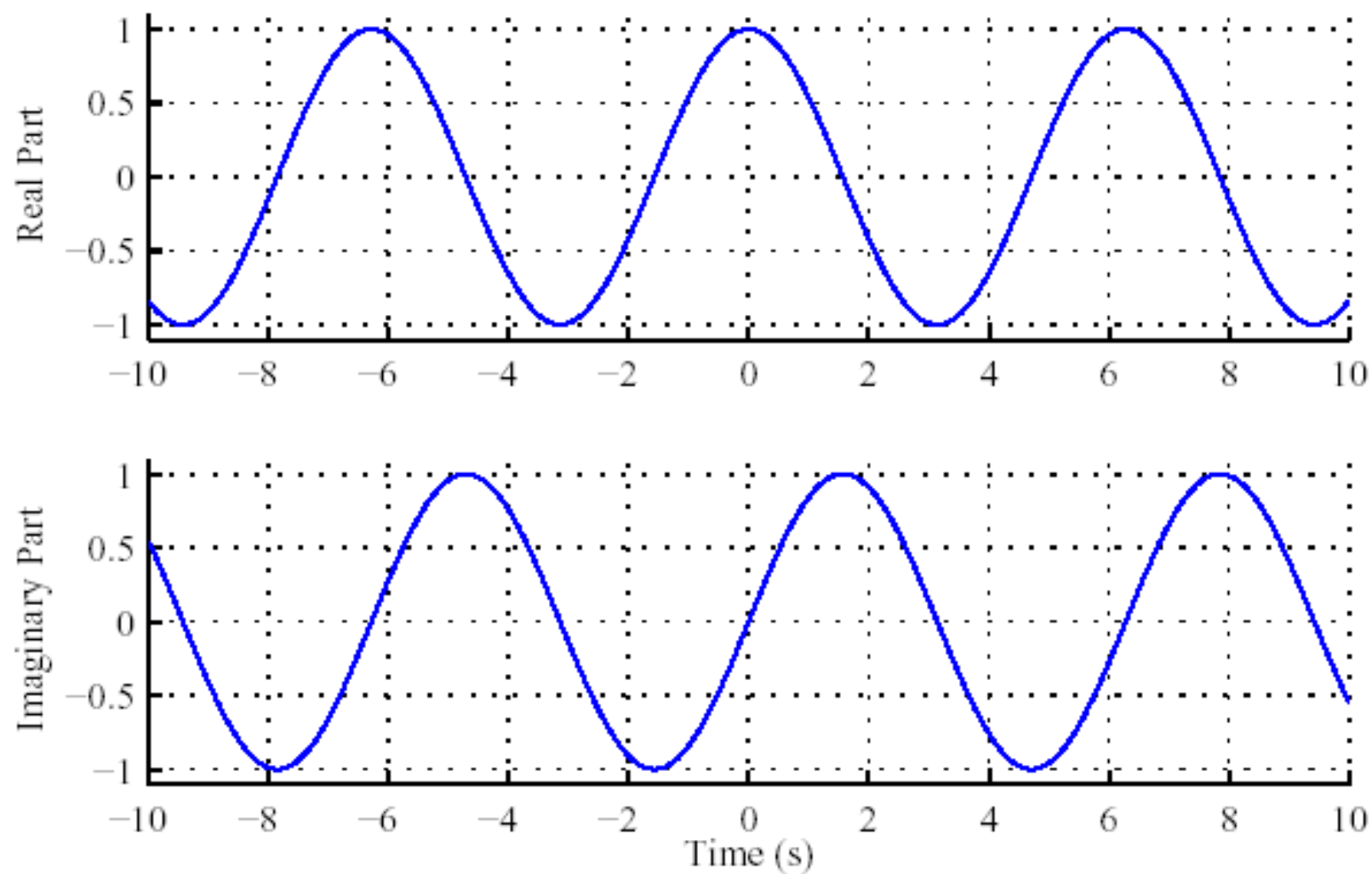
or alternatively,

$$A \cos(\omega t + \varphi) = (A/2) [e^{j\omega t} e^{j\varphi} + e^{-j\omega t} e^{-j\varphi}]$$

Note the periodic nature of the complex exponential,

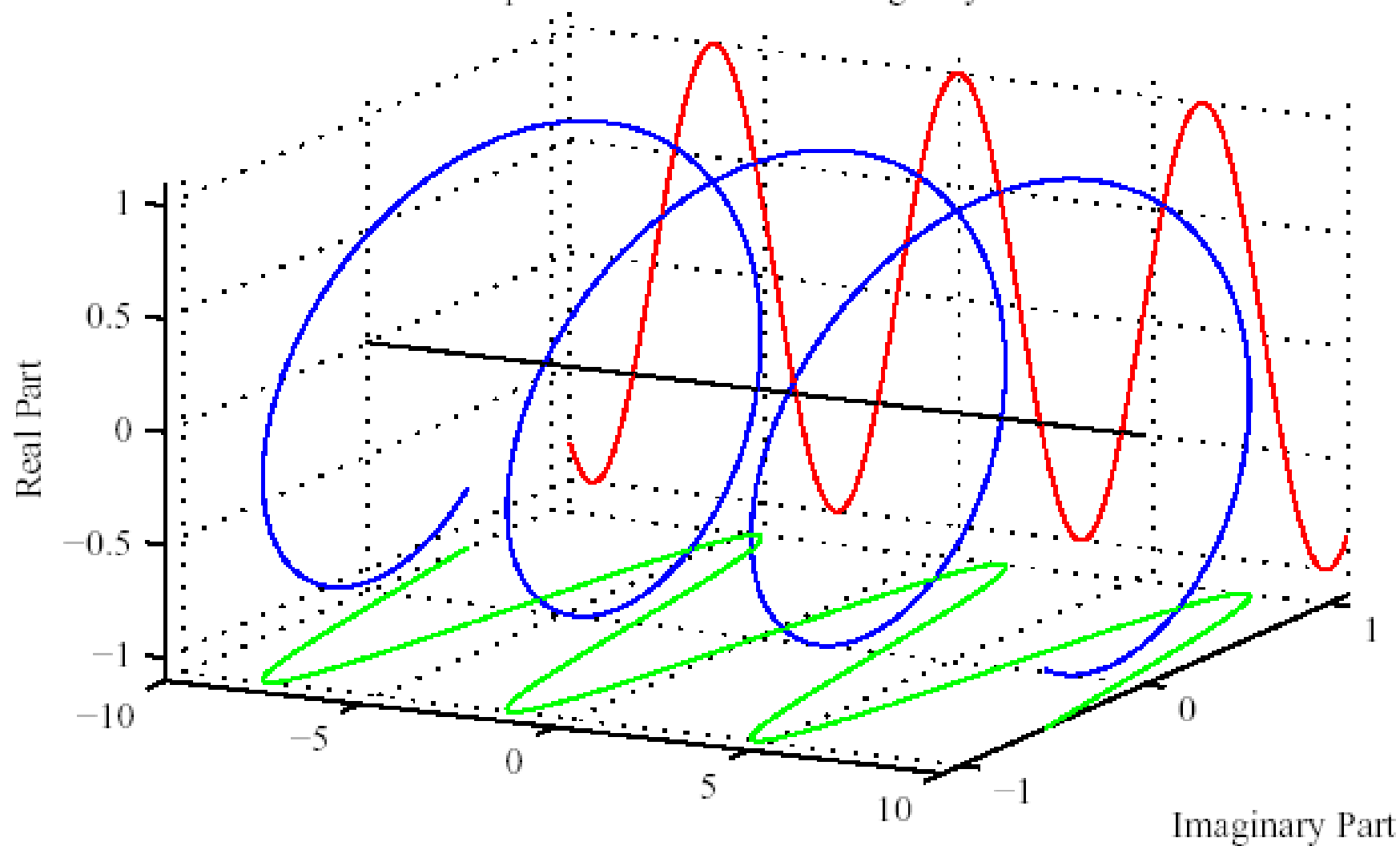
$$e^{j(\omega t + \varphi + 2k\pi)} = e^{j(\omega t + \varphi)}$$

Example 1: $x(t) = e^{jt}$



Example 1: $x(t) = e^{jt}$

Complex:Blue Real:Red Imaginary:Green



Similar to the CT case, the periodic DT signal in the form of a complex exponential is expressed as

$$\begin{aligned} x(n) &= A \cos(2\pi n/N) = \operatorname{Re}\{Ae^{j(2\pi n/N)}\} = \operatorname{Re}\{Ae^{j\Omega n}\} \\ \text{or} \quad x(n) &= A \sin(2\pi n/N) = \operatorname{Im}\{Ae^{j(2\pi n/N)}\} = \operatorname{Im}\{Ae^{j\Omega n}\} \end{aligned}$$

where $\Omega = 2\pi/N$ is the angular frequency in radians. Note that the period N is an integer and dimensionless quantity.

With a phase shift we have,

$$A \cos(\Omega n + \varphi) = (A/2)\{e^{j\Omega n} e^{j\varphi} + e^{-j\Omega n} e^{-j\varphi}\}$$

Note that a distinct property of the DT complex exponential is

$$x(n) = e^{j(\Omega+2\pi)n} = e^{j\Omega n} e^{j2\pi n} = e^{j\Omega n}$$

Need to consider only 2π interval; e.g. $0 \leq \Omega < 2\pi$ or $-\pi \leq \Omega < \pi$

A second distinct property is related to the periodicity of DT complex exponential; with period $N > 0$,

$$e^{j\Omega(n+N)} = e^{j\Omega n} \Rightarrow e^{j\Omega N} = 1 \Rightarrow \Omega N = 2\pi m \Rightarrow \Omega/2\pi = m/N$$

where m is also an integer, i.e. $\Omega/2\pi$ is a rational number.

If m and N have no common factors then the fundamental frequency is

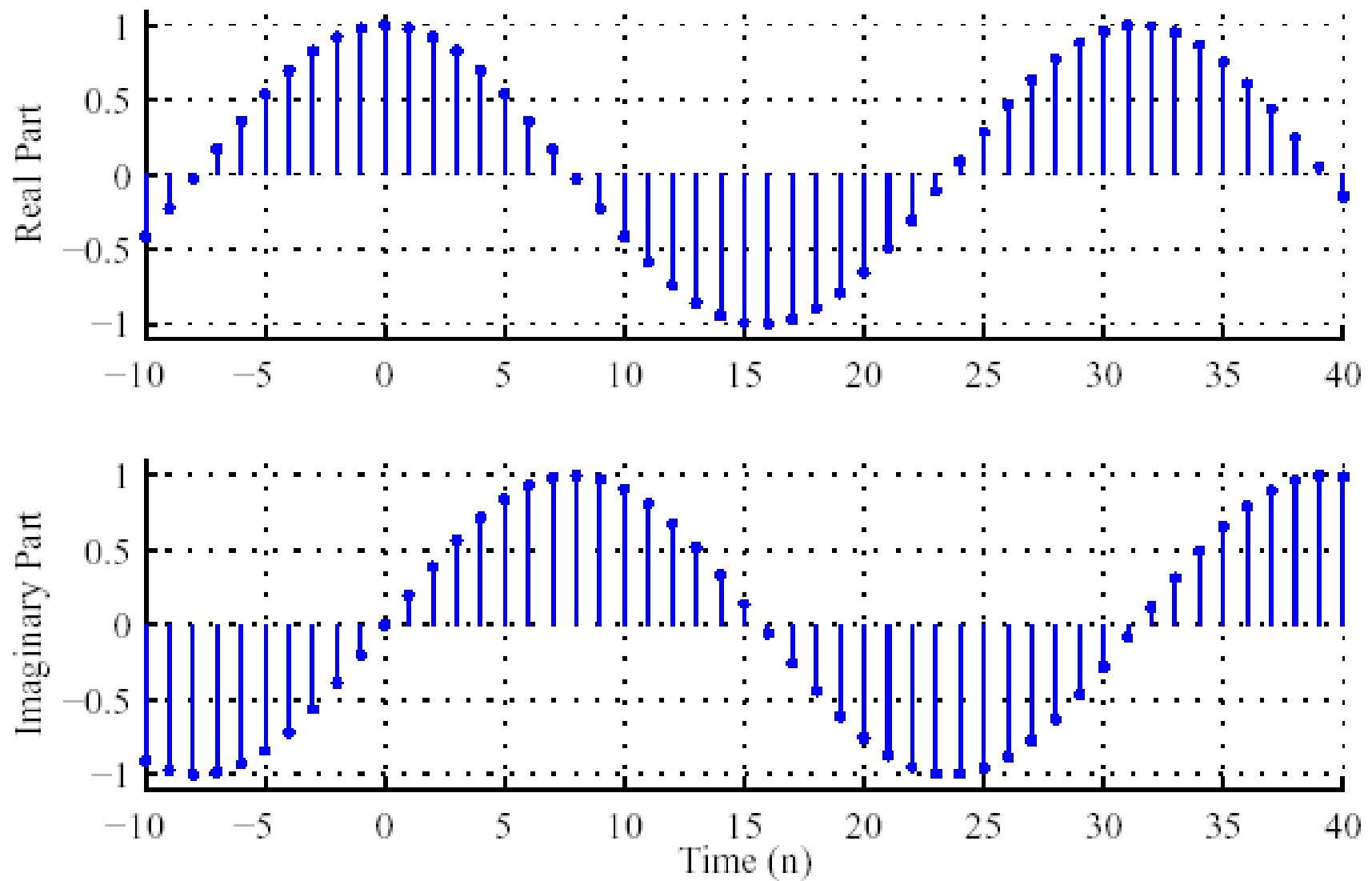
$$2\pi/N = \Omega/m$$

and the fundamental period is given by

$$N = m (2\pi/\Omega)$$

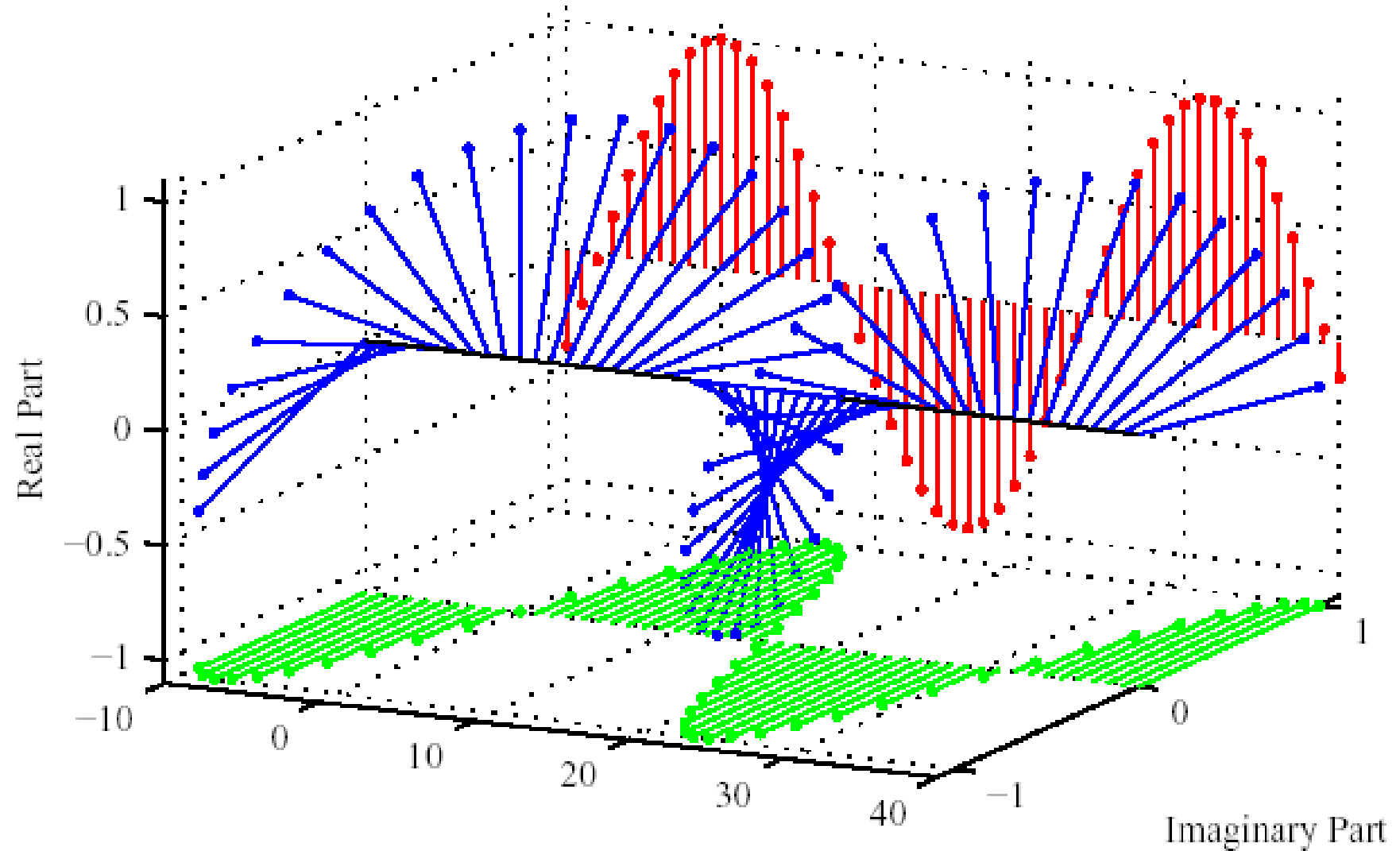
Note that CT $x(t)=x(t+mT)$ is periodic for integer m and any real T (need not be integer) whereas DT $x(n)=x(n+mN)$ is periodic for integers m and N .

Example 2: $x[n] = e^{j0.2n}$



Example 2: $x[n] = e^{j0.2n}$

Complex:Blue Real:Red Imaginary:Green

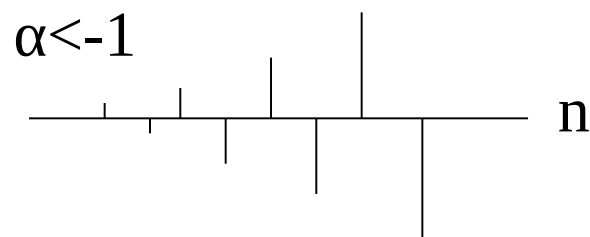
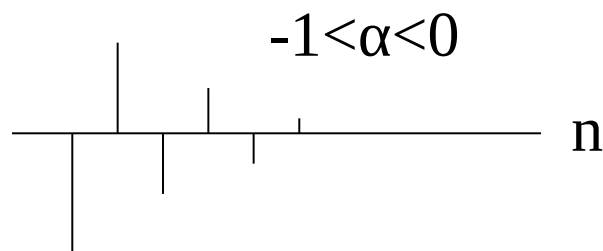
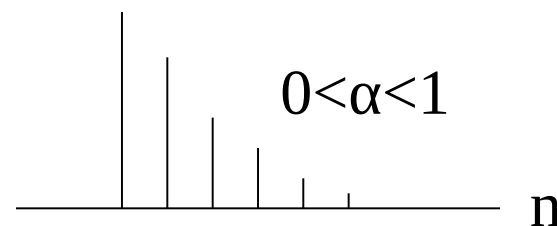
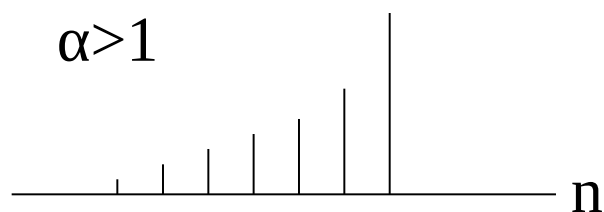


DT **complex** exponential sequence is defined as

$$x(n) = C \alpha^n \quad \text{or} \quad x(n) = C e^{\beta n}$$

where $\alpha = e^{\beta}$ and C, α are, in general complex.

Real exponential sequence: C, α are real quantities.



Some Common Signals

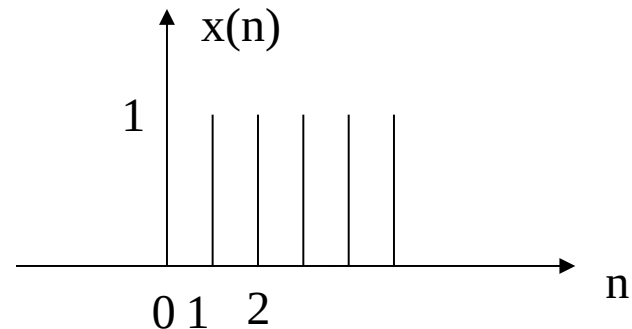
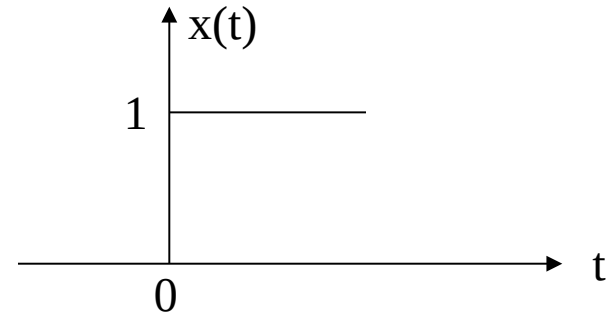
The **unit step** signal

$$u(t) = \begin{cases} 0 & t < 0 \\ 1 & t > 0 \end{cases}$$

At $t=0$ the value of function jumps 0 to 1; it is discontinuous.

The **unit step** sequence

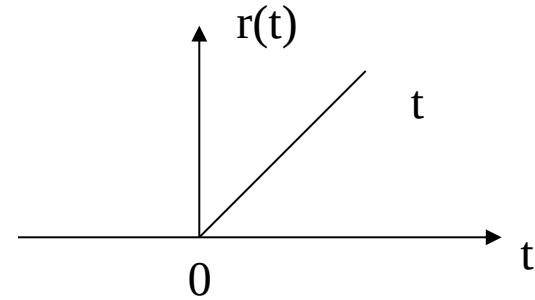
$$u(n) = \begin{cases} 0 & n < 0 \\ 1 & n \geq 0 \end{cases}$$



Integration of unit step gives **unit ramp** function $r(t)$ as

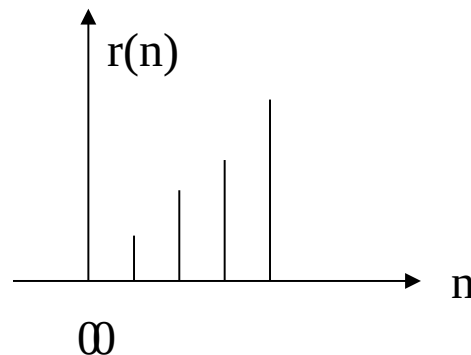
$$r(t) = t \quad \text{for } t \geq 0$$

A ramp signal with slope m is $mr(t)$.



Note that some other signals may generated by these singularity functions, namely $u(t)$ and $r(t)$.

DT **unit ramp** function is defined as $r(n) = n$ for $n \geq 0$.

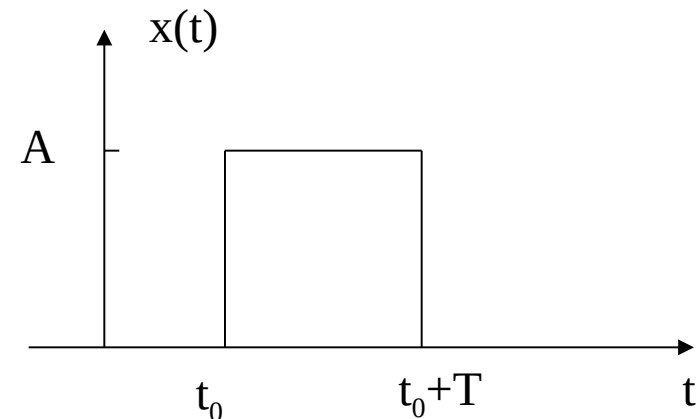


Example

Signals generated by singularity functions.

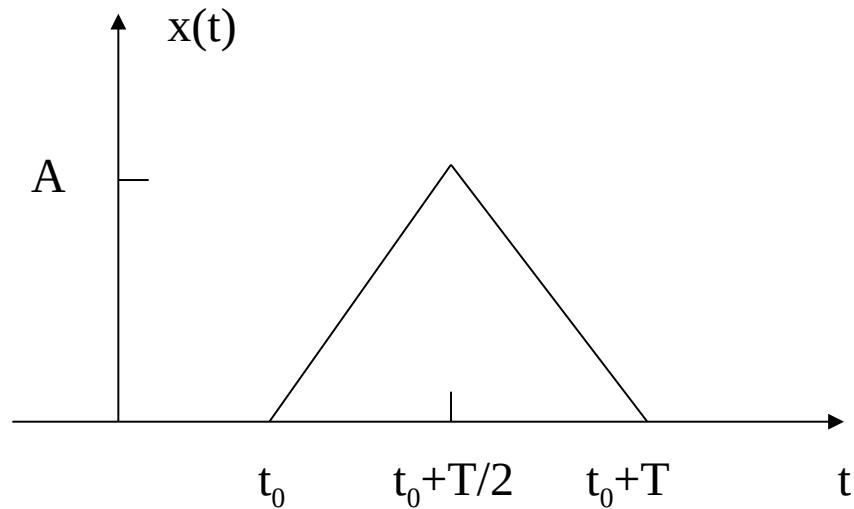
(a) Rectangular pulse with duration T and height A starting at $t = t_0$ can be expressed as

$$x(t) = A u(t-t_0) - A u(t-t_0 - T)$$



(b) Triangular pulse is expressed by

$$x(t) = (2A/T) [r(t - t_0) - 2r(t - t_0 - T/2) + r(t - t_0 - T)]$$



Unit Impulse

An important singularity function is the **impulse** or **delta** function denoted by $\delta(t)$, integration of which gives $u(t)$;

$$u(t) = \int_{-\infty}^t \delta(\tau) d\tau \quad \text{running integral}$$

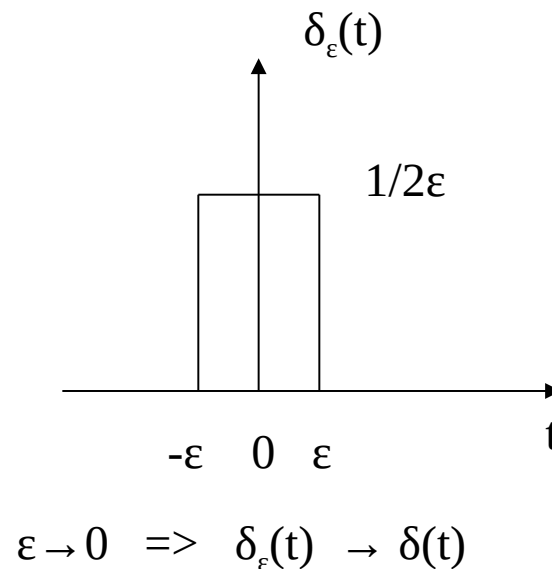
Alternatively, the unit impulse is given by the time derivative of the unit step; $\delta(t) = du(t)/dt$

Properties of the impulse:

1. $\delta(t) = 0$ for $t \neq 0$

2. $\int_{-\infty}^{\infty} \delta(\tau) d\tau = 1$

The area under the impulse
is unity



DT unit impulse sequence, $\delta(n)$

The process of integration is replaced with summation,

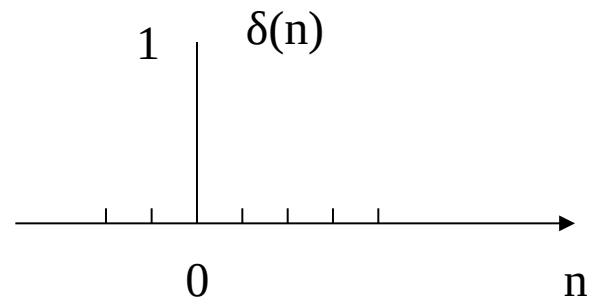
$$u(n) = \sum_{-\infty}^n \delta(m) \quad \text{(running sum)}$$

DT unit impulse is the first difference of the step sequence;

$$\delta(n) = u(n) - u(n-1)$$

DT unit impulse defined as

$$\delta(n) = \begin{cases} 0 & n \neq 0 \\ 1 & n = 0 \end{cases}$$



Power and Energy Signals

The energy content of a signal $x(t)$ is given by

$$E = \lim_{T \rightarrow \infty} \int_{-T}^T x^2(t) dt$$

and the average power by

$$P = \lim_{T \rightarrow \infty} \frac{1}{2T} \int_{-T}^T x^2(t) dt$$

Note that if $x(t)$ is the current (A) flowing in a 1 Ohm resistor then E is the total energy (Joules) dissipated and P is the average power (Watts) content in the signal

If the average power content is finite and positive, its energy content will be infinite.

Similarly, if the energy content is finite, then its average power content must be zero:

(i) $x(t)$ is a **power** signal if and only if its average power content P is finite and positive, implying that its energy content is infinite.

(ii) $x(t)$ is an **energy** signal if and only if its energy content E is finite implying that its power content is zero.

(iii) If $x(t)$ does not satisfy either of the two conditions stated above, then it is neither a power nor an energy signal.

Example: Consider, $x(t) = A e^{at} u(t)$ where $u(t)$ is the unit step and A, a are constants with $a < 0$.

$$\begin{aligned} E &= \lim_{T \rightarrow \infty} \int_{-T}^T [A e^{at} u(t)]^2 dt = \lim_{T \rightarrow \infty} A^2 \int_0^T e^{2at} dt = \lim_{T \rightarrow \infty} \left. \frac{A^2 e^{2at}}{2a} \right|_0^T \\ &= \lim_{T \rightarrow \infty} \left[\frac{A^2 e^{2aT}}{2a} - \frac{A^2}{2a} \right] = -\frac{A^2}{2a} \end{aligned}$$

Note that $a < 0$ and E is +ve and finite. So this is an energy signal.

If $a=0$, $x(t)=A u(t)$ and

$$\begin{aligned}
 E &= \lim_{T \rightarrow \infty} \int_{-T}^T [Au(t)]^2 dt = \lim_{T \rightarrow \infty} A^2 \int_0^T 1 dt = \lim_{T \rightarrow \infty} A^2 [t]_0^T \\
 &= \lim_{T \rightarrow \infty} A^2 T = \infty
 \end{aligned}$$

and

$$\begin{aligned}
 P &= \lim_{T \rightarrow \infty} \frac{1}{2T} \int_{-T}^T [Au(t)]^2 dt = \lim_{T \rightarrow \infty} \frac{A^2}{2T} [t]_0^T \\
 &= \lim_{T \rightarrow \infty} \frac{A^2}{2T} (T) = \frac{A^2}{2}
 \end{aligned}$$

As P is finite, Au(t), i.e. the step function is a power signal.

The average power content of a periodic signal $x(t)$ with period T can be determined by performing the integration over one complete cycle,

$$P = \frac{1}{T} \int_0^T x^2(t) dt$$

Example: Consider

$$x(t) = 30 \cos 4\pi t [u(t) - u(t-1)]$$

$$E = \lim_{T \rightarrow \infty} \int_{-T}^T x^2(t) dt = \lim_{T \rightarrow \infty} \int_0^1 30^2 \cos^2 4\pi t dt$$

$$= \lim_{T \rightarrow \infty} \frac{900}{2} \int_0^1 (1 + \cos 8\pi t) dt = \lim_{T \rightarrow \infty} 450 \left[t + \frac{\sin 8\pi t}{8\pi} \right] \Big|_0^1$$

$$= \lim_{T \rightarrow \infty} 450 \left[1 + \frac{\sin 8\pi}{8\pi} - 0 \right] = 450$$

As E is finite, the considered x(t) is a energy signal.

