

HANDOUT E.2 - EXAMPLES ON FOURIER TRANSFORMS

Example 1:

a) Evaluate the Fourier transform of the following continuous signal

$$f(x) = U(x)e^{-ax},$$

for $a > 0$.

The function $U(x)$ is defined as follows.

$$U(x) = \begin{cases} 1, & x \geq 0 \\ 0, & x < 0 \end{cases}.$$

The Fourier transform of a signal $f(x)$ is given by

$$F(\omega) = \int_{-\infty}^{\infty} f(x)e^{-j\omega x} dx$$

Therefore,

$$F(\omega) = \int_{-\infty}^{\infty} U(x)e^{-ax}e^{-j\omega x} dx,$$

$$\Rightarrow F(\omega) = \int_0^{\infty} e^{-(a+j\omega)x} dx,$$

$$\Rightarrow F(\omega) = -\left. \frac{e^{-(a+j\omega)x}}{(a+j\omega)} \right|_0^{\infty} = \frac{1}{(a+j\omega)}.$$

b) Evaluate the Fourier transform of the following discrete time signal.

$$x(n) = a^n u(n), \quad -1 < a < 1.$$

$u(n)$ is defined as follows.

$$u(n) = \begin{cases} 1, & n \geq 0 \\ 0, & n < 0 \end{cases}.$$

The Fourier transform of a discrete time signal is given by

$$X(\omega) = \sum_{n=-\infty}^{\infty} x(n) e^{-j\omega n}.$$

Therefore,

$$\begin{aligned} X(\omega) &= \sum_{n=-\infty}^{\infty} a^n u(n) e^{-j\omega n}, \\ \Rightarrow X(\omega) &= \sum_{n=0}^{\infty} a^n e^{-j\omega n} = \sum_{n=0}^{\infty} (ae^{-j\omega})^n, \end{aligned}$$

Since $|ae^{-j\omega}| = |a| < 1$, using the geometric formula yields

$$X(\omega) = \frac{1}{1 - ae^{-j\omega}}.$$

Example 2:

Determine the Fourier transform of the signal

$$x(n) = a^{|n|}, \quad -1 < a < 1.$$

First the signal can be expressed as

$$x(n) = x_1(n) + x_2(n)$$

where

$$x_1(n) = \begin{cases} a^n, & n \geq 0 \\ 0, & n < 0 \end{cases}$$

and

$$x_2(n) = \begin{cases} a^{-n}, & n < 0 \\ 0, & n \geq 0 \end{cases}$$

From the definition of Fourier transform, we have

$$X_1(\omega) = \sum_{n=-\infty}^{\infty} x_1(n) e^{-j\omega n} = \sum_{n=0}^{\infty} a^n e^{-j\omega n} = \sum_{n=0}^{\infty} (ae^{-j\omega})^n$$

The summation is a geometric series that converges to

$$X_1(\omega) = \frac{1}{1 - ae^{-j\omega}}.$$

Similarly, the Fourier transform of $x_2(n)$ is

$$\begin{aligned} X_2(\omega) &= \sum_{n=-\infty}^{\infty} x_2(n) e^{-j\omega n} = \sum_{n=-\infty}^{-1} a^{-n} e^{-j\omega n} = \sum_{n=-\infty}^{-1} (ae^{j\omega})^{-n} \\ \Rightarrow X_2(\omega) &= \sum_{k=1}^{\infty} (ae^{j\omega})^k = \frac{ae^{j\omega}}{1 - ae^{j\omega}}. \end{aligned}$$

By combining these two transforms, the Fourier transform of $x(n)$ is obtained as

$$X(\omega) = X_1(\omega) + X_2(\omega) = \frac{1 - a^2}{1 - 2a \cos \omega + a^2}.$$

Assignment

1) Evaluate the Fourier transform of the following signals.

a) $g(n) = \{-2, -1, 0, 1, 2\}$

b) $h(n) = \begin{cases} 2 - \left(\frac{1}{2}\right)^n, & -4 \leq n \leq 4 \\ 0, & \text{elsewhere} \end{cases}.$