

EXAMPLES ON SAMPLING AND ALIASING PHENOMENA

Example 1:

Consider the following two analog signals

$$x_1(t) = \cos 2\pi (10)t,$$

$$x_2(t) = \cos 2\pi (50)t,$$

which are sampled at a rate of $F_s = 40$ Hz. The corresponding discrete time signals are

$$x_1(n) = \cos 2\pi \left(\frac{10}{40}\right)n = \cos \frac{\pi}{2} n,$$

$$x_2(n) = \cos 2\pi \left(\frac{50}{40}\right)n = \cos \frac{5\pi}{2} n.$$

However,

$$\cos \frac{5\pi}{2} n = \cos \left(2\pi n + \frac{\pi}{2} n \right) = \cos \frac{\pi}{2} n.$$

Hence $x_1(n) = x_2(n)$. Thus the signals are identical and consequently indistinguishable.

If we are given the sampled values generated by $\cos \frac{\pi}{2} n$, there is some ambiguity as to whether these sampled values correspond to $x_1(t)$ or $x_2(t)$. Since $x_2(t)$ yields exactly the same values as $x_1(t)$ when the two are sampled at $F_s = 40$ samples per second, we say that the frequency $F_2 = 50$ Hz is an *alias* of the frequency $F_1 = 10$ Hz at the sampling rate of 40 samples per second.

Example 2:

Consider the analog signal

$$x_a(t) = 3 \cos 100\pi t.$$

- Determine the minimum sampling rate required to avoid aliasing.
- Suppose that the signal is sampled at the rate of $F_s = 200$ Hz, what is the discrete-time signal obtained after sampling?
- Suppose that the signal is sampled at the rate of $F_s = 75$ Hz, what is the discrete-time signal obtained after sampling?
- What is the frequency of a sinusoid that yields samples identical to those obtained in part (c)?

a) The frequency of the analog signal can be calculated as

$$2\pi F_1 = 100\pi,$$

$$\Rightarrow F_1 = 50.$$

Hence the minimum sampling rate required to avoid aliasing is $F_s = 100$ Hz.

b) If the signal is sampled at $F_s = 200$ Hz, then the discrete-time signal is

$$x(n) = 3 \cos\left(\frac{100\pi}{200}\right)n = 3 \cos\left(\frac{\pi}{2}\right)n.$$

c) If the signal is sampled at $F_s = 75$ Hz, then the discrete-time signal is

$$\begin{aligned} x(n) &= 3 \cos\left(\frac{100\pi}{75}\right)n = 3 \cos\left(\frac{4\pi}{3}\right)n \\ &= 3 \cos\left(2\pi - \frac{2\pi}{3}\right)n \\ &= 3 \cos\left(\frac{2\pi}{3}\right)n. \end{aligned}$$

d) For the sampling rate of $F_s = 75$ Hz, we have

$$F_2 = fF_s = 75f$$

The frequency of the sinusoid in part (c) is $f = \frac{1}{3}$. Hence

$$F_2 = 75f = \frac{75}{3} = 25.$$

Clearly, the sinusoidal signal

$$\begin{aligned} y(t) &= 3 \cos 2\pi F_2 t \\ &= 3 \cos 50\pi t \end{aligned}$$

sampled at $F_s = 75$ samples per second yields identical samples. Hence $F_1 = 50$ Hz is an alias of $F_2 = 25$ Hz for the sampling rate $F_s = 75$ Hz.

Example 3:

Consider the analog signal

$$x_a(t) = 3 \cos 50\pi t + 10 \sin 300\pi t - \cos 100\pi t.$$

What is the Nyquist rate for this signal?

The frequencies present in the signal above are

$$F_1 = 25,$$

$$F_2 = 150,$$

$$F_3 = 50.$$

Thus $F_{\max} = 150$ Hz and according to the sampling theorem

$$F_s > 2F_{\max} = 300 \text{ Hz}.$$

The Nyquist rate is $F_N = 2F_{\max}$. Hence

$$F_N = 300 \text{ Hz}.$$

Discussion

It should be observed that the signal component $10 \sin 300\pi t$, sampled at the Nyquist rate $F_N = 300$ Hz, results in samples $10 \sin \pi n$, which are identically zero. In other words, we are sampling the analog signal at its zero-crossing points, and hence we miss the signal component completely. This situation would not occur if the sinusoid is offset in phase by some amount θ . In such a case we have $10 \sin(300\pi t + \theta)$ sampled at the Nyquist rate $F_N = 300$ samples per second, which yields the samples

$$\begin{aligned} 10 \sin(\pi n + \theta) &= 10(\sin \pi n \cos \theta + \cos \pi n \sin \theta) \\ &= 10 \sin \theta \cos \pi n \\ &= (-1)^n 10 \sin \theta. \end{aligned}$$

Thus if $\theta \neq 0$ or π , the samples of the sinusoid taken at Nyquist rate are not all zero. However, we still cannot obtain the correct amplitude from the samples when the phase θ is unknown. A simple remedy that avoids this potentially troublesome situation is to sample the analog signal at a rate higher than the Nyquist rate.

Example 4:

Consider the analog signal

$$x_a(t) = 3 \cos 2000\pi t + 5 \sin 6000\pi t + 10 \cos 12000\pi t.$$

- a) What is the Nyquist rate for this signal?
- b) Suppose that the signal is sampled at the rate of $F_s = 5000$ samples per second, what is the discrete-time signal obtained after sampling?
- c) What is the analog signal we can reconstruct from the samples?

a) The frequencies existing in the above signal are

$$F_1 = 1000 \text{ Hz},$$

$$F_2 = 3000 \text{ Hz},$$

$$F_3 = 6000 \text{ Hz}.$$

Thus $F_{\max} = 6000$ Hz and according to the sampling theorem,

$$F_s > 2F_{\max} = 12000 \text{ Hz}.$$

The Nyquist rate is

$$F_N = 2F_{\max} = 12000 \text{ Hz}.$$

b) The discrete-time signal of the signal sampled at 5000 samples per second is

$$\begin{aligned} x(n) &= 3 \cos\left(\frac{2000\pi}{5000}n\right) + 5 \sin\left(\frac{6000\pi}{5000}n\right) + 10 \cos\left(\frac{12000\pi}{5000}n\right) \\ &= 3 \cos 2\pi \left(\frac{1}{5}\right)n + 5 \sin 2\pi \left(\frac{3}{5}\right)n + 10 \cos 2\pi \left(\frac{6}{5}\right)n \\ &= 3 \cos 2\pi \left(\frac{1}{5}\right)n + 5 \sin 2\pi \left(1 - \frac{2}{5}\right)n + 10 \cos 2\pi \left(1 + \frac{1}{5}\right)n \\ &= 3 \cos 2\pi \left(\frac{1}{5}\right)n + 5 \sin 2\pi \left(-\frac{2}{5}\right)n + 10 \cos 2\pi \left(\frac{1}{5}\right)n \end{aligned}$$

Finally, we obtain

$$x(n) = 13 \cos 2\pi \left(\frac{1}{5}\right)n - 5 \sin 2\pi \left(\frac{2}{5}\right)n.$$

Since $F_s = 5000$ Hz, the folding frequency is $F_s/2 = 2500$ Hz. This is the maximum frequency that can be represented uniquely by the sampled signal.

c) Since only the frequency components at 1000 Hz and 2000 Hz are present in the sampled signal, the analog signal that can be recovered is

$$y(t) = 13 \cos 2\pi (1000)t - 5 \sin 2\pi (2000)t$$

which is obviously different from the original signal $x(t)$. This distortion of the original analog signal was caused by the aliasing effect, due to the low sampling rate used.

Assignment

1) Consider the analog signal

$$x(t) = 3 \cos 600\pi t + 2 \cos 1800\pi t .$$

- Determine the minimum sampling rate required to avoid aliasing.
- What is the Nyquist rate for the signal $x(t)$?
- Suppose that the signal is sampled at $F_s = 2000$ Hz, what is the discrete-time signal obtained after the sampling? What are the frequencies in the resulting discrete-time signal?
- If the sampling rate is $F_s = 600$ Hz, then what is the maximum frequency that can be recovered from the discrete-time signal?

2) An analog signal contains frequencies up to 10 kHz.

- What range of sampling frequencies allows exact reconstruction of the signal from its samples?
- What is the Nyquist rate for the signal?
- If the sampling rate F_s is chosen as 10 kHz, then what is the maximum frequency that can be represented uniquely by the sampled signal?