

HANDOUT ON FOURIER ANALYSIS OF DYNAMIC SIGNALS

Signals and Signal processing

A *signal* is defined as any physical quantity that varies with time, space or any other independent variable or variables. Mathematically, we describe a signal as a function of one or more independent variables. For example, the functions

$$s_1(t) = 5t$$

$$s_2(t) = 20t^2$$

describe two signals, one that varies linearly with the independent variable 't' (time) and a second that varies quadratically with 't'. As another example consider the function

$$s(x, y) = 3x + 2xy + 10y^2.$$

This function describes a signal of two independent variables 'x' and 'y' that could represent the two spatial coordinates in a plane. Some of the examples of signals are speech signals, speed of the car, temperature of a room, etc.

A *system* may be defined as a physical device that performs an operation on a signal. For example, a filter used to reduce the noise and interference corrupting a desired information-bearing signal is called a system.

When we pass a signal through a system, as in filtering, we say that we have processed the signal. In this case the processing of the signal involves removing the noise and interference from the desired signal. In general, a system is characterized by the type of operation that it performs on the signal. For example, if the operation is linear, the system is called linear. If the operation on the signal is nonlinear, the system is said to be nonlinear, and so forth. Such operations are usually referred to as *signal processing*.

Basic elements of a digital signal processing system

The block diagram of a digital signal processing system is as shown below.

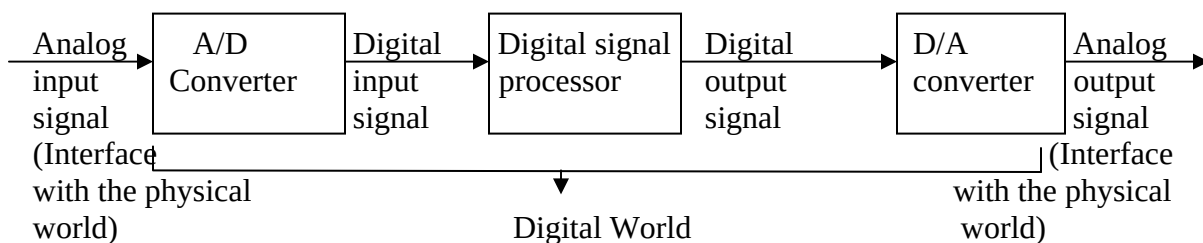


Fig1. Block diagram of a digital signal processing system

Classification of signals

Signals can be classified into three different categories depending on the characteristics of the time (independent) variable and the values they take.

- *Continuous-time signals or analog signals:* Analog signals are defined for every value of time they take on in a continuous interval (a, b). For example, the signal $x(t) = \sin(t)$, $-\infty < t < \infty$ is an analog signal. In other words, at any given instant an analog signal can take any value.
- *Discrete-time signals:* These signals are defined only at specific values of time. These time instants need not be equidistant, but in practice they are usually taken at equally spaced intervals. For example, $x(t_n) = \sin(nt)$, $n = 0, \pm 1, \pm 2, \dots$ is a discrete-time signal. In other words, the time variable of the signal can take only certain values. The amplitude of the signal can be continuous i.e., can take any value. The process of converting an analog signal to discrete-time signal is called *sampling*. A discrete-time signal is sometimes called a *sampled signal*.
- *Discrete-valued signals or digital signal:* Digital signals arise when the discrete signals are *quantized*. A quantized signal assumes only discrete amplitude values, for example only 0's and 1's. In other words, in these signals both the amplitude and time variable can take only certain values.

The difference between the signals is explained with the help of the following figures.

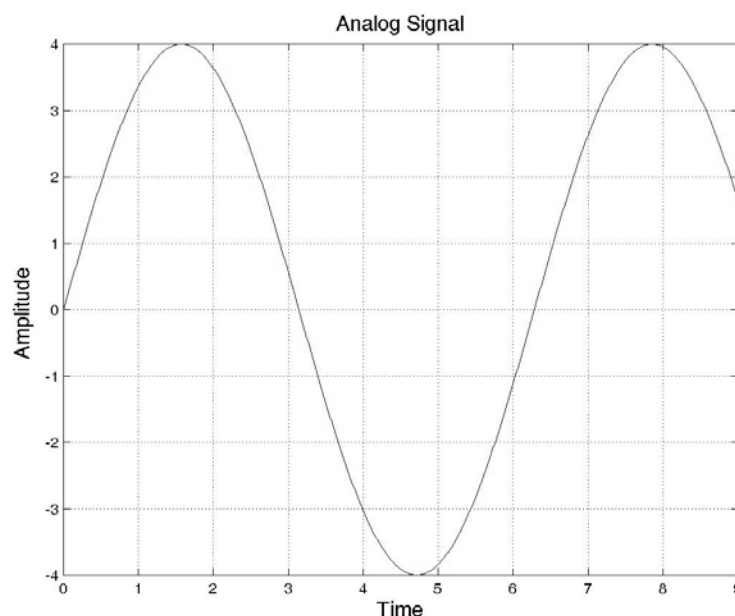


Fig 2. Analog signal

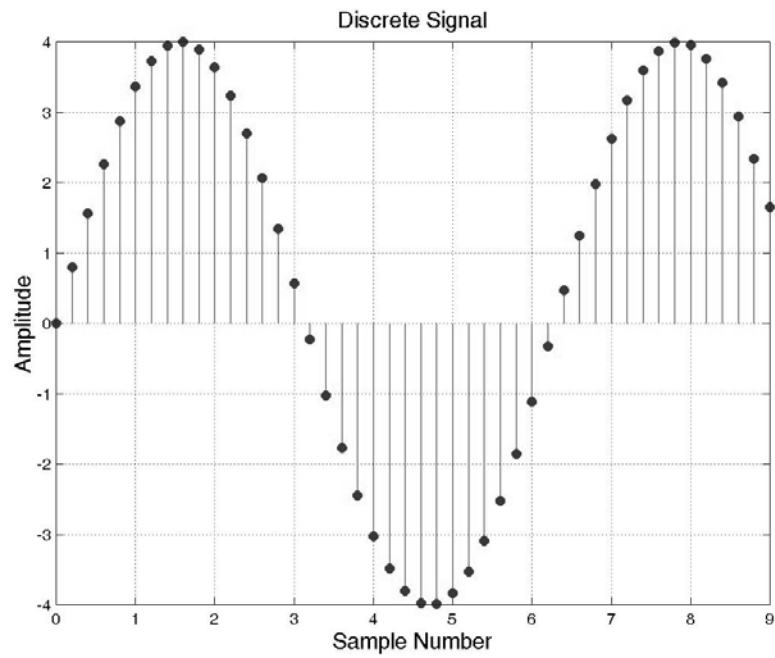


Fig3. Discrete signal

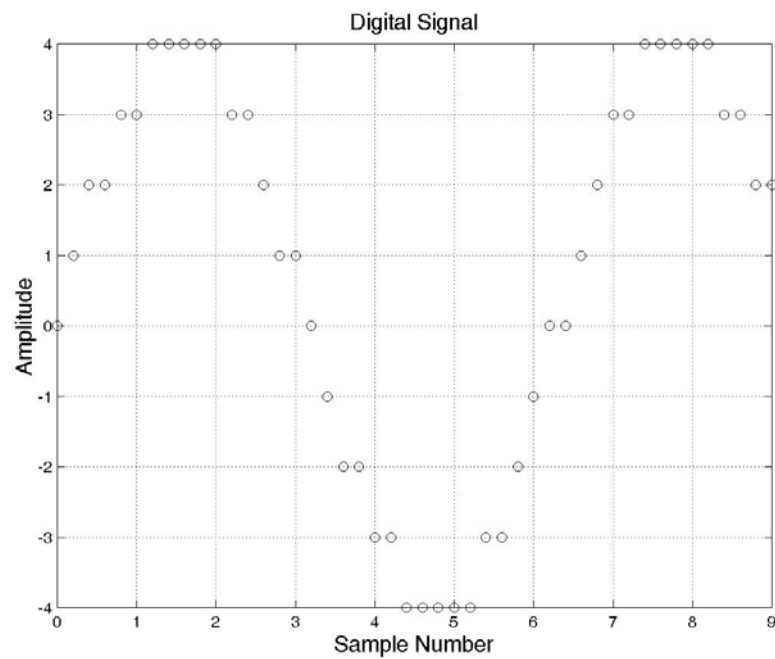


Fig4. Digital signal

Frequency analysis of signals

The Fourier transform is one of the several mathematical tools that is useful in the analysis and design of (linear time invariant) LTI systems. Another is the Fourier series. These representations basically involve the decomposition of the signals in terms of sinusoidal (or complex exponential) components. With such a decomposition, a signal is said to be represented in the frequency domain. For the class of periodic signals, such a decomposition is called the *Fourier series*. For the class of finite energy signals or those that are not necessarily periodic, the decomposition is called the *Fourier transform*.

The Fourier series for continuous-time periodic signals

Sinusoidal signals and complex exponentials play a major role in the analysis of signals and systems. In some cases we deal with sets of *harmonically related* complex exponentials (or sinusoids). These are sets of periodic complex exponentials with fundamental frequencies that are multiples of a single positive frequency.

The basic signal for continuous-time, harmonically related exponentials is

$$s_k(t) = e^{j2\pi k F_0 t} = e^{j2\pi f t} = e^{j\omega t}, \quad k = 0, \pm 1, \pm 2, \dots, \text{ where 'f' is the fundamental frequency.}$$

We note that for each value of 'k', $s_k(t)$ is periodic with fundamental frequency 'kF₀' or fundamental period $1/(kF_0) = T_p/k$. Since a signal that is periodic with period 'T_p/k' is also periodic with period $k(T_p/k) = T_p$ for any positive integer 'k', we see that all of the $s_k(t)$ have a common period of 'T_p'.

From the above basic signal, a linear combination of harmonically related complex exponentials of the form

$$x_a(t) = \sum_{k=-\infty}^{\infty} c_k s_k(t) = \sum_{k=-\infty}^{\infty} c_k e^{j2\pi k F_0 t} \quad (1)$$

can be formed, where c_k , $k = 0, \pm 1, \pm 2, \dots$ are arbitrary complex constants. This signal is periodic with fundamental period 'T_p = 1/F₀'.

Suppose that we are given a periodic signal $x(t)$ with period T_p. We can represent the periodic signal by the series given by equation(1), called the *Fourier series*, where the fundamental frequency 'F₀' is selected to be the reciprocal of the given period 'T_p'. To determine the expression for the coefficients {c_k}, we first multiply both sides of equation (1) by the complex exponential

$$e^{-j2\pi h F_0 t},$$

where 'h' is an integer and then integrate both sides of the resulting equation over a single period, say from 0 to T_p, or more generally from t₀ to t₀ + T_p, where t₀ is an arbitrary starting value. Thus we obtain

$$\int_{t_0}^{t_0+T_p} x(t) e^{-j2\pi h F_0 t} dt = \int_{t_0}^{t_0+T_p} e^{-j2\pi h F_0 t} \left(\sum_{k=-\infty}^{\infty} c_k e^{j2\pi k F_0 t} \right) dt. \quad (2)$$

To evaluate the integral on the right-hand side of equation (2), we interchange the order of summation and integration and combine the two exponentials. Hence

$$\sum_{k=-\infty}^{\infty} c_k \int_{t_0}^{t_0+T_p} e^{j2\pi F_0 (k-h)t} dt = \sum_{k=-\infty}^{\infty} c_k \left[\frac{e^{j2\pi F_0 (k-h)t}}{j2\pi F_0 (k-h)} \right]_{t_0}^{t_0+T_p}. \quad (3)$$

For $k \neq h$, the right-hand side of equation (3) evaluated at the lower and the upper limits yields zero. On the other hand, if $k = h$, we have

$$\int_{t_0}^{t_0+T_p} dt = t \Big|_{t_0}^{t_0+T_p} = T_p.$$

Therefore equation (2) reduces to

$$\int_{t_0}^{t_0+T_p} x(t) e^{-j2\pi h F_0 t} dt = c_h T_p.$$

Therefore the expression for the Fourier coefficients in terms of the given periodic signal becomes

$$c_h = \frac{1}{T_p} \int_{t_0}^{t_0+T_p} x(t) e^{-j2\pi h F_0 t} dt. \quad (4)$$

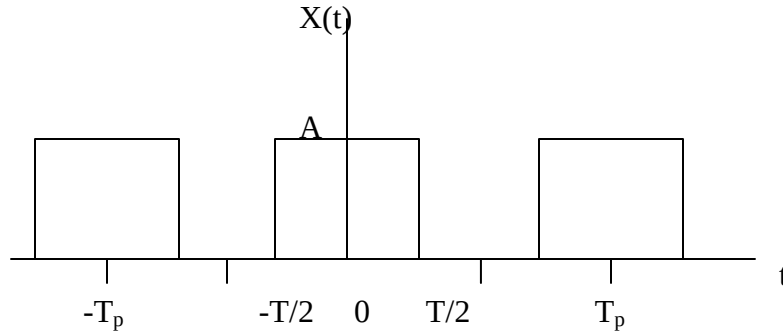
Since t_0 is arbitrary, this integral can be evaluated over any interval of length T_p , that is over any interval equal to the period of the signal $x(t)$. Consequently, the integral for the Fourier series coefficients can be written as

$$c_h = \frac{1}{T_p} \int_{T_p} x(t) e^{-j2\pi F_0 h t} dt. \quad (5)$$

In general, the Fourier coefficients ' c_k ' are complex valued.

Example 1

Determine the Fourier series of the rectangular pulse train signal as shown below.



Since $x(t)$ is an even signal, it is convenient to select the integration interval from $-T/2$ to $T/2$. Equation (5) evaluated for $h = 0$ yields

$$c_0 = \frac{1}{T_p} \int_{-T/2}^{T/2} x(t) dt = \int_{-T/2}^{T/2} A dt = \frac{AT}{T_p}$$

The term c_0 represents the average value (dc component) of the signal $x(t)$. For $h \neq 0$ we have,

$$\begin{aligned} c_h &= \frac{1}{T_p} \int_{-T/2}^{T/2} A e^{-j2\pi F_0 h t} dt = \frac{A}{T_p} \left[\frac{e^{-j2\pi F_0 h t}}{-j2\pi F_0 h} \right]_{-T/2}^{T/2} \\ &= \frac{A}{\pi F_0 h T_p} \frac{e^{-j\pi F_0 h T} - e^{j\pi F_0 h T}}{(-2j)} \\ &= \frac{AT}{T_p} \frac{\sin(\pi F_0 h T)}{(\pi F_0 h T)}, \quad h = \pm 1, \pm 2, \dots \end{aligned}$$

Therefore the above defined rectangular pulse train can be written as

$$\begin{aligned}
x(t) &= \sum_{k=-\infty}^{\infty} c_k e^{j2\pi F_0 k t} \\
&= \dots + c_0 + c_1 e^{j2\pi F_0 t} + c_2 e^{j2\pi (2F_0) t} + \dots \\
&= \dots + \frac{AT}{T_p} + \frac{AT}{T_p} \frac{\sin(\pi F_0 T)}{(\pi F_0 T)} e^{j2\pi F_0 t} + \frac{AT}{T_p} \frac{\sin(2\pi F_0 T)}{(2\pi F_0 T)} e^{j2\pi (2F_0) t} + \dots \\
&= \dots + \frac{AT}{T_p} \left(1 + \frac{\sin(\pi F_0 T)}{(\pi F_0 T)} e^{j2\pi F_0 t} + \frac{\sin(2\pi F_0 T)}{(2\pi F_0 T)} e^{j2\pi (2F_0) t} + \dots \right)
\end{aligned}$$

From the above result it can be concluded that, the second term corresponds to a frequency component of ' F_0 ', the third term corresponds to a frequency component of ' $2F_0$ ' and so on. So it can be seen that, the rectangular pulse consists of infinite frequency components.

The Fourier transform for continuous-time periodic signals

If the period of a periodic signal becomes infinite, the signal becomes aperiodic. The Fourier series expansion of the periodic signal $x_p(t)$ is given by equation (1) and the Fourier coefficients are given by equation (5). But for an aperiodic signal, since the period approaches infinity, equation (4) can be modified as

$$c_h = \frac{1}{T_p} \int_{-\infty}^{\infty} x(t) e^{-j2\pi h F_0 t} dt. \quad (6)$$

Let us now define a function $X(F)$, called the *Fourier transform* of $x(t)$, as

$$X(F) \equiv \int_{-\infty}^{\infty} x(t) e^{-j2\pi F t} dt. \quad (7)$$

$X(F)$ is a function of the continuous variable ' F '. It does not depend on ' T_p ' or ' F_0 '. However, by comparing equations (6) and (7), we get

$$c_h = \frac{1}{T_p} X(kF_0)$$

Therefore the Fourier coefficients are the samples of $X(F)$ taken at multiples of F_0 and scaled by $1/T_p$.

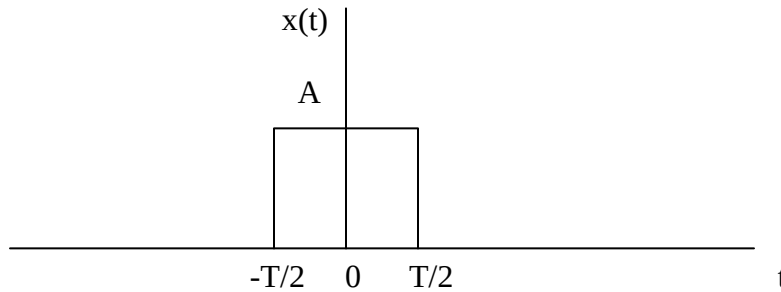
If the Fourier transform $X(F)$ is known, then the aperiodic signal $x(t)$ can be obtained as

$$x(t) = \int_{-\infty}^{\infty} X(F) e^{j2\pi F t} dF.$$

The above integral relationship is called the *inverse Fourier transform*.

Example 2

Determine the Fourier transform of a rectangular pulse signal as shown.



$$x(t) = \begin{cases} A, & |t| \leq T/2 \\ 0, & |t| > T/2 \end{cases}$$

The Fourier transform of the above aperiodic signal is given by

$$X(F) = \int_{-T/2}^{T/2} A e^{-j2\pi F t} dt = AT \frac{\sin(\pi FT)}{(\pi FT)}.$$

Note that in the above example, the signal considered is aperiodic. But in this handout we deal primarily with periodic signals. The Fourier transform is just an extension of Fourier series and hence can also be used for periodic signals.

The Fourier series for discrete-time periodic signals

If a continuous time signal is given by

$x(t) = a \cos(2\pi F t)$, is sampled at a rate of F_s , then the discrete-time signal is given by

$$x(n) = a \cos(2\pi \frac{F}{F_s} n) = a \cos(2\pi f n).$$

where ' f ' is the frequency of the discrete-time signal.

Therefore, a discrete-time periodic sequence can be represented as

$$x(n) = \sum_{k=0}^{N-1} c_k e^{j2\pi kn/N} \quad (8)$$

where N is the period of the sequence and $\{c_k\}$ are the coefficients in the series representation. The above equation is called the *discrete-time Fourier series* (DTFS).

The expression for the Fourier coefficients 'c_k' can be obtained by multiplying both sides of equation (8) by the exponential $e^{-j2\pi hn/N}$ and summing the product from $n = 0$ to $n = N-1$. Thus

$$\sum_{n=0}^{N-1} x(n)e^{-j2\pi hn/N} = \sum_{n=0}^{N-1} \sum_{k=0}^{N-1} c_k e^{j2\pi (k-h)n/N} \quad (9)$$

If the summation over n is performed first, in the right-hand side then

$$\sum_{n=0}^{N-1} e^{j2\pi (k-h)n/N} = \begin{cases} N, & k-h = 0, \pm N, \pm 2N, \dots \\ 0, & \text{otherwise} \end{cases}$$

Therefore equation (9) reduces to

$$\begin{aligned} \sum_{n=0}^{N-1} x(n)e^{-j2\pi hn/N} &= Nc_h \\ \Rightarrow c_h &= \frac{1}{N} \sum_{n=0}^{N-1} x(n)e^{-j2\pi hn/N} \end{aligned} \quad (10)$$

Thus we have the desired expression for the Fourier coefficients in terms of the signal $x(n)$.

Example 3

Determine the Fourier series of the following discrete-time signal.

$x(n) = \{1, 1, 0, 0\}$. $x(n)$ is a periodic signal with period $N = 4$.

From the second equation of equation (10), we get

$$\begin{aligned} c_h &= \frac{1}{4} \sum_{n=0}^3 x(n)e^{-j2\pi hn/4}, \quad h = 0, 1, 2, 3 \\ \Rightarrow c_h &= \frac{1}{4} (1 + e^{-j\pi h/2}), \quad h = 0, 1, 2, 3 \end{aligned}$$

For $h = 0, 1, 2, 3$ we obtain

$$\begin{aligned}
c_0 &= \frac{1}{2} \\
c_1 &= \frac{1}{4}(1-j) \\
c_2 &= 0 \\
c_3 &= \frac{1}{4}(1+j)
\end{aligned}$$

Therefore the above discrete-time signal can be represented in the form of equation (8) as

$$\begin{aligned}
x(n) &= \sum_{h=0}^3 c_h e^{-j2\pi hn/4} \\
\Rightarrow x(n) &= c_0 + c_1 e^{-j\pi n/2} + c_2 e^{-jn\pi} + c_3 e^{-j3\pi n/2} \\
\Rightarrow x(n) &= \frac{1}{2} + \frac{1}{4}(1-j)e^{-j\pi n/2} + \frac{1}{4}(1+j)e^{-j3\pi n/2}
\end{aligned}$$

Fourier transform of discrete-time periodic signals

The Fourier transform of a finite energy discrete-time signal $x(n)$ is defined as

$$X(\omega) = \sum_{n=-\infty}^{\infty} x(n) e^{-j\omega n} \quad (11)$$

Physically, $X(\omega)$ represents the frequency content of the signal $x(n)$. In other words, $X(\omega)$ is a decomposition of $x(n)$ into its frequency components.

There are two basic differences between the Fourier transform of discrete-time signal and the Fourier transform of a continuous time signal. First, for continuous-time signals, the Fourier transform has a frequency range of $(-\infty, \infty)$. In contrast, the frequency range for a discrete-time signal is unique over the frequency interval of $(-\pi, \pi)$ or equivalently, $(0, 2\pi)$.

The second basic difference is that, since the signal is discrete in time, the Fourier transform of the signal involves a summation of terms instead of an integral, as in the case of continuous-time signal.

Since $X(\omega)$ is a periodic function of the frequency variable, ω , it has a Fourier series expansion. From the definition of the Fourier transform $X(\omega)$ of the sequence $x(n)$, we observe that $X(\omega)$ has the form of the Fourier series. The Fourier coefficients in this series expansion are the values of the sequence $x(n)$. Therefore

$$x(n) = \frac{1}{2\pi} \int_{-\pi}^{\pi} X(\omega) e^{j\omega n} d\omega \quad (12)$$

Example 4

Determine the Fourier transform of the following signal

$$x(n) = \begin{cases} A, & 0 \leq n \leq L-1 \\ 0, & \text{otherwise} \end{cases}$$

The Fourier transform of the above signal is given by

$$\begin{aligned} X(\omega) &= \sum_{n=0}^{L-1} A e^{-j\omega n} \\ &= A \frac{1 - e^{-j\omega L}}{1 - e^{-j\omega}} = A e^{-j(\omega/2)(L-1)} \frac{\sin(\omega L/2)}{\sin(\omega/2)} \end{aligned}$$

Fast Fourier transform

As we have observed in the preceding sections that the Fourier series expansion and the Fourier transforms play an important role in many applications of digital signal processing. Fast Fourier transforms (FFT) are more computationally efficient algorithms for evaluating the discrete Fourier transforms. The two most commonly used algorithms are the *Radix-2 FFT* algorithm and the *Radix-4 FFT* algorithm. These algorithms are computationally very intensive and hence MATLAB is used to obtain the FFT of the various signals.

Using MATLAB to obtain the FFT

The `'fft'` command in MATLAB can be used to obtain the discrete Fourier transforms. This command uses the Radix-2 FFT algorithm. The online help of the `'fft'` command displays the following message.

FFT Discrete Fourier transform.

FFT(X) is the discrete Fourier transform (DFT) of vector X. If the length of X is a power of two, a fast radix-2 fast-Fourier transform algorithm is used. If the length of X is not a power of two, a slower non-power-of-two algorithm is employed. For matrices, the FFT operation is applied to each column. For N-D arrays, the FFT operation operates on the first non-singleton dimension.

FFT(X,N) is the N-point FFT, padded with zeros if X has less than N points and truncated if it has more.

FFT(X,[],DIM) or FFT(X,N,DIM) applies the FFT operation across the dimension DIM.

For length N input vector x, the DFT is a length N vector X,

with elements

$$X(k) = \sum_{n=1}^N x(n) \exp(-j2\pi(k-1)(n-1)/N), \quad 1 \leq k \leq N.$$

The inverse DFT (computed by IFFT) is given by

$$x(n) = (1/N) \sum_{k=1}^N X(k) \exp(j2\pi(k-1)(n-1)/N), \quad 1 \leq n \leq N.$$

The relationship between the DFT and the Fourier coefficients a and b in

$$x(n) = \sum_{k=1}^{N/2} a(k) \cos(2\pi k t(n)/(N dt)) + b(k) \sin(2\pi k t(n)/(N dt))$$

is

$a_0 = X(1)/N$, $a(k) = 2 \cdot \text{real}(X(k+1))/N$, $b(k) = -2 \cdot \text{imag}(X(k+1))/N$,
where x is a length N discrete signal sampled at times t with spacing dt .

The use of the above command is explained with the help of the following example.

Example 5

Evaluate the FFT of the following sinusoid.

$$x(n) = \cos(\omega_n nT)$$

where

ω_n = angular frequency given by $2\pi f$, ' f ' = 0.25 cycles per second.

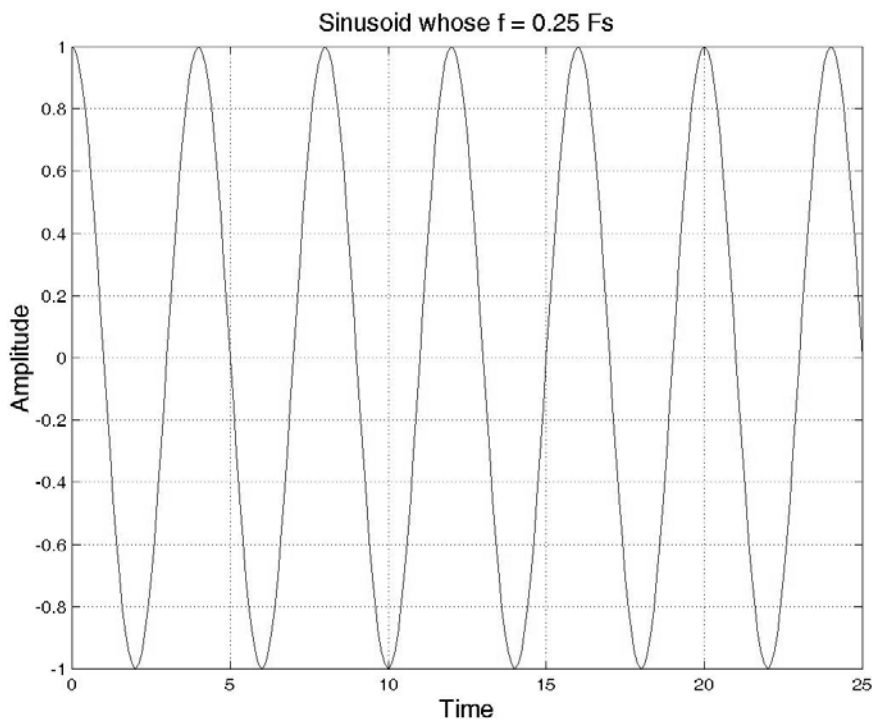
$T = 1/F_s$, where F_s is the sampling rate and is chosen as 1 cycles per sample.

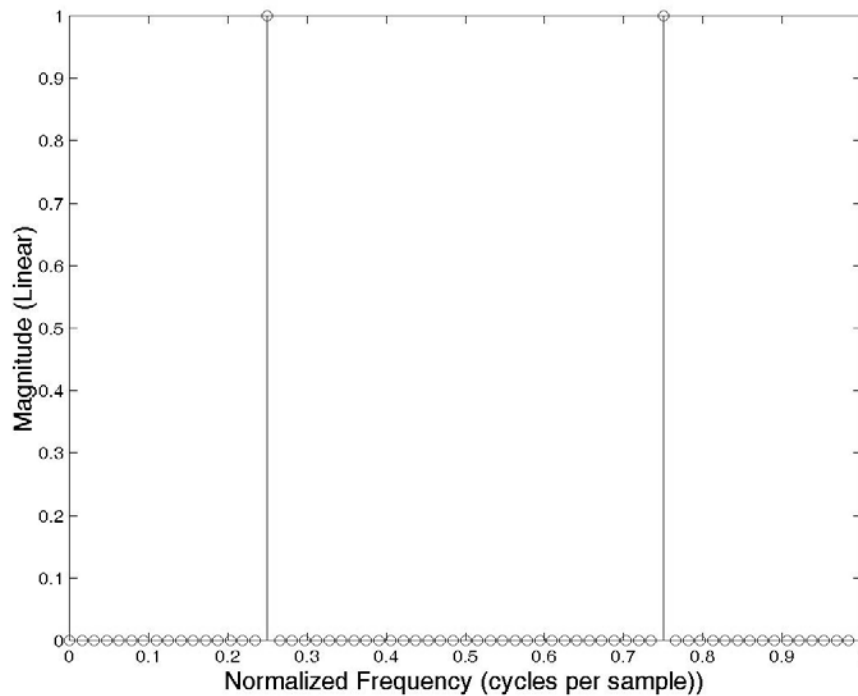
Since we are using the command '`fft`', the signal length must be a power of 2 and so N is chosen to be 64. The MATLAB code is given below.

```
% Parameters:
N = 64;                % Must be a power of two
T = 1;                 % Set sampling rate to 1
f = 0.25;              % Sinusoidal frequency in cycles per sample
A = 1;                 % Sinusoidal amplitude
phi = 0;               % Sinusoidal phase
n = [0:N-1];
x = cos(2*pi*n*f*T);   % Signal to analyze
X = fft(x);            % Spectrum

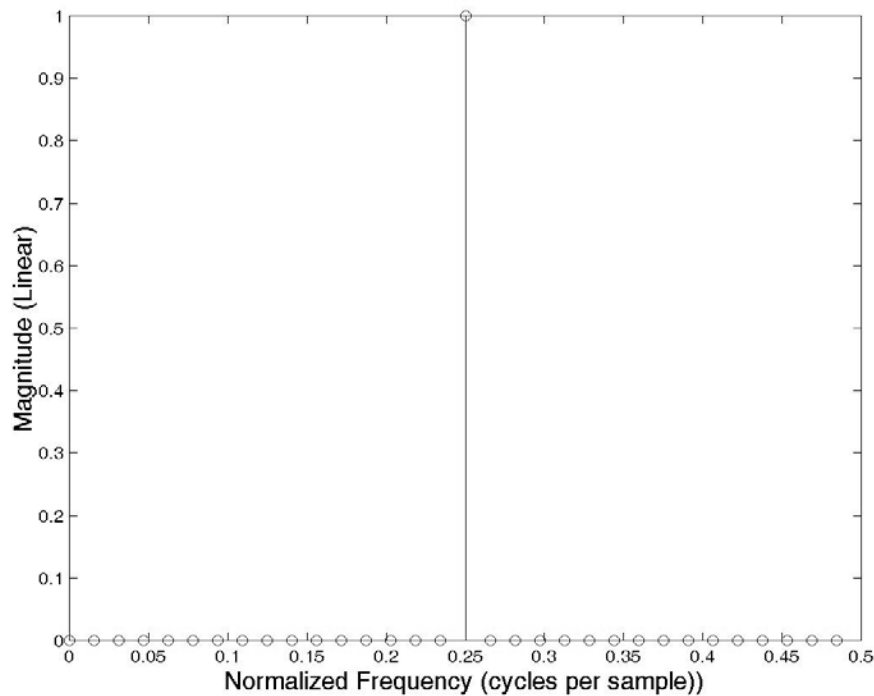
% plotting the sinusoid
ni = [0:.1:N-1];
xi = cos(2*pi*ni*f*T);
figure(1)
plot(ni,xi)
grid;
xlabel('Time');
ylabel('Amplitude');
```

```
title('Sinusoid whose f = 0.25 Fs');  
  
% plotting the FFT  
  
magX = abs(X);  
mag = (magX*2)/N;  
fn = [0:1.0/N:1-1.0/N]; % Normalized frequency axis  
  
figure(2)  
stem(fn,mag)  
xlabel('Normalized Frequency (cycles per sample)');  
ylabel('Magnitude (Linear)');  
  
% Plotting FFT up to half the sampling rate  
  
mag1 = mag(1:(size(mag,2)/2));  
fn1 = fn(1:(size(fn,2)/2));  
  
figure(3)  
stem(fn1,mag1)  
xlabel('Normalized Frequency (cycles per sample)');  
ylabel('Magnitude (Linear)');  
The plots are given below
```





From the above figure it can be seen that there are two peaks in the magnitude spectrum, whose magnitude is equal to the magnitude of the sinusoid and located at normalized frequencies $f = 0.25$ and $f = 0.75$. Since there is only one frequency present in the signal that is equal to 0.25 Hz, we should expect to see only one peak after the FFT analysis. The reason why we see two peaks is that, the FFT is symmetric about the folding frequency, which is actually half the sampling frequency. In this case since the sampling frequency is chosen to be 1 sample per second, the FFT is symmetric about 0.5 Hz. So it is enough if we plot the FFT up to 0.5 Hz, to get the entire information about the signal. The plot is shown below.



In other words, by obtaining the FFT of a signal, one is able to get information about the frequency component of the signal and as well as the magnitude of the signal.

The `fft` command returns complex numbers. So, we can find both the magnitude and the phase associated at each frequency component. Note that, in the MATLAB code given above, the magnitude is obtained by using the `abs` command. The phase can also be calculated, but is not of importance here.

Also note that the FFT of a signal will give spectral peaks only at the frequency of the signal. For example, in the above problem the first peak is obtained at the frequency of 0.25 cycles per second, which is nothing but the frequency of the sinusoid.

Assignment

1) Compute the Fourier transform of the following signals.

$$a) \quad x(n) = \begin{cases} 2 - \frac{1}{2}n, & |n| \leq 4 \\ 0, & \text{elsewhere} \end{cases}$$

2) Compute the Fourier series of the following signal

$$a) \quad x(n) = \{-2, -1, 0, 1, 2\}$$

3) Evaluate the FFT of the following signal using MATLAB and discuss the results obtained. (Hint: verify that the peak obtained is at the frequency of the signal.)

$$x(n) = \sin(\omega_n nT)$$

where the frequency ' f ' of the sinusoid is 0.5 cycles per second, the sampling rate $F_s = 1$ sample per cycle. Also plot both the sinusoid and also the magnitude spectrum of the FFT.