

Texas A & M University
Department of Mechanical Engineering
MEEN 651 Control System Design

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Fall 2003

Lecture 1B: Fourier Analysis and Fourier Transforms

The objective of this lecture is to present to you some of the mathematics involved in the Fourier analysis of periodic functions and present the Fourier transform. The latter is widely used in various signal acquisition and processing functions, as you will witness in the lab part of the course.

Basic Continuous Time Signals

Complex Exponential and Sinusoidal Signals

The complex exponential signal is given by

$$x(t) = C e^{at}, \quad (1)$$

where C and a are, in general, complex numbers. If these are real numbers, then $x(t)$ is called a real exponential and depending on the sign of a it can become either a growing exponential ($a > 0$) or a decaying exponential ($a < 0$). If $a = 0$ then $x(t)$ is the constant C .

An equally important class of complex exponential signals is obtained when a is purely imaginary. In this case, we obtain

$$x(t) = e^{j\omega_0 t}. \quad (2)$$

An important property of this signal is that it is periodic.

A class of signals closely related to the complex exponential is the sinusoidal signal

$$x(t) = A \cos(\omega_0 t + \phi), \quad (3)$$

where it is common to write $\omega_0 = 2\pi f_0$, where f_0 has the units of cycles per second or Hertz (Hz). The fundamental period of the sinusoid is given by

$$T_0 = \frac{2\pi}{|\omega_0|}. \quad (4)$$

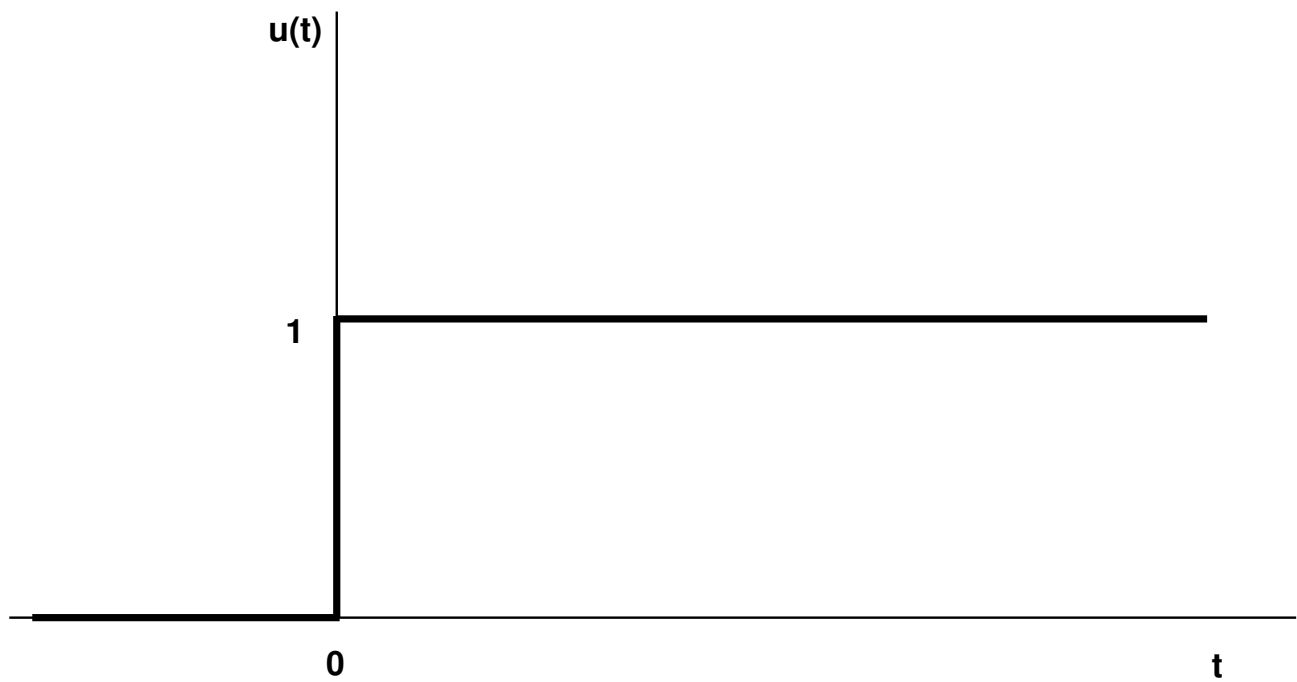


Figure 1: Continuous time Unit Step Function.

The sinusoid of equation (3) can be expressed in terms of the complex exponential signal as

$$A \cos(\omega_0 t + \phi) = A \Re\{e^{j(\omega_0 t + \phi)}\}, \quad (5)$$

where the right-hand-side denotes the real part of a complex number.

Unit Step and Unit Impulse Functions

The unit step function is defined as

$$u(t) = \begin{cases} 0; & t < 0 \\ 1; & t > 0 \end{cases}, \quad (6)$$

shown in Figure 1. Note that it is discontinuous at $t = 0$.

The unit impulse function $\delta(t)$ is related to the unit step function by the equation

$$u(t) = \int_{-\infty}^t \delta(\tau) d\tau. \quad (7)$$

Therefore, this suggests that

$$\delta(t) = \frac{du(t)}{dt}. \quad (8)$$

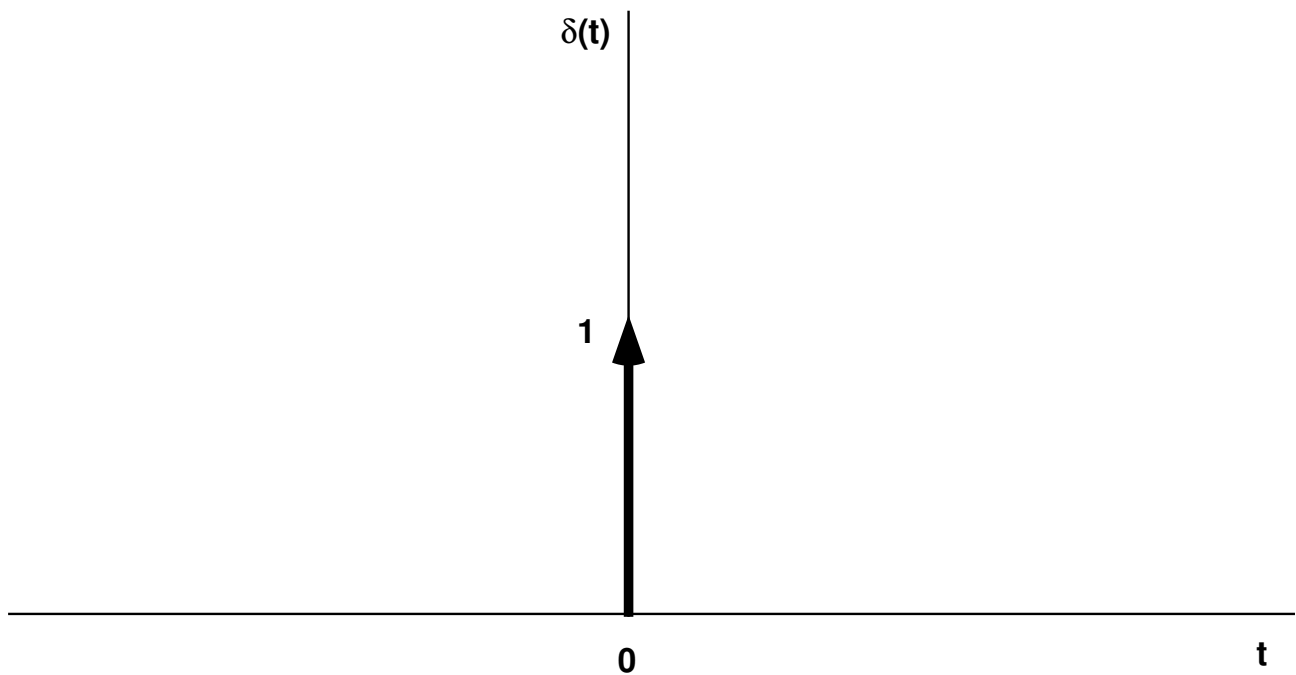


Figure 2: Unit Impulse Function.

A unit impulse is shown in Figure 2. Although the “value” at $t = 0$ is infinite, the height of the arrow used to depict the scaled impulse will be chosen to be representative of its area.

Continuous Time Fourier Series and the Continuous Fourier Transform

Recall that a signal is periodic if for some positive, nonzero value T ,

$$x(t) = x(t + T), \quad (9)$$

for all t . The fundamental period T_0 is the minimum value of the of T for which equation (9) is satisfied. Further $\frac{2\pi}{T_0}$ is called the fundamental frequency.

We also introduced the periodic complex exponential

$$x(t) = e^{j\omega_0 t}. \quad (10)$$

The signal (10) has fundamental frequency ω_0 and fundamental period $T_0 = \frac{2\pi}{\omega_0}$. Now we can speak of the collection of *harmonically related* complex exponentials

$$\phi_k(t) = e^{jk\omega_0 t}, \quad k = 0, \pm 1, \pm 2, \dots, \quad (11)$$

Each of these signals has a fundamental frequency that is a multiple of ω_0 . Therefore, a linear combination of harmonically related complex exponentials of the form

$$x(t) = \sum_{k=-\infty}^{+\infty} a_k e^{jk\omega_0 t}, \quad (12)$$

is also periodic with period T_0 . In this infinite sum the term for $k = 0$ is a dc or a constant term. The two terms for $k = +1$ and $k = -1$ both have fundamental period equal to T_0 and are collectively referred to as the *fundamental components* or the *first harmonic components*. More generally, the components for $k = +N$ and $k = -N$ are referred to as N^{th} harmonic components. The representation of equation (12) is called the *Fourier Series* representation.

Having stated the Fourier Series expansion, it is now important to derive an expression for the expansion coefficients a_k . Multiplying both sides of equation (12) by $e^{-jn\omega_0 t}$ we obtain

$$x(t)e^{-jn\omega_0 t} = \sum_{k=-\infty}^{+\infty} a_k e^{jk\omega_0 t} e^{-jn\omega_0 t}. \quad (13)$$

Integrating both sides from 0 to $T_0 = \frac{2\pi}{\omega_0}$, one fundamental period, we have

$$\int_0^{T_0} x(t)e^{-jn\omega_0 t} dt = \int_0^{T_0} \sum_{k=-\infty}^{+\infty} a_k e^{jk\omega_0 t} e^{-jn\omega_0 t} dt. \quad (14)$$

Interchanging the order of integration and summation yields

$$\int_0^{T_0} x(t)e^{-jn\omega_0 t} dt = \sum_{k=-\infty}^{+\infty} a_k \left[\int_0^{T_0} e^{j(k-n)\omega_0 t} dt \right]. \quad (15)$$

The bracketed integral can be expressed as¹

$$\int_0^{T_0} e^{j(k-n)\omega_0 t} dt = \begin{cases} T_0, & k = n \\ 0, & k \neq n \end{cases}. \quad (16)$$

Therefore, the right-hand-side of equation (15) reduces to $T_0 a_k$.

If we denote integration over any interval of length T_0 by $\int_{T_0}$, then we can summarize the Fourier series representation of a periodic signal as

$$x(t) = \sum_{k=-\infty}^{+\infty} a_k e^{jk\omega_0 t}, \quad (17)$$

and

$$a_k = \frac{1}{T_0} \int_{T_0} x(t) e^{-jk\omega_0 t} dt. \quad (18)$$

¹Prove this to yourselves as an exercise.

Equation (17) is referred to as the synthesis equation and equation (18) is referred to as the analysis equation. The coefficients $\{a_k\}$ are often called the Fourier series coefficients or the spectral coefficients of $x(t)$. These complex coefficients indicate the portion of the signal $x(t)$ that is at each harmonic of the fundamental component. For $k = 0$ the coefficient a_0 is given by

$$a_0 = \frac{1}{T_0} \int_{T_0} x(t) dt, \quad (19)$$

which represents the average value of the signal $x(t)$ over one period.

The continuous time Fourier transform can be obtained by considering the behavior of the Fourier series as the fundamental period $T_0 \rightarrow \infty$ or as the fundamental frequency $\omega_0 \rightarrow 0$. If we define the envelope $X(\omega)$ as

$$X(\omega) = T_0 a_k, \quad (20)$$

then the Fourier series equations, (17) and (18), can be expressed as

$$x(t) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} X(\omega) e^{j\omega t} d\omega, \quad (21)$$

and

$$X(\omega) = \int_{-\infty}^{+\infty} x(t) e^{-j\omega t} dt, \quad (22)$$

where for the limiting case of $T_0 \rightarrow \infty$, the summations are replaced with integrals. Equations (21) and (22) are referred to as the Fourier transform pair with equation (22) called the Fourier transform or the Fourier integral and equation (21) called the inverse Fourier transform. The transform $X(\omega)$ of any signal $x(t)$ is also commonly referred to as the spectrum of $x(t)$, because it provides us with information regarding how $x(t)$ is composed of sinusoidal signals at different frequencies.

Basic Discrete Time Signals

Unit Step and Unit Impulse Sequences

The unit step function in discrete time is defined as

$$u[n] = \begin{cases} 0; & n < 0 \\ 1; & n \geq 0 \end{cases}, \quad (23)$$

shown in Figure 3.

The unit impulse (or unit sample) sequence $\delta[n]$ is defined as

$$\delta[n] = \begin{cases} 0; & n \neq 0 \\ 1; & n = 0 \end{cases}. \quad (24)$$

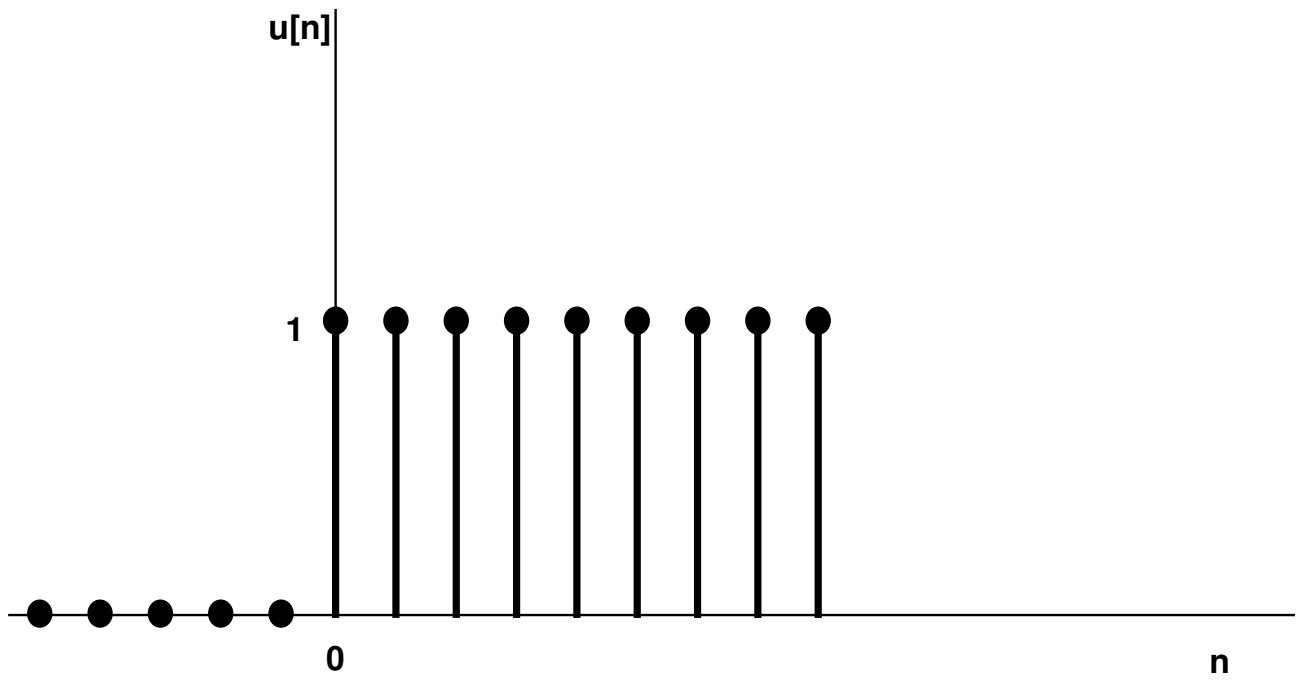


Figure 3: Discrete time Unit Step Sequence.

A unit impulse is shown in Figure 4.

Complex Exponential and Sinusoidal Signals

As in the continuous time, an important signal in discrete time is the complex exponential signal or sequence, defined by

$$x[n] = C\alpha^n, \quad (25)$$

where C and α are, in general, complex numbers.

If C and α are real, we can have one of several types of behavior. If $|\alpha| > 1$, the signal grows exponentially with n , while if $|\alpha| < 1$, we have a decaying exponential. Furthermore, if $\alpha > 0$, all of the values of $C\alpha^n$ are of the same sign, but if $\alpha < 0$, then the sign of $C\alpha^n$ alternates. If $\alpha = 1$ then $x[n]$ is constant, whereas if $\alpha = -1$, then $x[n]$ alternates in value between $+C$ and $-C$.

Another important class of complex exponential signals is obtained if we let $\alpha = e^{j\omega}$ and

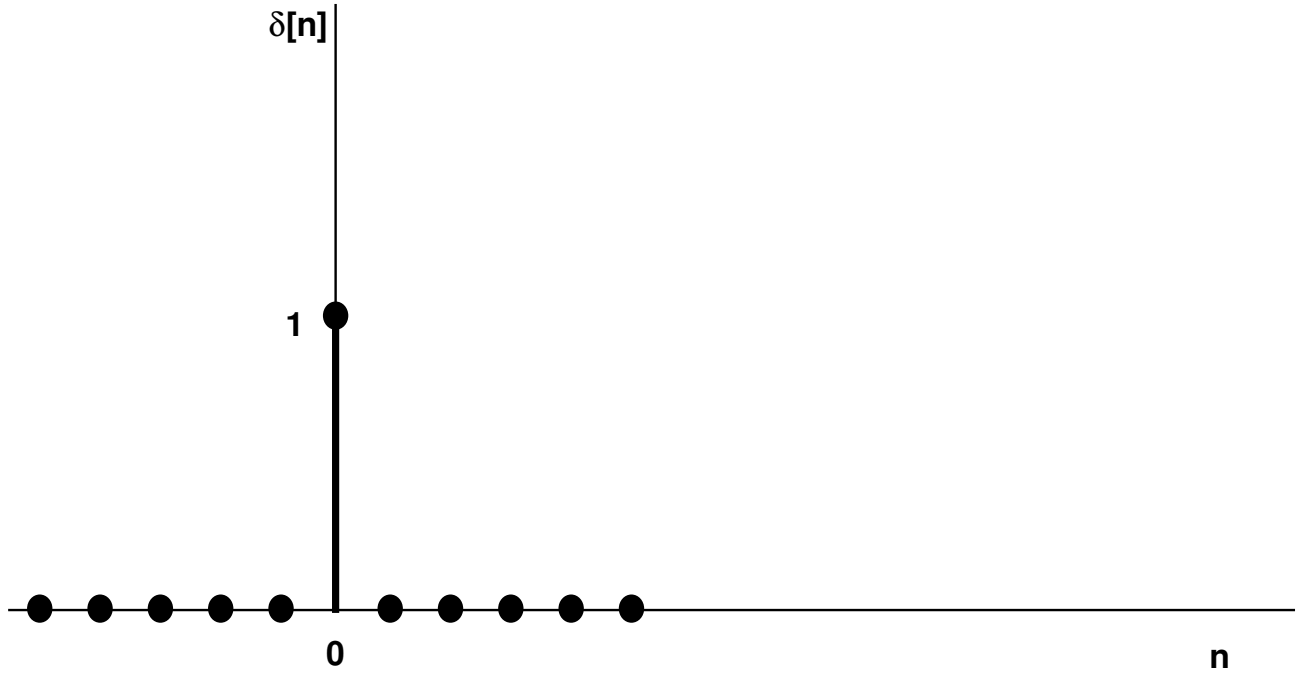


Figure 4: Unit Impulse Function.

let β be purely imaginary. In this case, we obtain

$$x[n] = e^{j\Omega_0 n}. \quad (26)$$

A class of signals closely related to the complex exponential is the sinusoidal signal

$$x[n] = A \cos(\Omega_0 n + \phi), \quad (27)$$

where if we take n to be dimensionless, then both Ω_0 and ϕ have units of radians. The sinusoid of equation (27) can be related to the complex exponential signal as follows

$$A \cos(\Omega_0 n + \phi) = \frac{A}{2} e^{j\phi} e^{j\Omega_0 n} + \frac{A}{2} e^{-j\phi} e^{-j\Omega_0 n}. \quad (28)$$

There are some important differences between the complex exponential signals in continuous and discrete time regarding their range of definition and periodicity. Recall that in the continuous time case the larger the magnitude of ω_0 the higher the rate of oscillation in the signal. Furthermore, the function $e^{j\omega_0 t}$ is periodic for any value of ω_0 .

In the discrete time case the complex exponential is only defined in the range

$$0 \leq \Omega_0 < 2\pi, \text{ or } -\pi \leq \Omega_0 < +\pi, \quad (29)$$

Table 1: Differences Between Continuous and Discrete Exponential Signals

$e^{j\omega_0 t}$	$e^{j\Omega_0 n}$
Distinct signals for distinct values of ω_0	Identical signals for exponentials at frequencies separated by 2π
Periodic for any choice of ω_0	Periodic only if $\Omega_0 = \frac{2\pi m}{N}$ for some integers $N > 0$ and m .
Fundamental frequency ω_0	Fundamental frequency $\frac{\Omega_0}{m}$
Fundamental period $\omega_0 = 0$: undefined $\omega_0 \neq 0$: $\frac{2\pi}{\omega_0}$	Fundamental period $\Omega_0 = 0$: undefined $\Omega_0 \neq 0$: $m(\frac{2\pi}{\Omega_0})$

and the signal $e^{j\Omega_0 n}$ is not periodic for arbitrary values of Ω_0 . Rather for the signal to have period $N > 0$, there must exist an integer m such that the following relation is true

$$\frac{\Omega_0}{2\pi} = \frac{m}{N}. \quad (30)$$

Table 1 summarizes the differences between the signals $e^{j\omega_0 t}$ and $e^{j\Omega_0 n}$.

Read handout A.1 for a presentation of the discrete time Fourier series and discrete Fourier transform.

Reading Assignment

Read “Handout A.1: Fourier Analysis and Fourier Transforms” and the examples Handout E.2 posted on the course web page.