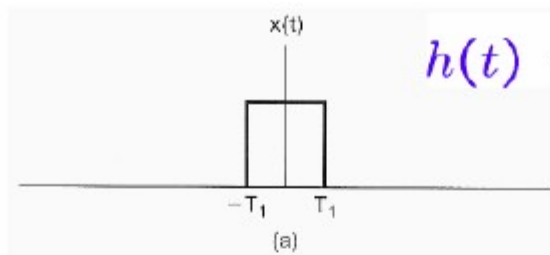
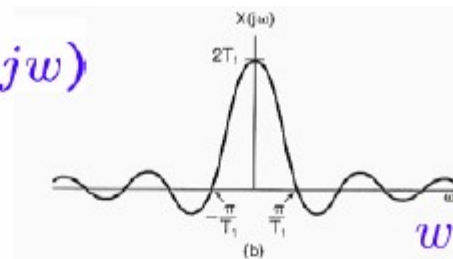


SAMPLING

- Fourier Series of CT Periodic Signals
 - Fourier Series of DT Periodic Signals
- CT Fourier Transform
 - DT Fourier Transform
- Laplace Transform
 - z-Transform
- Discrete Fourier Transform

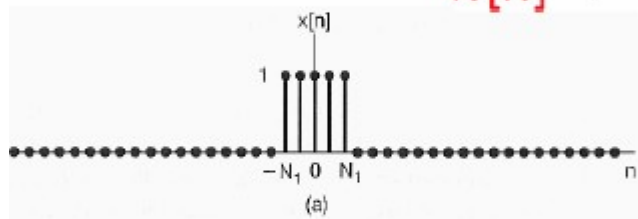


$$h(t) \xleftrightarrow{\text{C.T.F.T.}} H(j\omega)$$

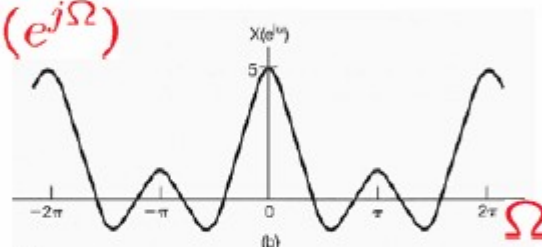


$$\omega_s = \frac{2\pi}{T} \quad t = nT$$

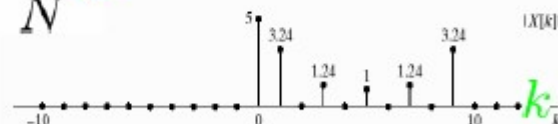
$$\Omega = \omega T$$



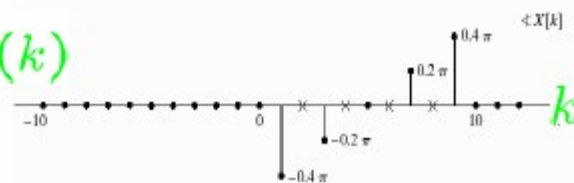
$$h[n] \xleftrightarrow{\text{D.T.F.T.}} H(e^{j\Omega})$$



$$\Omega = \frac{2\pi}{N} k$$

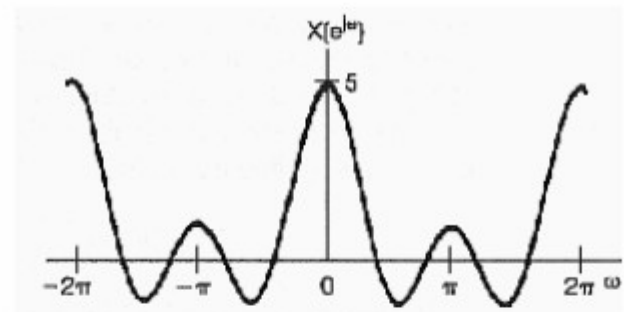
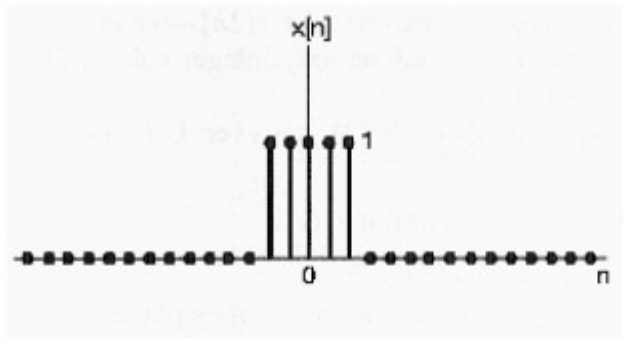


$$h[n] \xleftrightarrow{\text{D.F.T.}} H(k)$$



- Table of Content of “Discrete-Time Signal Processing”
by Oppenheim, Schaffer, and Buck, 1999
 1. Introduction
 2. Discrete-Time Signals and Systems
 3. The z-Transform
 4. Sampling of Continuous-Time Signals
 5. Transform Analysis of Linear Time-Invariant Systems
 6. Structures for Discrete-Time Systems
 7. Filter Design Techniques
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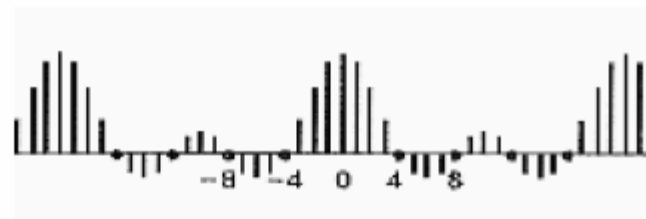
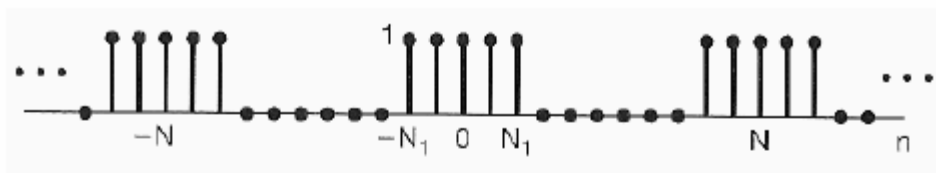
$$x[n] = \begin{cases} 1, & |n| \leq N_1 \\ 0, & |n| > N_1 \end{cases} \xleftrightarrow{\text{D.T.F.T.}} X(e^{j\omega}) = \frac{\sin\left(\omega\left(N_1 + \frac{1}{2}\right)\right)}{\sin(\omega/2)}$$

- The **synthesis** equation:

$$x[n] = \frac{1}{2\pi} \int_{2\pi} X(e^{j\omega}) e^{j\omega n} d\omega$$

- The **analysis** equation:

$$X(e^{j\omega}) = \sum_{n=-\infty}^{+\infty} x[n] e^{-j\omega n}$$



$$x[n] = \begin{cases} 1, & -N_1 \leq n \leq N_1 \\ 0, & \text{others in } \langle N \rangle \end{cases} \xleftrightarrow{\mathcal{F.S.}} a_k = \frac{1}{N} \sum_{n=-N_1}^{N_1} e^{-jk(2\pi/N)n}$$

- The **synthesis** equation:

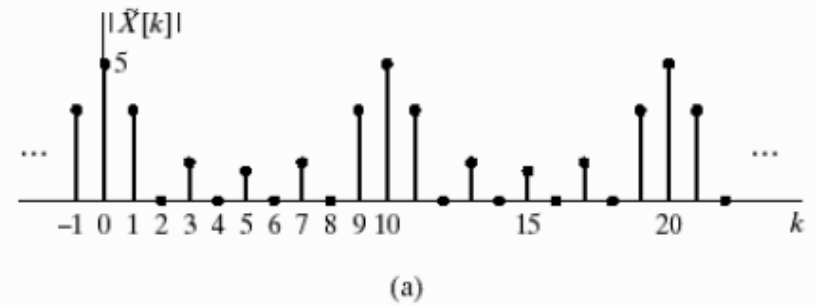
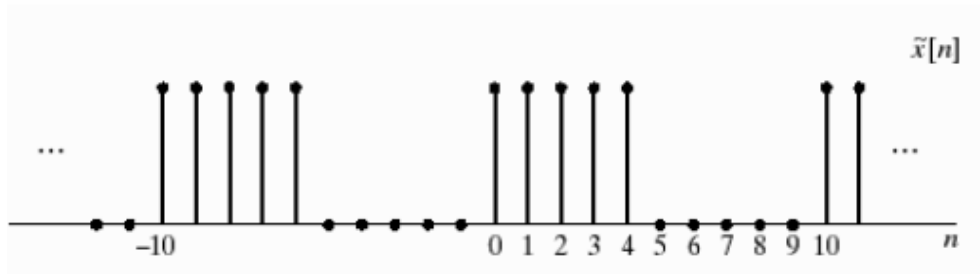
$$x[n] = \sum_{k=\langle N \rangle} a_k e^{jk\omega_0 n} = \sum_{k=\langle N \rangle} a_k e^{jk(2\pi/N)n}$$

- The **analysis** equation:

$$a_k = \frac{1}{N} \sum_{n=\langle N \rangle} x[n] e^{-jk\omega_0 n} = \frac{1}{N} \sum_{n=\langle N \rangle} x[n] e^{-jk(2\pi/N)n}$$

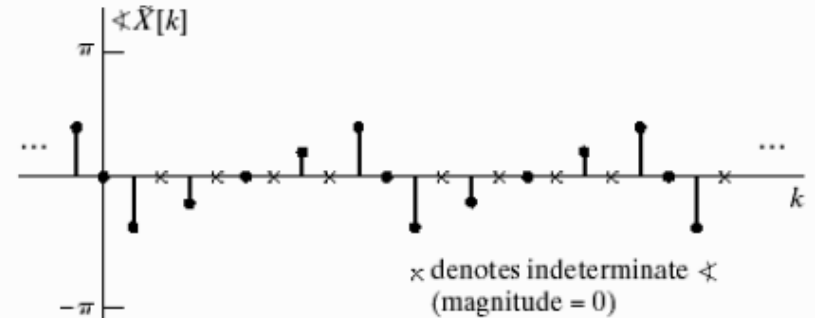
$$a_k = a_{k+N}$$

- Periodic and Finite-Length Sequences:



- The **synthesis** equation:

$$\tilde{x}[n] = \frac{1}{N} \sum_{k=0}^{N-1} \tilde{X}[k] (W_N)^{-kn}$$



- The **analysis** equation:

$$\tilde{X}[k] = \sum_{n=0}^{N-1} \tilde{x}[n] (W_N)^{kn}$$

$$W_N = e^{-j(2\pi/N)}$$

$$\tilde{x}[n] \xleftrightarrow{\mathcal{D.F.S.}} \tilde{X}[k]$$

■ Periodic Complex Exponentials:

$$W_N = e^{-j(\frac{2\pi}{N})}$$

$$e_k[n] = e^{-j(\frac{2\pi}{N})^{kn}} = (W_N)^{kn} = W_N^{kn}$$

$$e_k[n + rN] = e^{-j(\frac{2\pi}{N})k(n+rN)} = e^{-j(\frac{2\pi}{N})kn - j(2\pi r)}$$

$$= e^{-j(\frac{2\pi}{N})kn} e^{-j(2\pi r)} = e^{-j(\frac{2\pi}{N})kn} = e_k[n]$$

$$e_{k+lN}[n] = e^{-j(\frac{2\pi}{N})(k+lN)n}$$

$$= e^{-j(\frac{2\pi}{N})kn} e^{-j(2\pi l)} = e^{-j(\frac{2\pi}{N})kn} = e_k[n]$$

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■ Linearity:

- $\tilde{x}[n], \tilde{y}[n]$: periodic sequences with period N

$$\tilde{x}[n] \xleftrightarrow{\mathcal{DFS}} \tilde{X}[k]$$

$$\tilde{y}[n] \xleftrightarrow{\mathcal{DFS}} \tilde{Y}[k]$$

$$\Rightarrow \tilde{z}[n] = A\tilde{x}[n] + B\tilde{y}[n] \xleftrightarrow{\mathcal{DFS}} \tilde{Z}[k] = A\tilde{X}[k] + B\tilde{Y}[k]$$

- Shift of a Sequence:

$$W_N = e^{-j(\frac{2\pi}{N})}$$

$$\tilde{x}[n] \xleftrightarrow{\mathcal{DFS}} \tilde{X}[k]$$

$$\tilde{x}[n - m] \xleftrightarrow{\mathcal{DFS}} W_N^{km} \tilde{X}[k]$$

$$W_N^{-ln} \tilde{x}[n] \xleftrightarrow{\mathcal{DFS}} \tilde{X}[k - l]$$

■ Duality:

$$W_N = e^{-j(\frac{2\pi}{N})}$$

$$\tilde{x}[n] = \frac{1}{N} \sum_{k=0}^{N-1} \tilde{X}[k] W_N^{-kn}$$

$$\tilde{x}[n] \xleftrightarrow{\mathcal{DFS}} \tilde{X}[k]$$

$$\tilde{X}[k] = \sum_{n=0}^{N-1} \tilde{x}[n] W_N^{kn}$$

$$N\tilde{x}[-n] = \sum_{k=0}^{N-1} \tilde{X}[k] W_N^{kn}$$

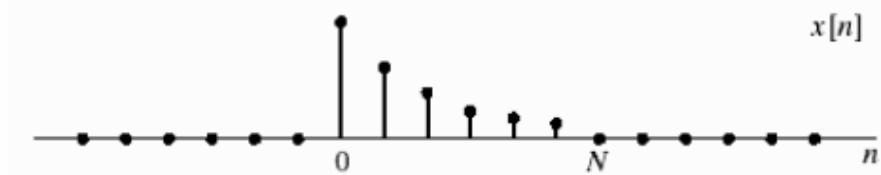
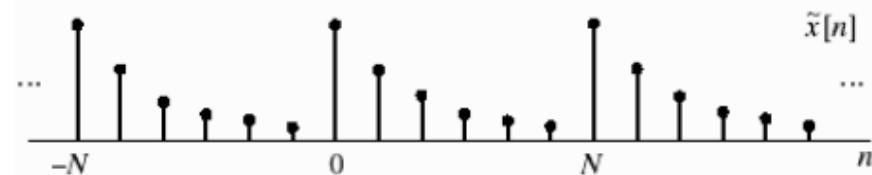
$$\tilde{X}[n] \xleftrightarrow{\mathcal{DFS}} N\tilde{x}[-k]$$

$$N\tilde{x}[-k] = \sum_{n=0}^{N-1} \tilde{X}[n] W_N^{kn}$$

TABLE 8.1 SUMMARY OF PROPERTIES OF THE DFS

Periodic Sequence (Period N)	DFS Coefficients (Period N)
1. $\tilde{x}[n]$	$\tilde{X}[k]$ periodic with period N
2. $\tilde{x}_1[n], \tilde{x}_2[n]$	$\tilde{X}_1[k], \tilde{X}_2[k]$ periodic with period N
3. $a\tilde{x}_1[n] + b\tilde{x}_2[n]$	$a\tilde{X}_1[k] + b\tilde{X}_2[k]$
4. $\tilde{X}[n]$	$N\tilde{x}[-k]$
5. $\tilde{x}[n - m]$	$W_N^{km} \tilde{X}[k]$
6. $W_N^{-\ell n} \tilde{x}[n]$	$\tilde{X}[k - \ell]$
7. $\sum_{m=0}^{N-1} \tilde{x}_1[m] \tilde{x}_2[n - m]$ (periodic convolution)	$\tilde{X}_1[k] \tilde{X}_2[k]$
8. $\tilde{x}_1[n] \tilde{x}_2[n]$	$\frac{1}{N} \sum_{\ell=0}^{N-1} \tilde{X}_1[\ell] \tilde{X}_2[k - \ell]$ (periodic convolution)
9. $\tilde{x}^*[n]$	$\tilde{X}^*[-k]$
10. $\tilde{x}^*[-n]$	$\tilde{X}^*[k]$
11. $\mathcal{R}e\{\tilde{x}[n]\}$	$\tilde{X}_e[k] = \frac{1}{2}(\tilde{X}[k] + \tilde{X}^*[-k])$
12. $j\mathcal{I}m\{\tilde{x}[n]\}$	$\tilde{X}_o[k] = \frac{1}{2j}(\tilde{X}[k] - \tilde{X}^*[-k])$
13. $\tilde{x}_e[n] = \frac{1}{2}(\tilde{x}[n] + \tilde{x}^*[-n])$	$\mathcal{R}e\{\tilde{X}[k]\}$
14. $\tilde{x}_o[n] = \frac{1}{2j}(\tilde{x}[n] - \tilde{x}^*[-n])$	$j\mathcal{I}m\{\tilde{X}[k]\}$
Properties 15–17 apply only when $x[n]$ is real.	
15. Symmetry properties for $\tilde{x}[n]$ real.	$\begin{cases} \tilde{X}[k] = \tilde{X}^*[-k] \\ \mathcal{R}e\{\tilde{X}[k]\} = \mathcal{R}e\{\tilde{X}^*[-k]\} \\ \mathcal{I}m\{\tilde{X}[k]\} = -\mathcal{I}m\{\tilde{X}^*[-k]\} \\ \tilde{X}[k] = \tilde{X}^*[-k] \\ \angle \tilde{X}[k] = -\angle \tilde{X}^*[-k] \end{cases}$
16. $\tilde{x}_e[n] = \frac{1}{2}(\tilde{x}[n] + \tilde{x}[-n])$	$\mathcal{R}e\{\tilde{X}[k]\}$
17. $\tilde{x}_o[n] = \frac{1}{2j}(\tilde{x}[n] - \tilde{x}[-n])$	$j\mathcal{I}m\{\tilde{X}[k]\}$

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$$\tilde{x}[n] \xleftrightarrow{\text{D.F.S.}} \tilde{X}[k]$$

$$x[n] \xleftrightarrow{\text{D.T.F.T.}} X(e^{jw})$$

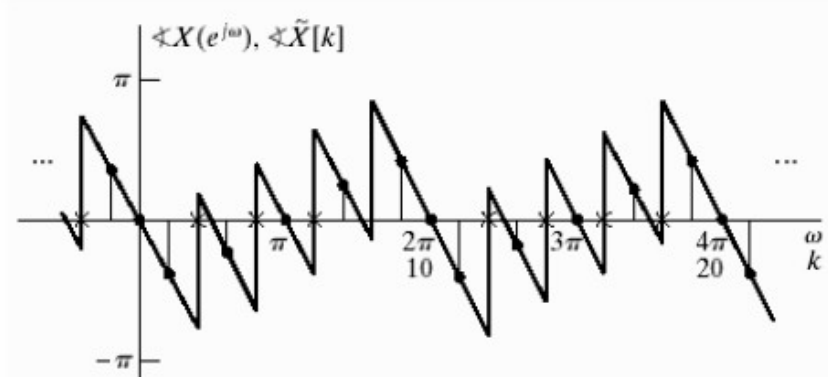
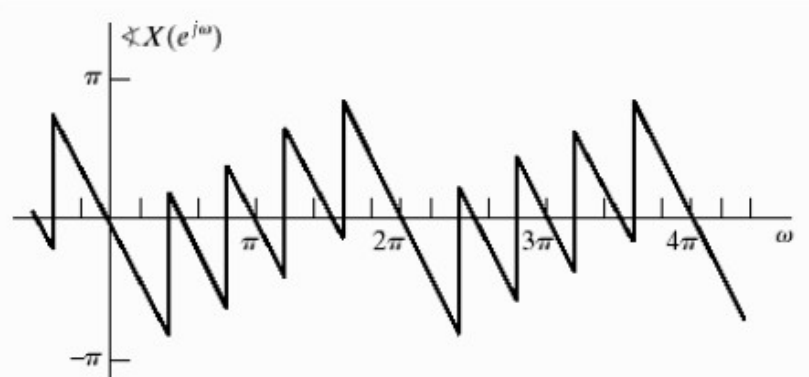
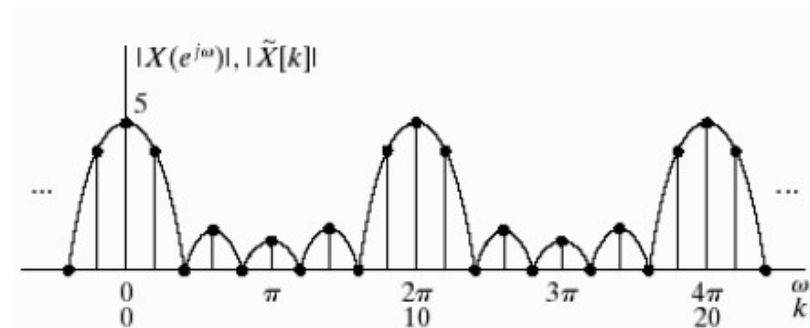
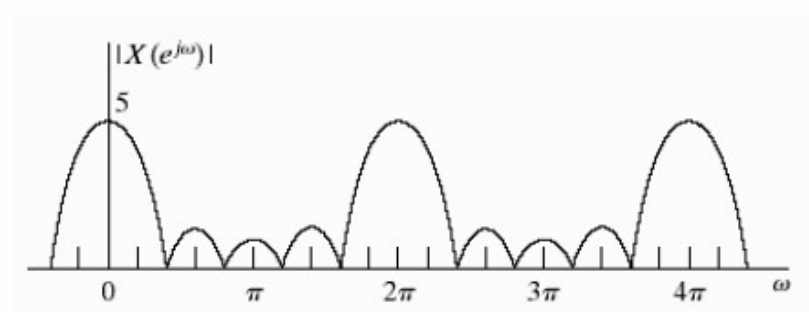
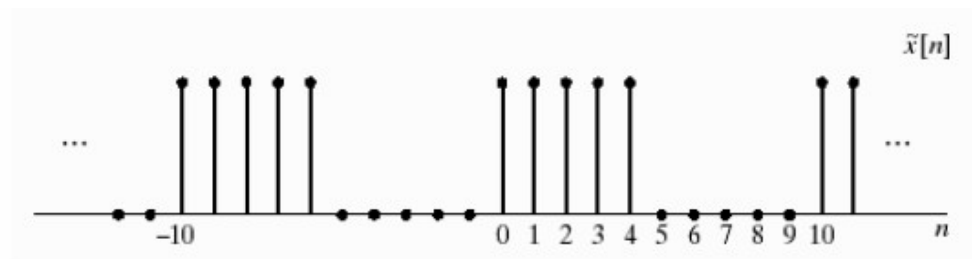
$$\tilde{x}[n] = \frac{1}{N} \sum_{k=0}^{N-1} \tilde{X}[k] e^{j\left(\frac{2\pi}{N}\right)kn}$$

$$x[n] = \frac{1}{2\pi} \int_{2\pi} X(e^{jw}) e^{jwn} dw$$

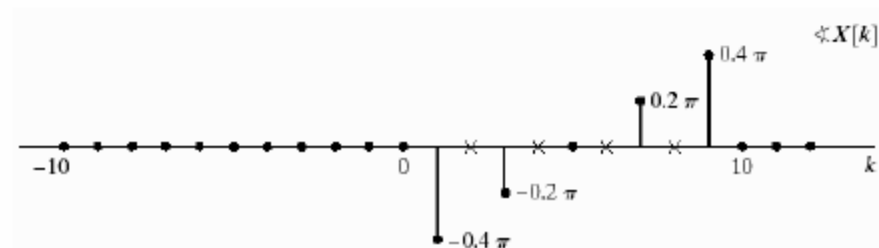
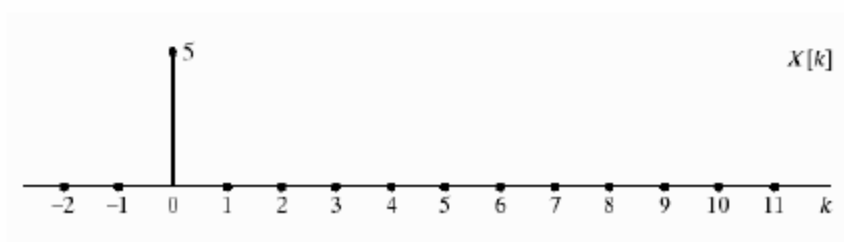
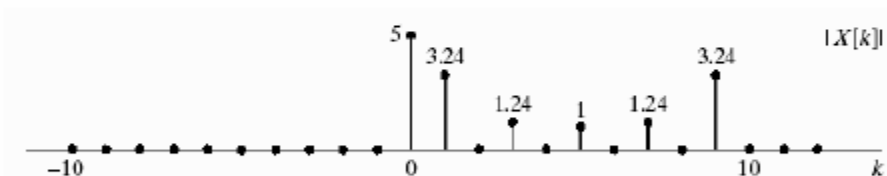
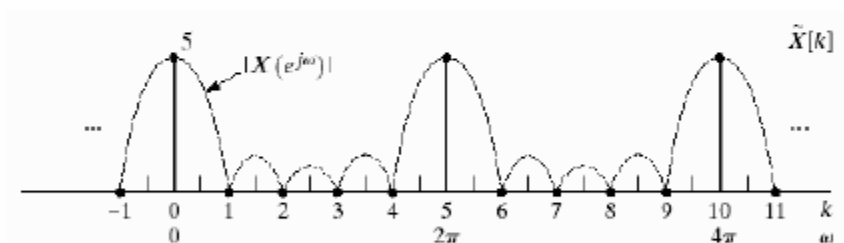
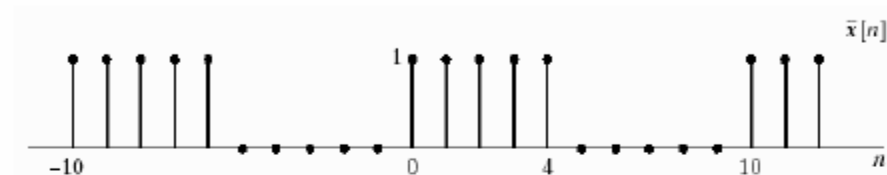
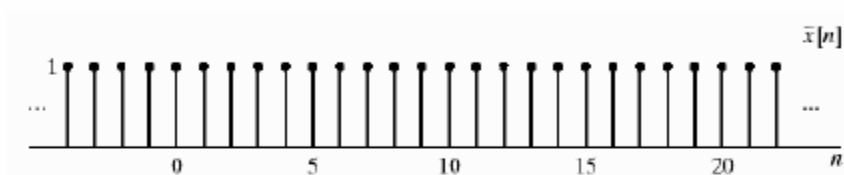
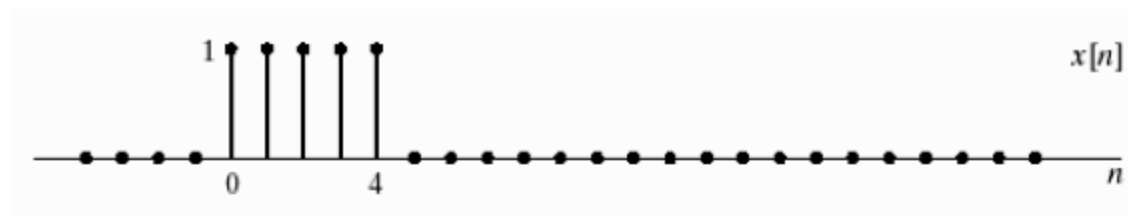
$$\tilde{X}[k] = \sum_{n=0}^{N-1} \tilde{x}[n] e^{-j\left(\frac{2\pi}{N}\right)kn}$$

$$X(e^{jw}) = \sum_{n=-\infty}^{+\infty} x[n] e^{-jwn}$$

$$\tilde{X}[k] = X\left(e^{j\left(\frac{2\pi}{N}\right)k}\right) = X(e^{jw}) \Big|_{w=\left(\frac{2\pi}{N}\right)k}$$



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$$\tilde{X}[k] = \sum_{n=0}^{N-1} \tilde{x}[n] W_N^{kn} \quad W_N = e^{-j(\frac{2\pi}{N})}$$

$$\tilde{x}[n] = \frac{1}{N} \sum_{k=0}^{N-1} \tilde{X}[k] W_N^{-kn}$$

$$x[n] \xleftrightarrow{\mathcal{D.F.T.}} X[k]$$

$$X[k] = \begin{cases} \tilde{X}[k], & 0 \leq k \leq N-1 \\ 0, & \text{otherwise} \end{cases}$$

$$x[n] = \begin{cases} \tilde{x}[n], & 0 \leq n \leq N-1 \\ 0, & \text{otherwise} \end{cases}$$

$$X[k] = \begin{cases} \sum_{n=0}^{N-1} x[n] W_N^{kn}, & 0 \leq k \leq N-1 \\ 0, & \text{otherwise} \end{cases}$$

$$x[n] = \begin{cases} \frac{1}{N} \sum_{k=0}^{N-1} X[k] W_N^{-kn}, & 0 \leq n \leq N-1 \\ 0, & \text{otherwise} \end{cases}$$

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■ Linearity:

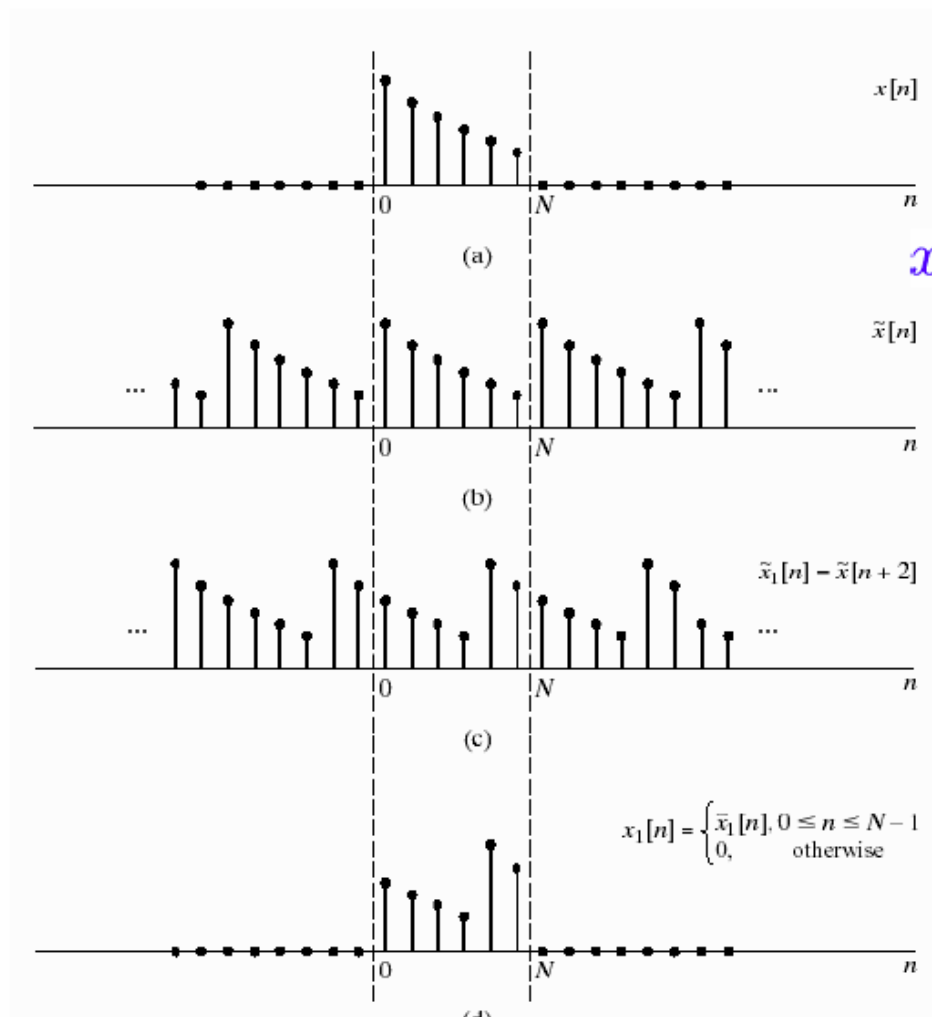
- $x[n]$, $y[n]$: periodic sequences with period N

$$x[n] \xleftrightarrow{\mathcal{DFT}} X[k]$$

$$y[n] \xleftrightarrow{\mathcal{DFT}} Y[k]$$

$$\Rightarrow z[n] = Ax[n] + By[n] \xleftrightarrow{\mathcal{DFT}} Z[k] = AX[k] + BY[k]$$

■ Circular Shift of a Sequence:



$$x[((n - m))_N], \quad 0 \leq n \leq N - 1$$

$$\xleftrightarrow{\mathcal{DFT}} e^{-j(2\pi k/N)m} X[k]$$

- Circular Shift of a Sequence:

$$x[n] \xleftrightarrow{\mathcal{DFT}} X[k]$$

$$y[n] \xleftrightarrow{\mathcal{DFT}} Y[k] = e^{-j(2\pi k/N)m} X[k]$$

$$\tilde{x}[n] = x[((n))_N] \xleftrightarrow{\mathcal{DFS}} \tilde{X}[k] = X[((k))_N]$$

$$\tilde{y}[n] = y[((n))_N] \xleftrightarrow{\mathcal{DFS}} \tilde{Y}[k] = Y[((k))_N]$$

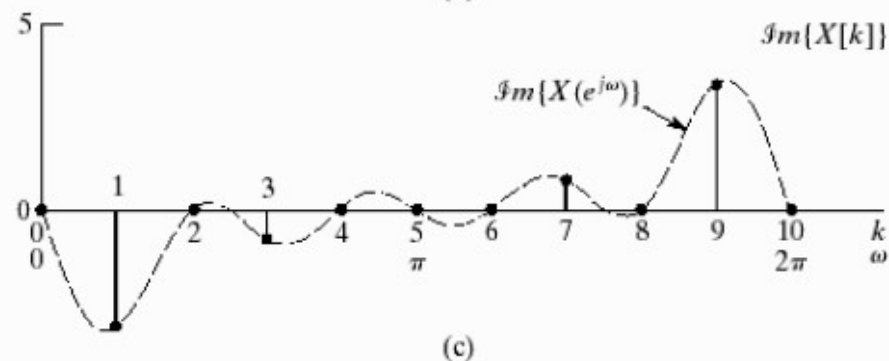
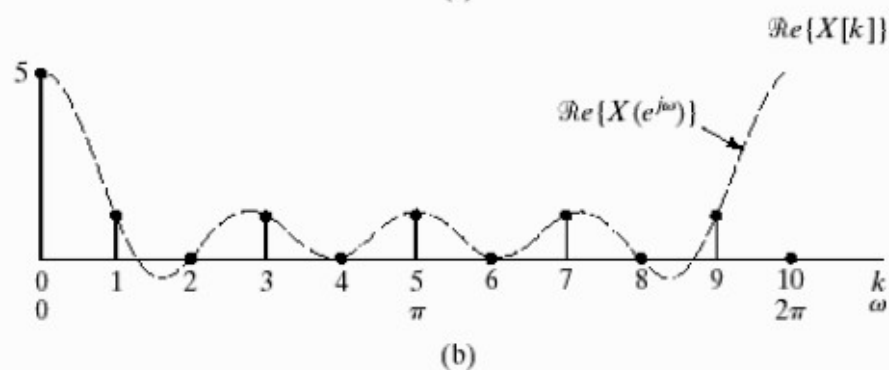
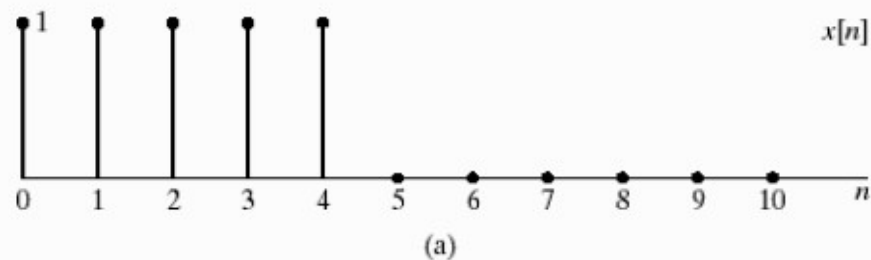
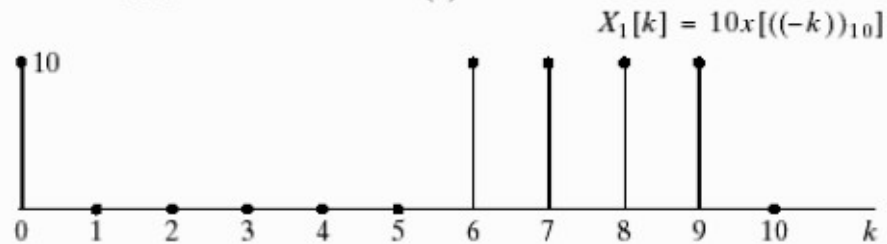
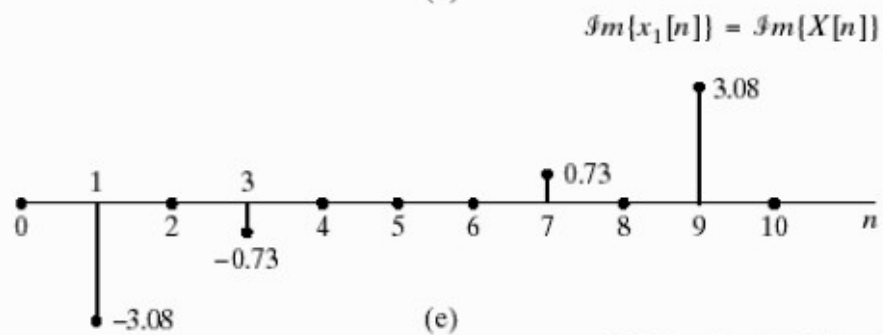
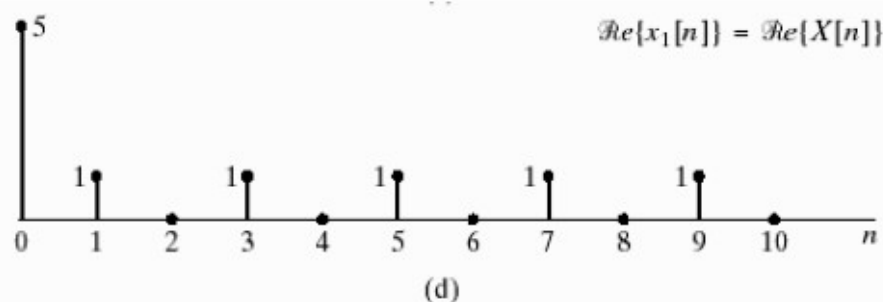
$$\begin{aligned} Y[k] = e^{-j(2\pi k/N)m} X[k] &\Leftrightarrow \tilde{Y}[k] = e^{-j(2\pi((k))_N/N)m} \tilde{X}[(k))_N] \\ &= e^{-j(2\pi k/N)m} \tilde{X}[k] \end{aligned}$$

$$y[n] = \begin{cases} \tilde{y}[n] = x[((n - m))_N], & 0 \leq n \leq N - 1 \\ 0, & \text{otherwise} \end{cases}$$

■ Duality:

$$\tilde{x}[n] \xleftrightarrow{\text{DFS}} \tilde{X}[k]$$

$$\tilde{X}[n] \xleftrightarrow{\text{DFS}} N\tilde{x}[-k]$$



- Duality:

$$\tilde{x}[n] = x[((n))_N]$$

$$\tilde{x}[\textcolor{red}{n}] \xleftrightarrow{\mathcal{DFS}} \tilde{X}[\textcolor{red}{k}]$$

$$\tilde{X}[k] = X[((k))_N]$$

$$\tilde{X}[\textcolor{red}{n}] \xleftrightarrow{\mathcal{DFS}} \textcolor{red}{N}\tilde{x}[-\textcolor{red}{k}]$$

$$\tilde{y}[n] = \tilde{X}[n] \quad \text{i.e.} \quad \textcolor{red}{y}[n] = X[n]$$

$$\Rightarrow \textcolor{red}{Y}[k] = N\tilde{x}[-k]$$

$$\Rightarrow \textcolor{red}{Y}[k] = \begin{cases} N\tilde{x}[-k], & 0 \leq n \leq N-1 \\ 0, & \text{otherwise} \end{cases}$$

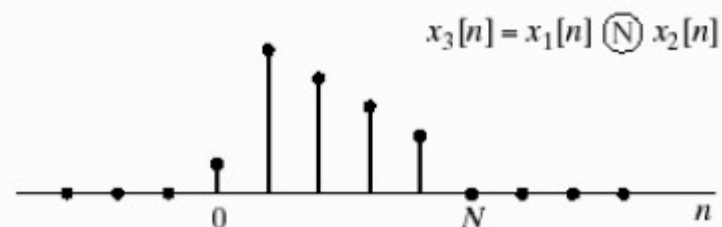
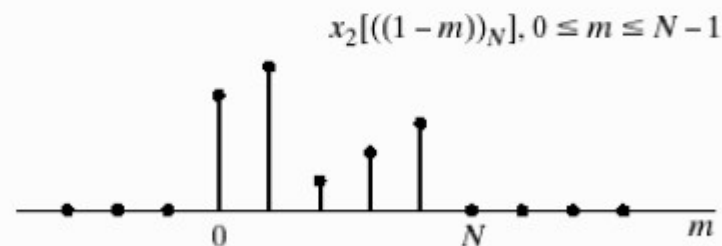
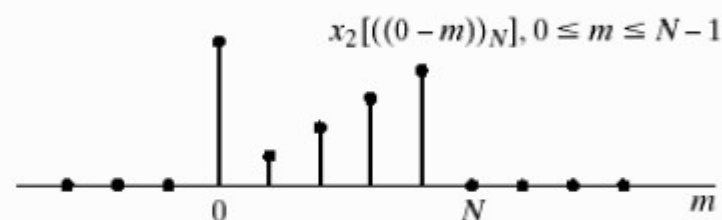
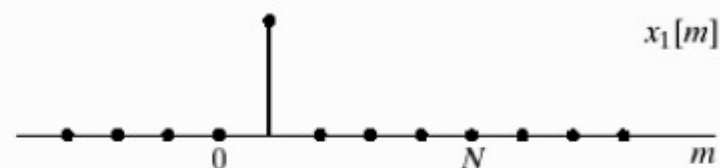
$$= \begin{cases} N\textcolor{blue}{x}[((-k))_N], & 0 \leq n \leq N-1 \\ 0, & \text{otherwise} \end{cases}$$

- Duality:

$$x[n] \xleftrightarrow{\mathcal{DFT}} X[k]$$

$$X[n] \xleftrightarrow{\mathcal{DFT}} Nx[((-k))_N], \quad 0 \leq k \leq N-1$$

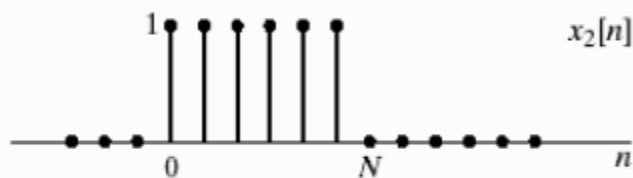
■ Circular Convolution:



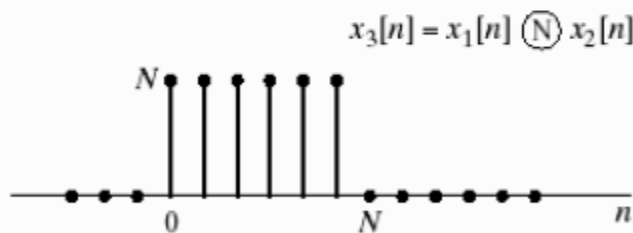
■ Circular Convolution:



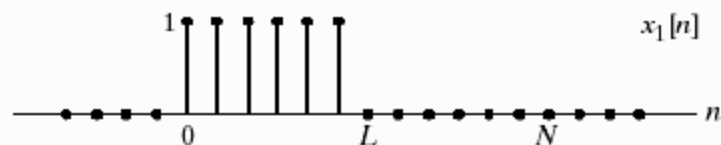
(a)



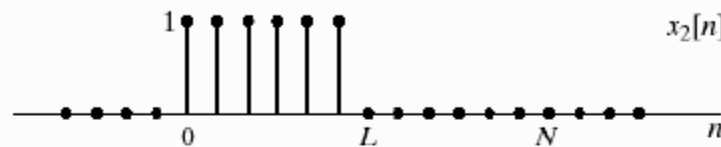
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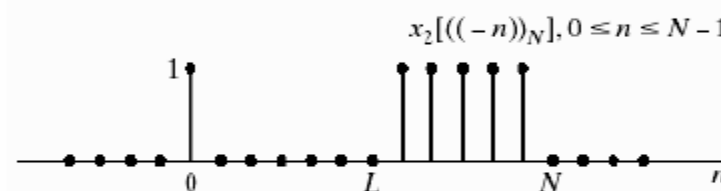
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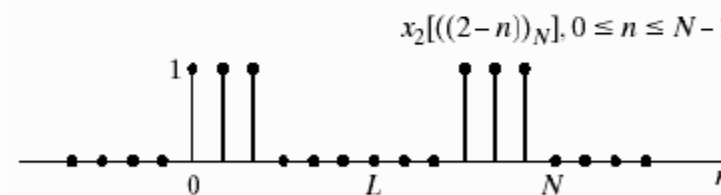
(a)



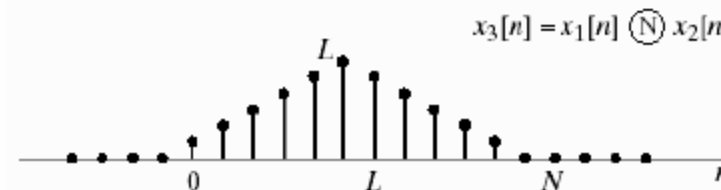
(b)



(c)



(d)



- Circular Convolution:

$$z[n] = \sum_{m=0}^{N-1} x[m]y[((n-m))_N] = x[n] \circledast y[n]$$

$$x[n] \circledast y[n] \xleftrightarrow{\mathcal{DFT}} X[k]Y[k]$$

$$x[n]y[n] \xleftrightarrow{\mathcal{DFT}} \frac{1}{N}X[k] \circledast Y[k]$$

TABLE 8.2

Finite-Length Sequence (Length N)	N -point DFT (Length N)
1. $x[n]$	$X[k]$
2. $x_1[n], x_2[n]$	$X_1[k], X_2[k]$
3. $ax_1[n] + bx_2[n]$	$aX_1[k] + bX_2[k]$
4. $X[n]$	$Nx[((-k))_N]$
5. $x[((n-m))_N]$	$W_N^{km} X[k]$
6. $W_N^{-\ell n} x[n]$	$X[((k-\ell))_N]$
7. $\sum_{m=0}^{N-1} x_1(m)x_2[((n-m))_N]$	$X_1[k]X_2[k]$
8. $x_1[n]x_2[n]$	$\frac{1}{N} \sum_{\ell=0}^{N-1} X_1(\ell)X_2[((k-\ell))_N]$
9. $x^*[n]$	$X^*[((-k))_N]$
10. $x^*[((-n))_N]$	$X^*[k]$
11. $\mathcal{R}e\{x[n]\}$	$X_{\text{cp}}[k] = \frac{1}{2}\{X[((k))_N] + X^*[(((-k))_N)]\}$
12. $j\mathcal{I}m\{x[n]\}$	$X_{\text{op}}[k] = \frac{1}{2}\{X[((k))_N] - X^*[(((-k))_N)]\}$
13. $x_{\text{cp}}[n] = \frac{1}{2}\{x[n] + x^*[(((-n))_N)]\}$	$\mathcal{R}e\{X[k]\}$
14. $x_{\text{op}}[n] = \frac{1}{2}\{x[n] - x^*[(((-n))_N)]\}$	$j\mathcal{I}m\{X[k]\}$
Properties 15–17 apply only when $x[n]$ is real.	
15. Symmetry properties	$\begin{cases} X[k] = X^*[(((-k))_N)] \\ \mathcal{R}e\{X[k]\} = \mathcal{R}e\{X[(((-k))_N)]\} \\ \mathcal{I}m\{X[k]\} = -\mathcal{I}m\{X[(((-k))_N)]\} \\ X[k] = X[(((-k))_N)] \\ \angle\{X[k]\} = -\angle\{X[(((-k))_N)]\} \end{cases}$
16. $x_{\text{cp}}[n] = \frac{1}{2}\{x[n] + x[(((-n))_N)]\}$	$\mathcal{R}e\{X[k]\}$
17. $x_{\text{op}}[n] = \frac{1}{2}\{x[n] - x[(((-n))_N)]\}$	$j\mathcal{I}m\{X[k]\}$

- The Discrete Fourier Series of Periodic Sequences
- Properties of the Discrete Fourier Series
- The Fourier Transform of Periodic Signals
- The Discrete Fourier Transform of
Finite-Duration Sequences
- Properties of the Discrete Fourier Transform
- The Discrete Cosine Transform

■ Discrete Cosine Transform:

$$\tilde{X}[k] = \sum_{n=0}^{N-1} \tilde{x}[n] W_N^{kn}$$

$$\tilde{x}[n] = \frac{1}{N} \sum_{k=0}^{N-1} \tilde{X}[k] W_N^{-kn}$$

$$W_N = e^{-j(\frac{2\pi}{N})}$$

$$A[k] = \sum_{n=0}^{N-1} x[n] \phi_k^*[n]$$

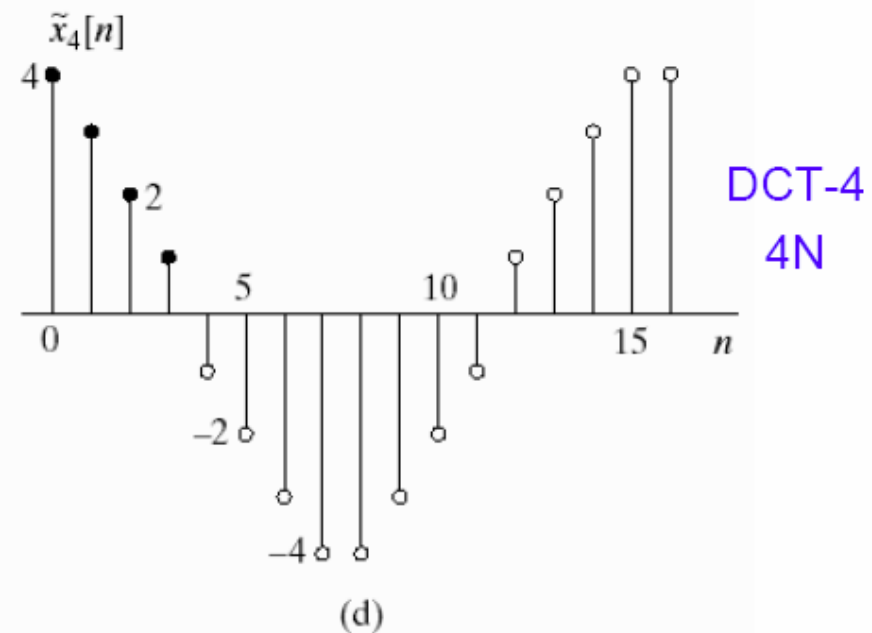
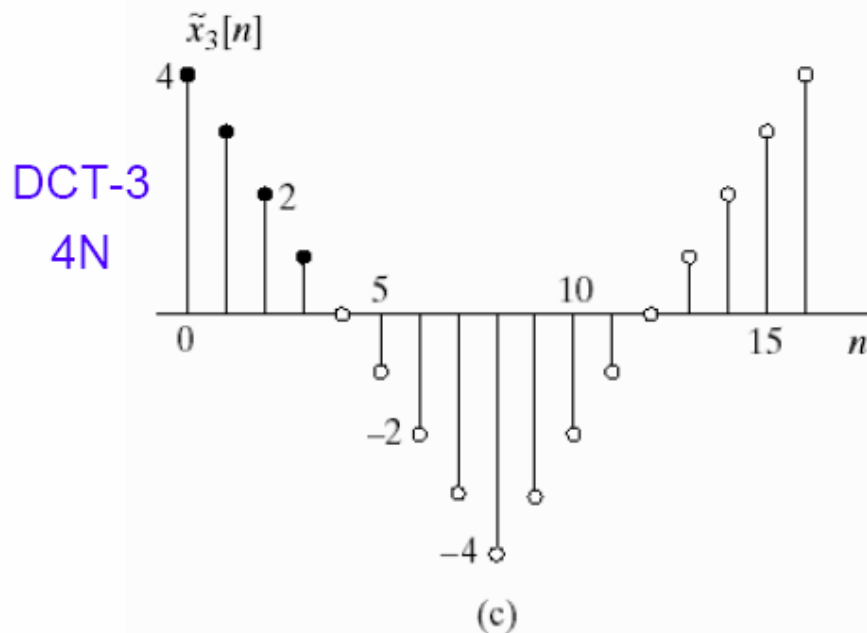
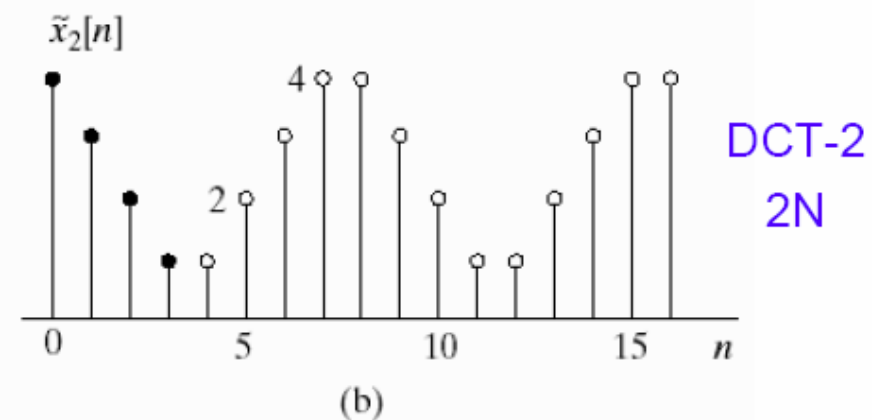
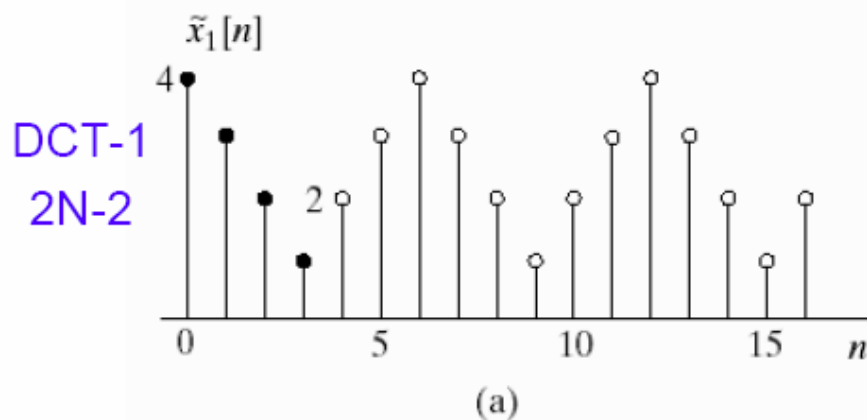
$$x[n] = \frac{1}{N} \sum_{k=0}^{N-1} A[k] \phi_k[n]$$

$$\frac{1}{N} \sum_{n=0}^{N-1} \phi_k[n] \phi_m^*[n] = \begin{cases} 1, & m = k \\ 0, & m \neq k \end{cases}$$

$\phi_k[n] = \cos(k, n)$: even symmetric and periodic

■ DCT-1, 2, 3, 4:

Figure 8.25 Four ways to extend a four-point sequence $x[n]$ both periodically and symmetrically. The finite-length sequence $x[n]$ is plotted with solid dots. (a) Type-1 periodic extension for DCT-1. (b) Type-2 periodic extension for DCT-2. (c) Type-3 periodic extension for DCT-3. (d) Type-4 periodic extension for DCT-4.



■ DCT-1:

$$A[k] = \sum_{n=0}^{N-1} x[n] \phi_k^*[n]$$

$$x[n] = \frac{1}{N} \sum_{k=0}^{N-1} A[k] \phi_k[n]$$

$$X^{c1}[k] = 2 \sum_{n=0}^{N-1} \alpha[n] x[n] \cos\left(\frac{\pi k n}{N-1}\right), \quad 0 \leq k \leq N-1$$

$$x[n] = \frac{1}{N-1} \sum_{k=0}^{N-1} \alpha[k] X^{c1}[k] \cos\left(\frac{\pi k n}{N-1}\right), \quad 0 \leq k \leq N-1$$

$$\alpha[n] = \begin{cases} \frac{1}{2}, & n = 0 \text{ and } N-1 \\ 1, & 1 \leq n \leq N-2 \end{cases}$$

■ DCT-2:

$$A[k] = \sum_{n=0}^{N-1} x[n] \phi_k^*[n]$$

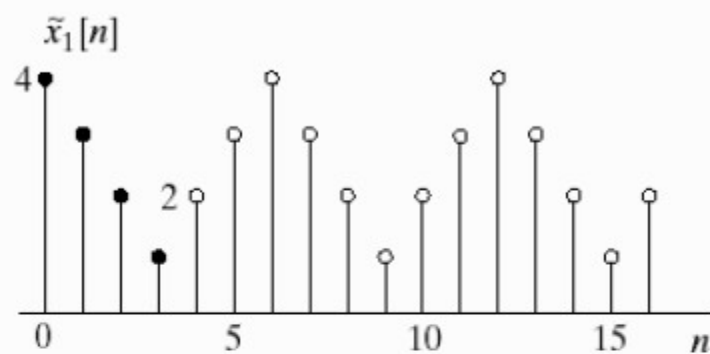
$$x[n] = \frac{1}{N} \sum_{k=0}^{N-1} A[k] \phi_k[n]$$

$$X^{c2}[k] = 2 \sum_{n=0}^{N-1} x[n] \cos\left(\frac{\pi k(2n+1)}{2N}\right), \quad 0 \leq k \leq N-1$$

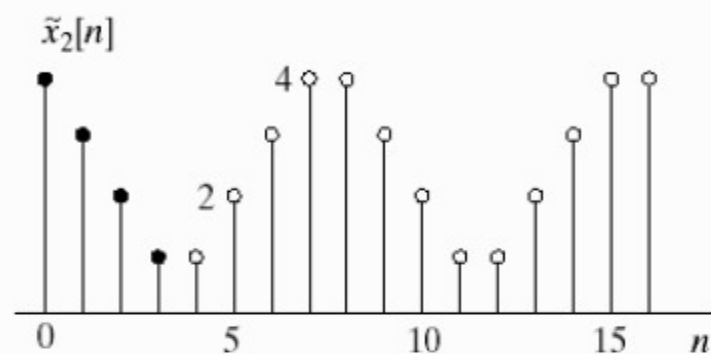
$$x[n] = \frac{1}{N} \sum_{k=0}^{N-1} \beta[k] X^{c2}[k] \cos\left(\frac{\pi k(2n+1)}{2N}\right), \quad 0 \leq k \leq N-1$$

$$\beta[k] = \begin{cases} \frac{1}{2}, & k = 0 \\ 1, & 1 \leq k \leq N-1 \end{cases}$$

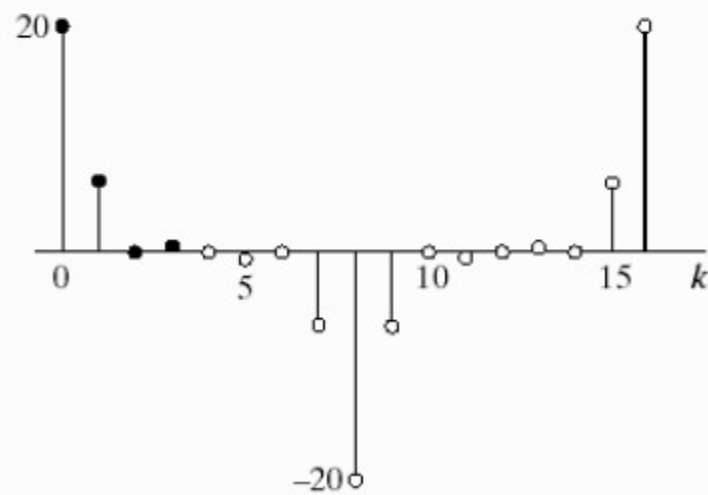
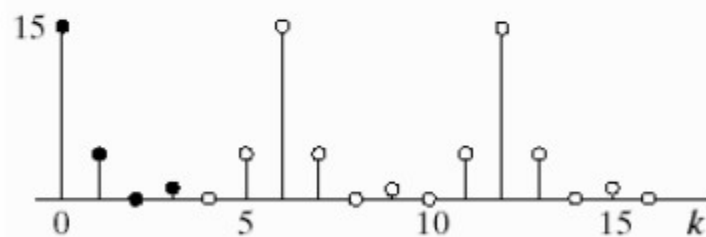
■ DCT-1 & 2:



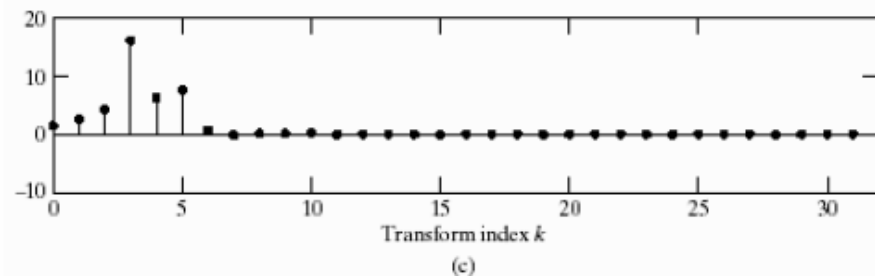
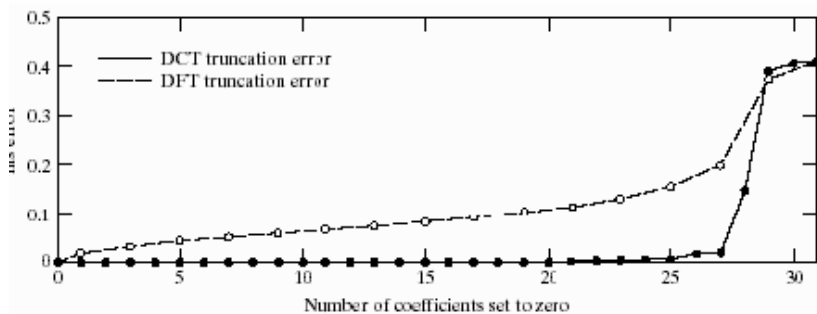
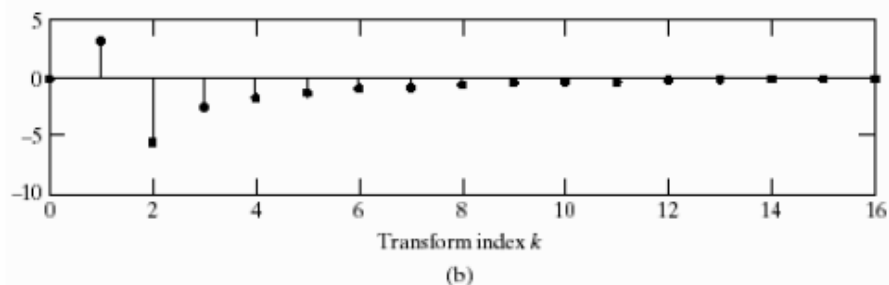
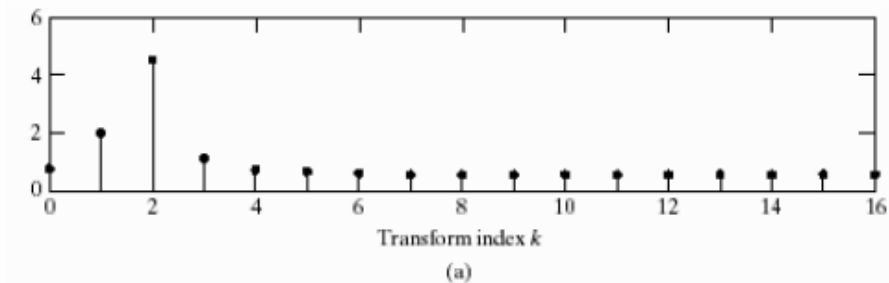
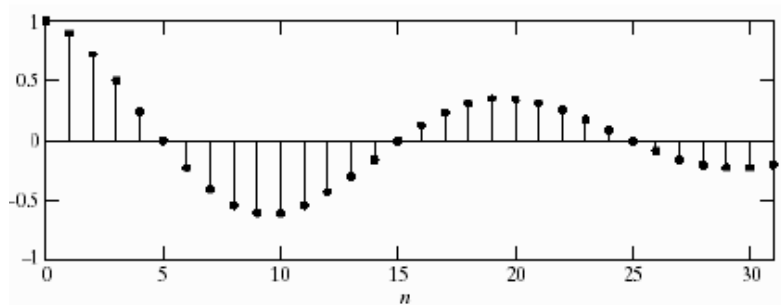
(a)



(b)



■ Energy Compaction Property of the DCT-2:



■ Energy Compaction Property of the DCT-2:

- Hence, the major application of the DCT-2 is in **signal compression**
- Due to its **energy concentration** property