

Chapter IX

Laplace Transform

- The Laplace Transform
- The Region of Convergence for Laplace Transforms
- The Inverse Laplace Transform
- Geometric Evaluation of the Fourier Transform
- Properties of the Laplace Transform
- Some Laplace Transform Pairs
- Analysis & Characterization of LTI Systems Using the Laplace Transform
- System Function Algebra and Block Diagram Representations
- The Unilateral Laplace Transform

- The Laplace transform of a general signal $x(t)$:

$$X(jw) \triangleq \int_{-\infty}^{\infty} x(t) e^{-jw t} dt$$

$$X(s) \triangleq \int_{-\infty}^{\infty} x(t) e^{-st} dt \quad s = \sigma + jw$$

$$x(t) \xleftrightarrow{\mathcal{F}} X(jw)$$

$$X(jw) = \mathcal{F}\{x(t)\}$$

$$x(t) = \mathcal{F}^{-1}\{X(jw)\}$$

$$x(t) \xleftrightarrow{\mathcal{L}} X(s)$$

$$X(s) = \mathcal{L}\{x(t)\}$$

$$x(t) = \mathcal{L}^{-1}\{X(s)\}$$

$$X(s) \Big|_{s=jw} = \mathcal{L}\{x(t)\} \Big|_{s=jw} = \mathcal{F}\{x(t)\} = X(jw)$$

- Laplace Transform & Fourier Transform:

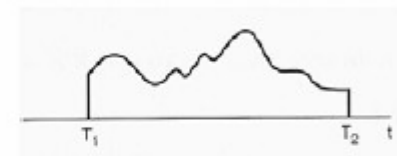
$$X(s) \Big|_{s=jw} = \mathcal{L}\{x(t)\} \Big|_{s=jw} = \mathcal{F}\{x(t)\} = X(jw)$$

$$\mathcal{L}\{x(t)\} = X(s) \quad s = \sigma + jw$$

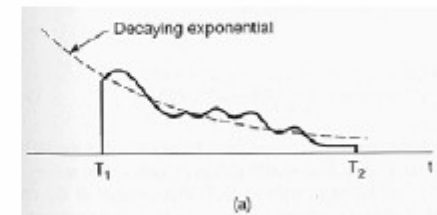
$$= X(\sigma + jw) = \int_{-\infty}^{\infty} x(t) e^{-(\sigma + jw)t} dt$$

$$= \int_{-\infty}^{\infty} [x(t) e^{-\sigma t}] e^{-jw t} dt$$

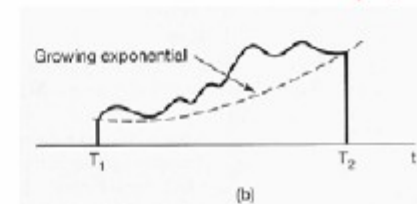
$$= \mathcal{F}\{x(t) e^{-\sigma t}\}$$



$\sigma > 0$

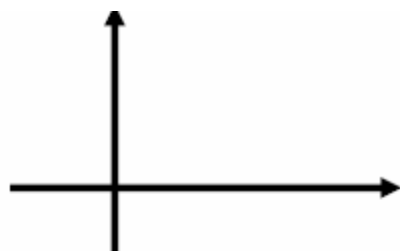


$\sigma < 0$



■ Example 9.1:

$$x(t) = e^{-at}u(t)$$



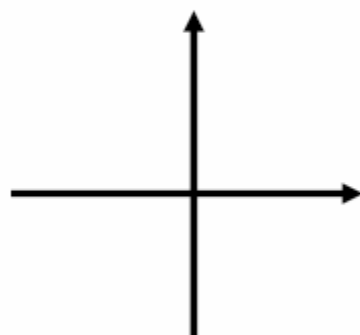
$$X(j\omega) \triangleq \int_{-\infty}^{\infty} x(t)e^{-j\omega t} dt$$

$$X(s) \triangleq \int_{-\infty}^{\infty} x(t)e^{-st} dt$$

$$X(j\omega) = \int_{-\infty}^{\infty} e^{-at}u(t)e^{-j\omega t} dt$$

$$= \int_0^{\infty} e^{-at}e^{-j\omega t} dt = \left. \frac{e^{-(a+j\omega)t}}{-(a+j\omega)} \right|_0^{\infty} = \frac{1}{j\omega + a}, \quad a > 0$$

$$X(s) = \int_{-\infty}^{\infty} e^{-at}u(t)e^{-st} dt = \int_0^{\infty} e^{-at}e^{-st} dt$$

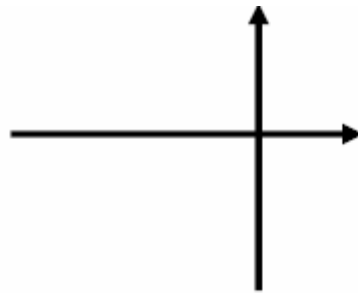


$$X(\sigma + j\omega) = \int_0^{\infty} e^{-(\sigma+a)t}e^{-j\omega t} dt = \frac{1}{(\sigma + a) + j\omega}, \quad \sigma + a > 0$$

$$= \frac{1}{s + a}, \quad \text{Re}\{s\} > -a$$

■ Example 9.2:

$$x(t) = -e^{-at}u(-t)$$

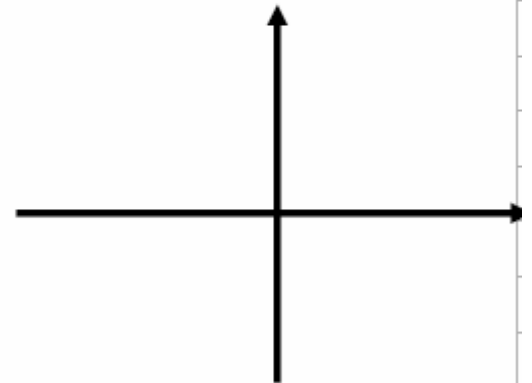


$$X(s) \triangleq \int_{-\infty}^{\infty} x(t)e^{-st}dt$$

$$X(s) = \int_{-\infty}^{\infty} -e^{-at}u(-t)e^{-st}dt$$

$$= - \int_{-\infty}^0 e^{-at}e^{-st}dt$$

$$= \frac{1}{s+a}, \quad \text{Re}\{s\} < -a$$



$$e^{-at}u(t) \xleftrightarrow{\mathcal{L}} \frac{1}{s+a}, \quad \text{Re}\{s\} > -a$$

$$-e^{-at}u(-t) \xleftrightarrow{\mathcal{L}} \frac{1}{s+a}, \quad \text{Re}\{s\} < -a$$

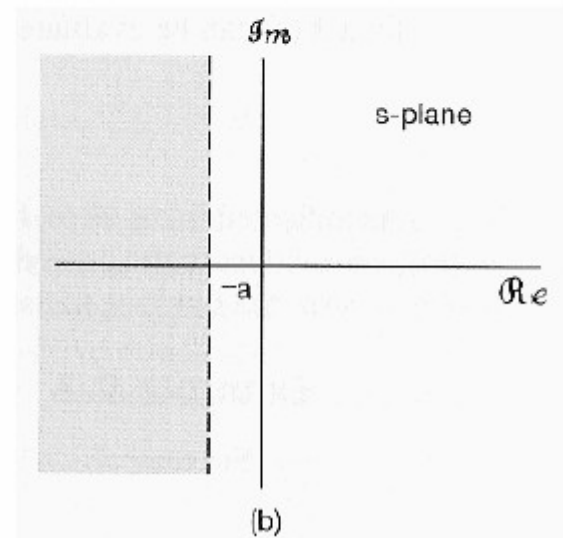
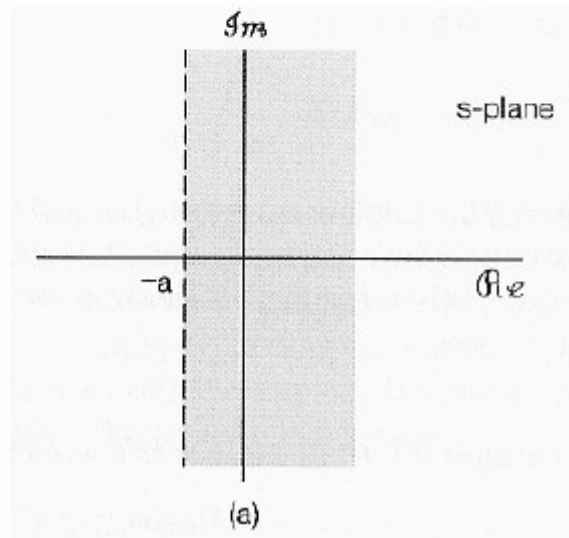
Region of Convergence
(ROC)

- Region of Convergence (ROC):

$$e^{-at}u(t) \xleftrightarrow{\mathcal{L}} \frac{1}{s+a}, \quad \mathcal{R}e\{s\} > -a$$

$$-e^{-at}u(-t) \xleftrightarrow{\mathcal{L}} \frac{1}{s+a}, \quad \mathcal{R}e\{s\} < -a$$

where Fourier transform of $x(t)e^{-\sigma t}$ converges



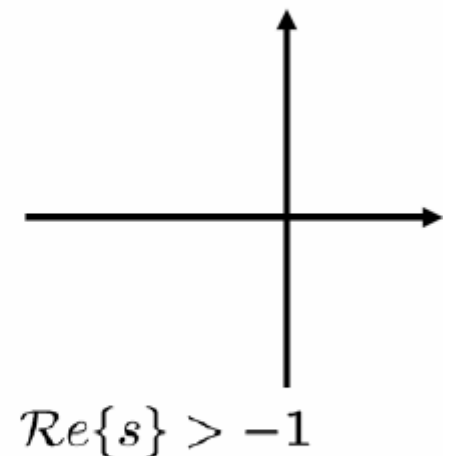
■ Example 9.3: $x(t) = 3e^{-2t}u(t) - 2e^{-t}u(t)$

$$X(s) = \int_{-\infty}^{\infty} [3e^{-2t}u(t) - 2e^{-t}u(t)] e^{-st} dt$$

$$= 3 \int_{-\infty}^{\infty} e^{-2t}u(t)e^{-st} dt - 2 \int_{-\infty}^{\infty} e^{-t}u(t)e^{-st} dt$$

$$= 3 \left(\frac{1}{s+2} \right) - 2 \left(\frac{1}{s+1} \right)$$

$$e^{-t}u(t) \xleftrightarrow{\mathcal{L}} \frac{1}{s+1},$$



$$\mathcal{R}e\{s\} > -1$$

$$e^{-2t}u(t) \xleftrightarrow{\mathcal{L}} \frac{1}{s+2},$$

$$\mathcal{R}e\{s\} > -2$$

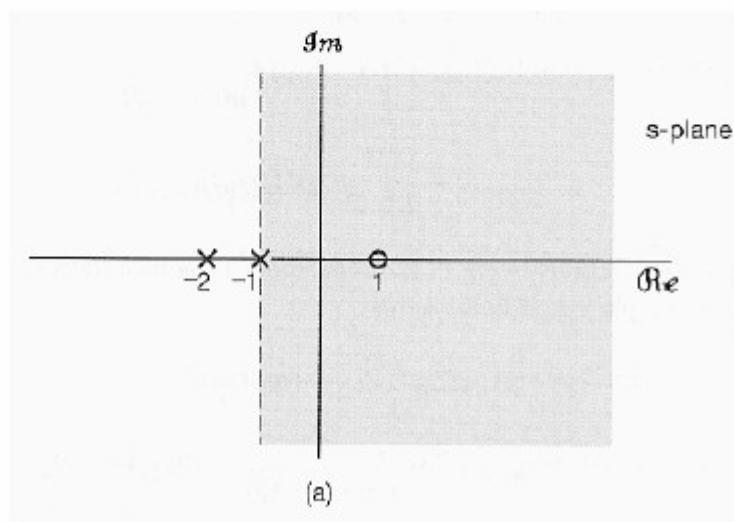
$$3e^{-2t}u(t) - 2e^{-t}u(t) \xleftrightarrow{\mathcal{L}} \frac{3}{s+2} - \frac{2}{s+1}, \quad \mathcal{R}e\{s\} > -1$$

■ Example 9.3:

$$\mathcal{R}e\{s\} > -2 \quad \mathcal{R}e\{s\} > -1$$

$$3e^{-2t}u(t) - 2e^{-t}u(t) \xleftrightarrow{\mathcal{L}} \frac{3}{s+2} - \frac{2}{s+1}, \quad \mathcal{R}e\{s\} > -1$$

$$\xleftrightarrow{\mathcal{L}} \frac{s-1}{(s+2)(s+1)}, \quad \mathcal{R}e\{s\} > -1$$



- The **jw-axis** is included in the **ROC**!
- **Fourier transform!**
 - $s = j\omega$

■ Example 9.4:

$$e^{-at}u(t) \xleftrightarrow{\mathcal{L}} \frac{1}{s+a}, \quad \operatorname{Re}\{s\} > -\operatorname{Re}\{a\}$$

$$x(t) = e^{-2t}u(t) + e^{-t}(\cos(3t))u(t) \quad \operatorname{Re}\{s\} > -2$$

$$= \left[e^{-2t} + \frac{1}{2}e^{-t} \left(e^{j3t} + e^{-j3t} \right) \right] u(t) \quad \operatorname{Re}\{s\} > -1$$

$$e^{-2t}u(t) \xleftrightarrow{\mathcal{L}} \frac{1}{s+2}, \quad \operatorname{Re}\{s\} > -2$$

$$e^{-(1-3j)t}u(t) \xleftrightarrow{\mathcal{L}} \frac{1}{s+(1-3j)}, \quad \operatorname{Re}\{s\} > -1$$

$$e^{-(1+3j)t}u(t) \xleftrightarrow{\mathcal{L}} \frac{1}{s+(1+3j)}, \quad \operatorname{Re}\{s\} > -1$$

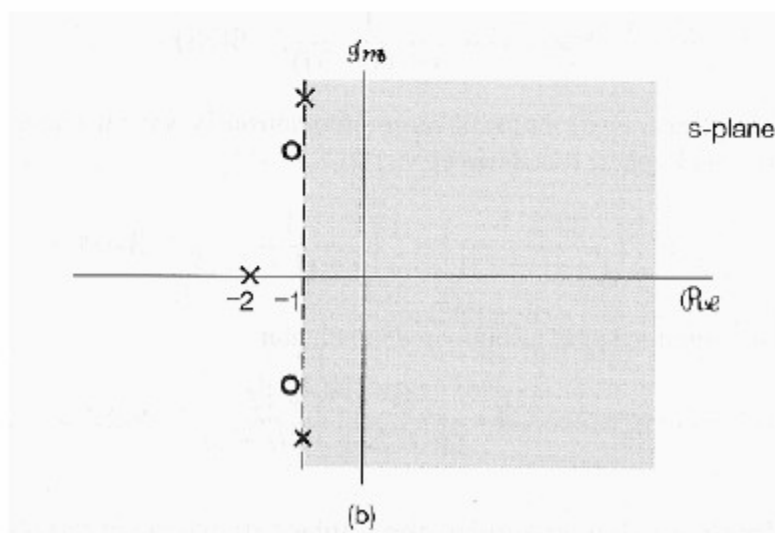
$$X(s) = \frac{1}{s+2} + \frac{1}{2} \left[\frac{1}{s+(1-3j)} + \frac{1}{s+(1+3j)} \right] = \frac{2s^2+5s+12}{(s^2+2s+10)(s+2)}$$

■ Example 9.4:

$$\mathcal{R}e\{s\} > -2 \quad \mathcal{R}e\{s\} > -1$$

$$e^{-2t}u(t) + e^{-t}(\cos(3t))u(t) \xleftrightarrow{\mathcal{L}} \frac{2s^2 + 5s + 12}{(s^2 + 2s + 10)(s + 2)}, \quad \mathcal{R}e\{s\} > -1$$

$$\frac{2(s + 1.25 - 2.11j)(s + 1.25 + 2.11j)}{(s + 1 - 3j)(s + 1 + 3j)(s + 2)}$$



- The $j\omega$ -axis is included in the ROC!
- Fourier transform!
 - $s = j\omega$

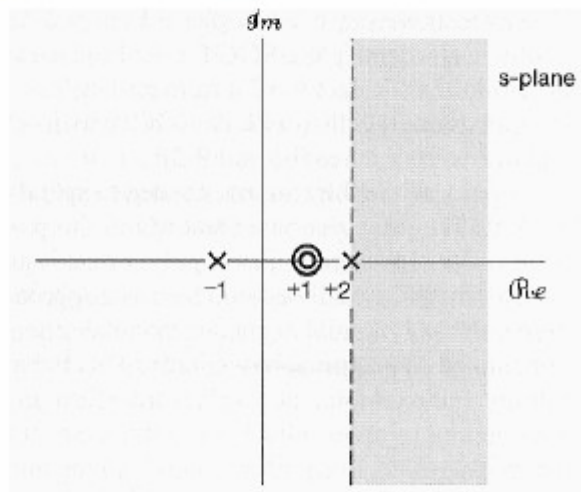
▪ Example 9.5:

$$\int_{-\infty}^{\infty} \delta(t) e^{-st} dt = 1$$

$$x(t) = \delta(t) - \frac{4}{3}e^{-t}u(t) + \frac{1}{3}e^{2t}u(t)$$

$$X(s) = 1 - \frac{4}{3} \frac{1}{s+1} + \frac{1}{3} \frac{1}{s-2}, \quad \mathcal{R}e\{s\} > 2$$

$$\delta(t) - \frac{4}{3}e^{-t}u(t) + \frac{1}{3}e^{2t}u(t) \xleftrightarrow{\mathcal{L}} \frac{(s-1)^2}{(s+1)(s-2)}, \quad \mathcal{R}e\{s\} > 2$$

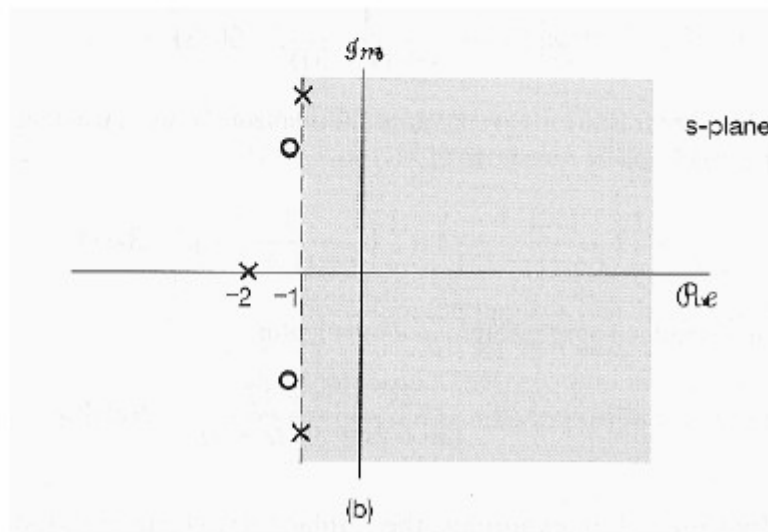


- The $j\omega$ -axis is not included in the ROC!
- Fourier transform?
- Why?

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■ Properties of ROC:

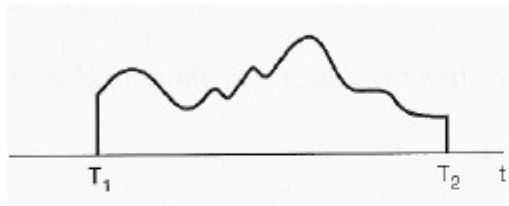
1. The ROC of $X(s)$ consists of strips parallel to the $j\omega$ -axis in the s -plane
2. For rational Laplace transforms, the ROC does not contain any poles



$$\frac{2(s+1.25-2.11j)(s+1.25+2.11j)}{(s+1-3j)(s+1+3j)(s+2)}$$

■ Properties of ROC:

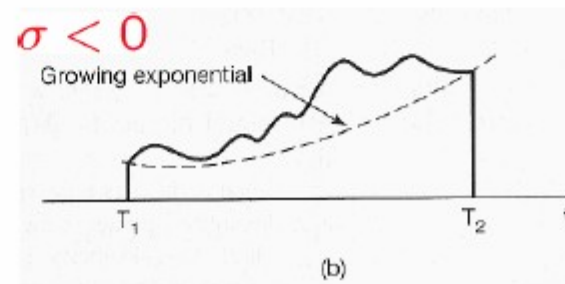
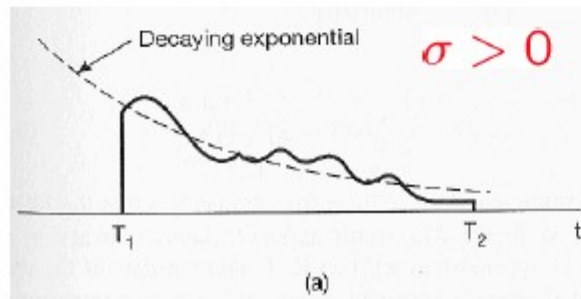
3. If $x(t)$ is of finite duration & is absolutely integrable, then the ROC is the entire s-plane



$$\int_{T_1}^{T_2} |x(t)| dt < \infty$$

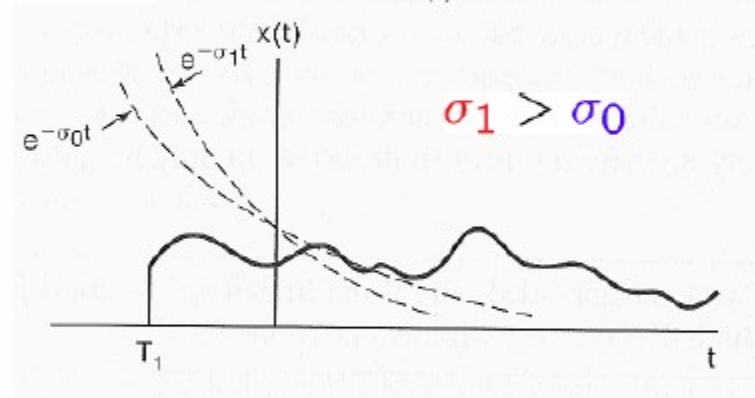
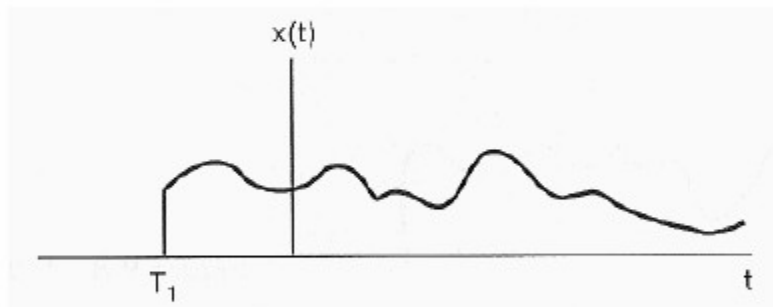
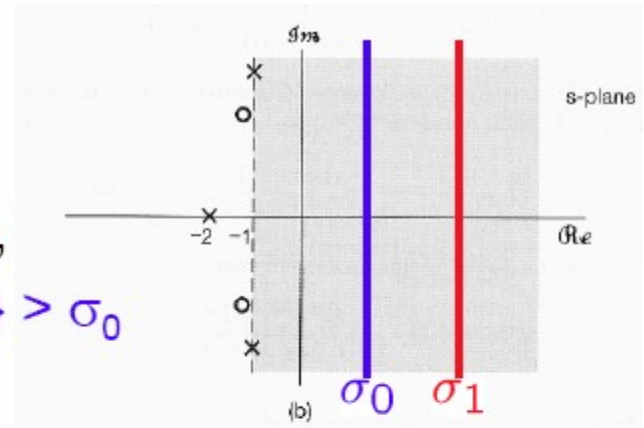
$$s = \sigma + jw$$

$$X(s) = \int_{-\infty}^{\infty} x(t)e^{-st} dt = \int_{T_1}^{T_2} x(t)e^{-st} dt < e^{-\sigma(T_1 \text{ or } T_2)} \int_{T_1}^{T_2} |x(t)| dt$$



■ Properties of ROC:

4. If $x(t)$ is right-sided, and if the line $\text{Re}\{s\} = \sigma_0$ is in the ROC, then all values of s for which $\text{Re}\{s\} > \sigma_0$ will also be in the ROC

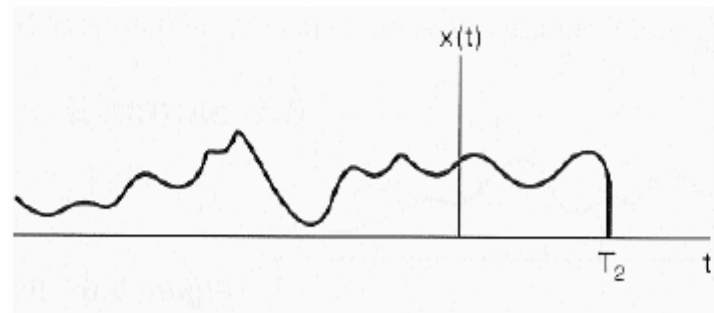


$$\int_{T_1}^{+\infty} |x(t)| e^{-\sigma_0 t} dt < \infty$$

$$\Rightarrow \int_{T_1}^{+\infty} |x(t)| e^{-\sigma_1 t} dt \leq e^{-(\sigma_1 - \sigma_0)T_1} \int_{T_1}^{+\infty} |x(t)| e^{-\sigma_0 t} dt$$

■ Properties of ROC:

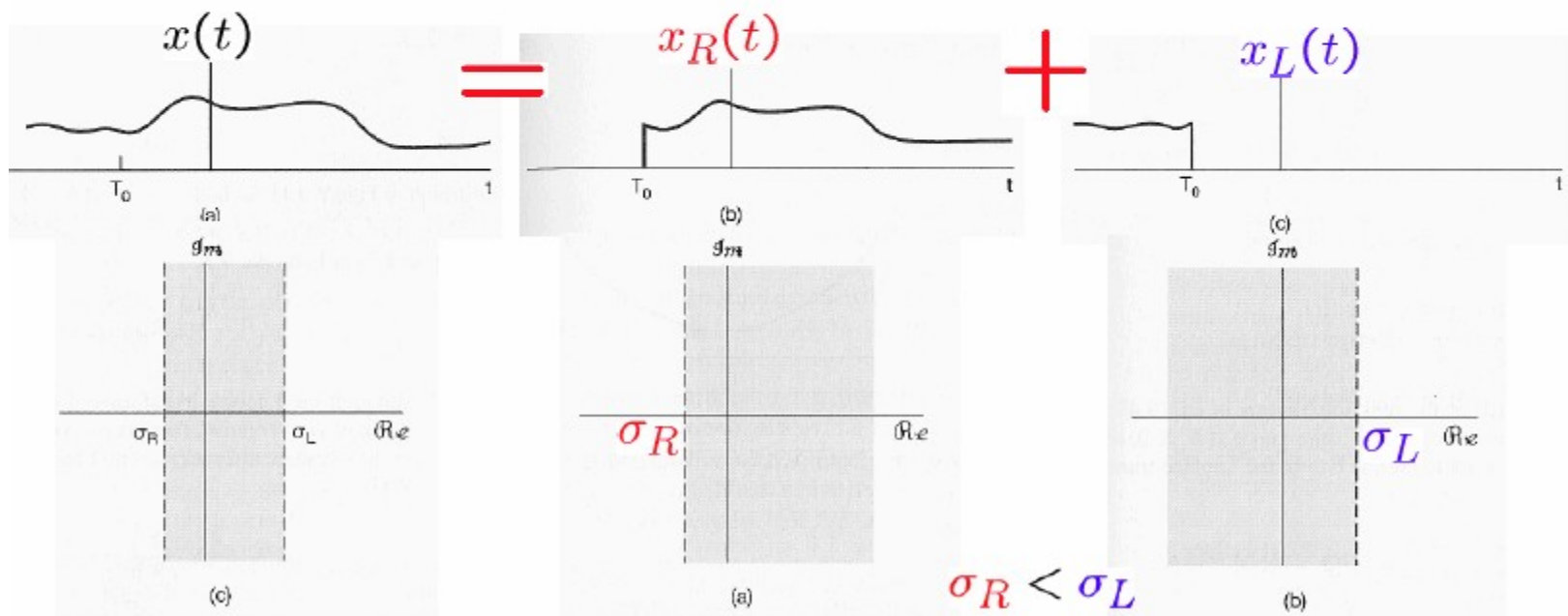
5. If $x(t)$ is left-sided, and
if the line $\text{Re}\{s\} = \sigma_0$ is in the ROC,
then all values of s for which $\text{Re}\{s\} < \sigma_0$
will also be in the ROC



The argument is the similar to that for Property 4.

- Properties of ROC:

6. If $x(t)$ is two-sided, and
if the line $\text{Re}\{s\} = \sigma_0$ is in the ROC,
then the ROC will consist of a strip in the s-plane
that includes the line $\text{Re}\{s\} = \sigma_0$



■ Example 9.7:

$$x(t) = e^{-b|t|} = e^{-bt}u(t) + e^{+bt}u(-t)$$

$$e^{-bt}u(t) \xleftrightarrow{\mathcal{L}} \frac{1}{s+b}, \quad \text{Re}\{s\} > -b$$

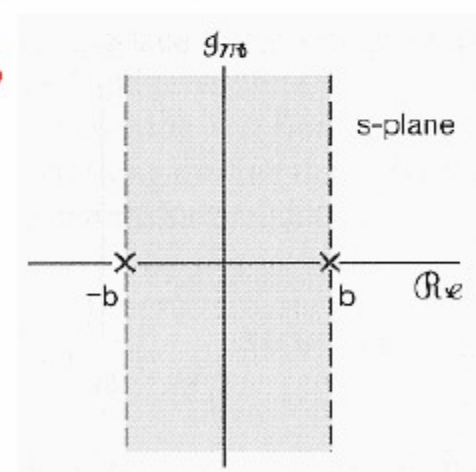
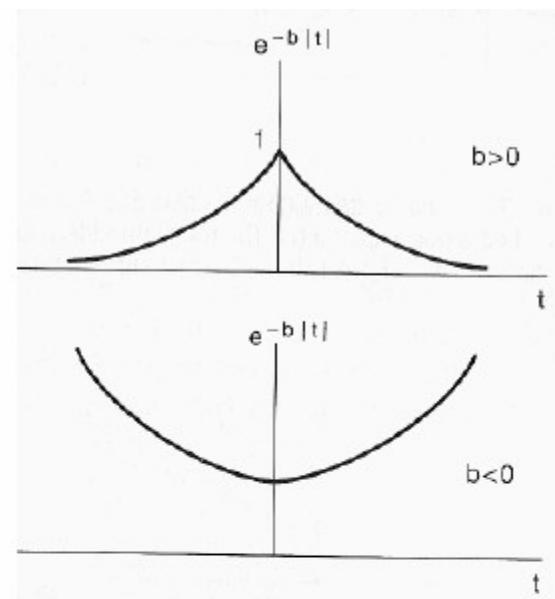
$$e^{+bt}u(-t) \xleftrightarrow{\mathcal{L}} \frac{-1}{s-b}, \quad \text{Re}\{s\} < +b$$

• $b > 0$:

$$\begin{aligned} e^{-b|t|} &\xleftrightarrow{\mathcal{L}} \frac{1}{s+b} + \frac{-1}{s-b}, \quad -b < \text{Re}\{s\} < +b \\ &= \frac{-2b}{(s+b)(s-b)} \end{aligned}$$

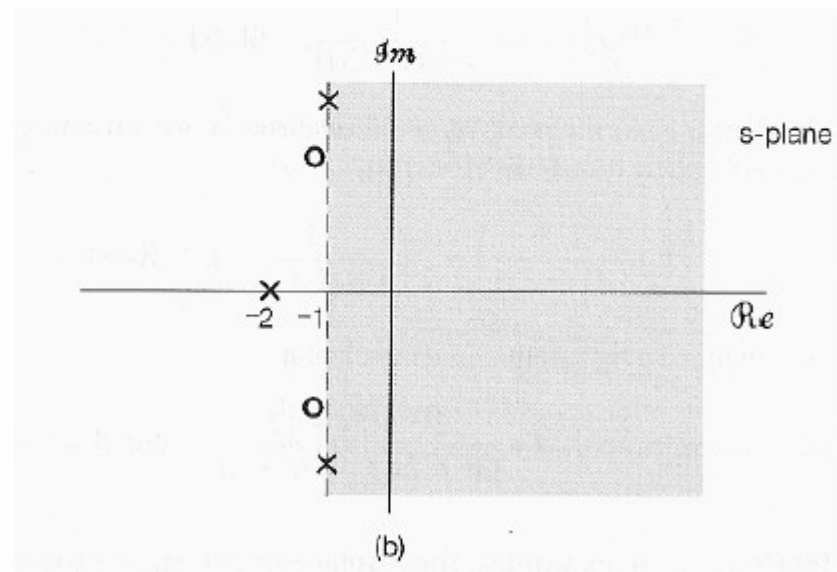
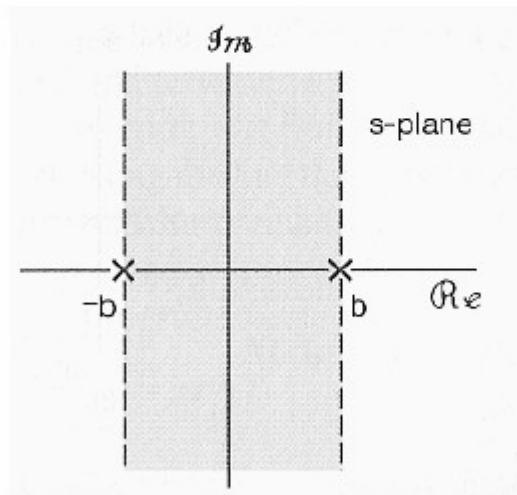
• $b \leq 0$: no common ROC

$x(t)$ has no Laplace transform



■ Properties of ROC:

7. If the Laplace transform $X(s)$ of $x(t)$ is rational, then its ROC is bounded by poles or extends to ∞ . In addition, no poles of $X(s)$ are contained in ROC



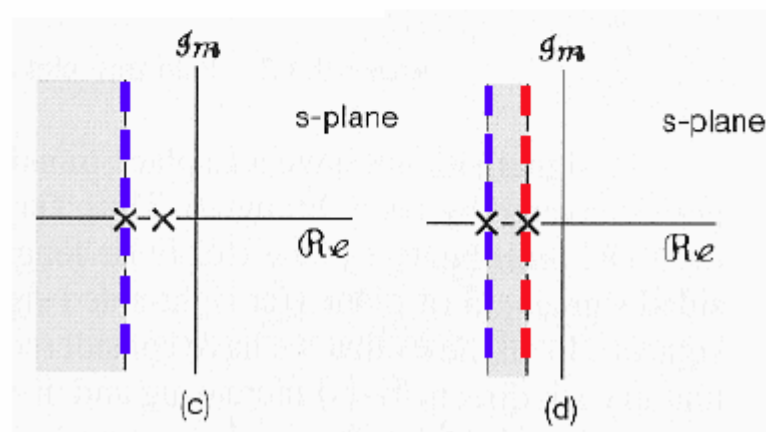
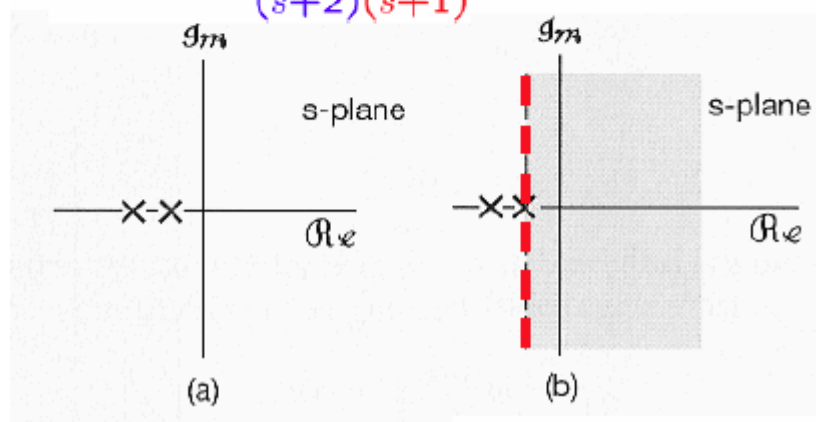
■ Properties of ROC:

8. If the Laplace transform $X(s)$ of $x(t)$ is rational

- If $x(t)$ is right-sided, the ROC is the region in the s-plane to the right of the rightmost pole
- If $x(t)$ is left-sided, the ROC is the region in the s-plane to the left of the leftmost pole

$$X(s) = \frac{1}{(s+2)(s+1)}$$

■ Which one has Fourier transform?



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▪ The Inverse Laplace Transform:

- By the use of contour integration

$$X(s) = \int_{-\infty}^{\infty} x(t)e^{-st} dt$$

$$X(\sigma + jw) = \mathcal{F} \left\{ x(t)e^{-\sigma t} \right\} = \int_{-\infty}^{\infty} x(t)e^{-\sigma t} e^{-jw t} dt$$

$\forall s = \sigma + jw$ in the ROC

$$x(t)e^{-\sigma t} = \mathcal{F}^{-1} \left\{ X(\sigma + jw) \right\} = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(\sigma + jw) e^{jw t} dw$$

$$\Rightarrow x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(\sigma + jw) e^{(\sigma + jw)t} dw \quad \begin{array}{l} s = \sigma + jw \\ ds = jdw \end{array}$$

$$\Rightarrow x(t) = \frac{1}{2\pi j} \int_{\sigma - j\infty}^{\sigma + j\infty} X(s) e^{st} ds$$

▪ The Inverse Laplace Transform:

- By the technique of partial fraction expansion

$$X(s) = \frac{A_1}{s + a_1} + \frac{A_2}{s + a_2} + \cdots + \frac{A_m}{s + a_m}$$

$$x(t) = A_1 e^{-a_1 t} u(t) - A_2 e^{-a_2 t} u(-t) + \cdots + x_m(t)$$

(if R.S.) (if L.S.)

$$e^{-at} u(t) \xleftrightarrow{\mathcal{L}} \frac{1}{s + a}, \quad \operatorname{Re}\{s\} > -a$$

$$-e^{-at} u(-t) \xleftrightarrow{\mathcal{L}} \frac{1}{s + a}, \quad \operatorname{Re}\{s\} < -a$$

■ Example 9.9:

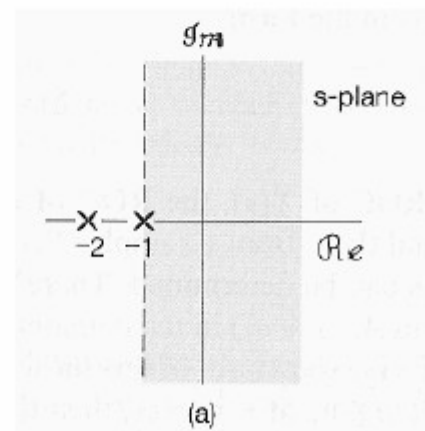
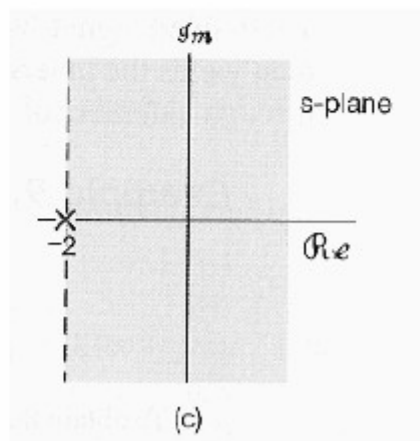
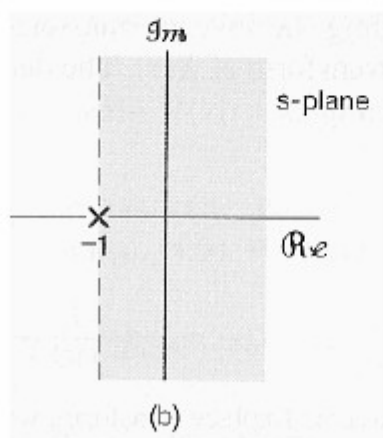
$$X(s) = \frac{1}{(s+1)(s+2)}, \quad \mathcal{R}e\{s\} > -1$$

$$= \frac{1}{(s+1)} - \frac{1}{(s+2)}$$

$$e^{-t}u(t) \xleftrightarrow{\mathcal{L}} \frac{1}{s+1}, \quad \mathcal{R}e\{s\} > -1$$

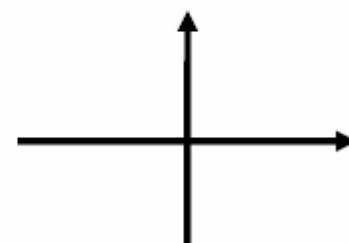
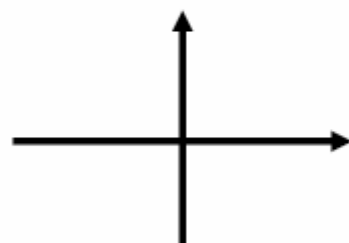
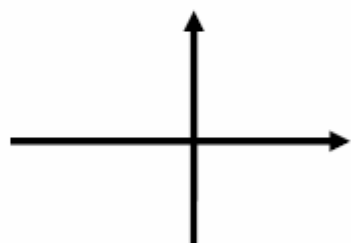
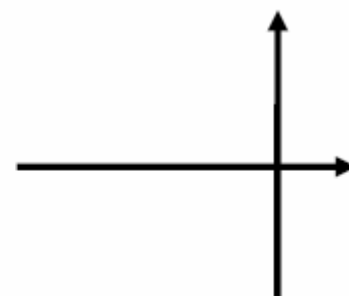
$$e^{-2t}u(t) \xleftrightarrow{\mathcal{L}} \frac{1}{s+2}, \quad \mathcal{R}e\{s\} > -2$$

$$[e^{-t} - e^{-2t}]u(t) \xleftrightarrow{\mathcal{L}} \frac{1}{(s+1)(s+2)}, \quad \mathcal{R}e\{s\} > -1$$



■ Examples 9.9, 9.10, 9.11:

	$\mathcal{R}e\{s\} < -1$	$-1 < \mathcal{R}e\{s\}$
$\frac{1}{(s+1)}$	$- e^{-t}u(-t)$	$e^{-t}u(t)$
	$\mathcal{R}e\{s\} < -2$	$-2 < \mathcal{R}e\{s\}$
$\frac{1}{(s+2)}$	$- e^{-2t}u(-t)$	$e^{-2t}u(t)$



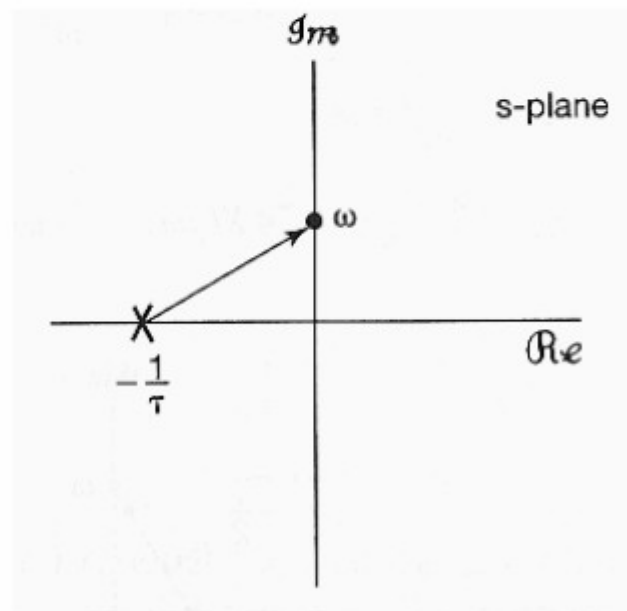
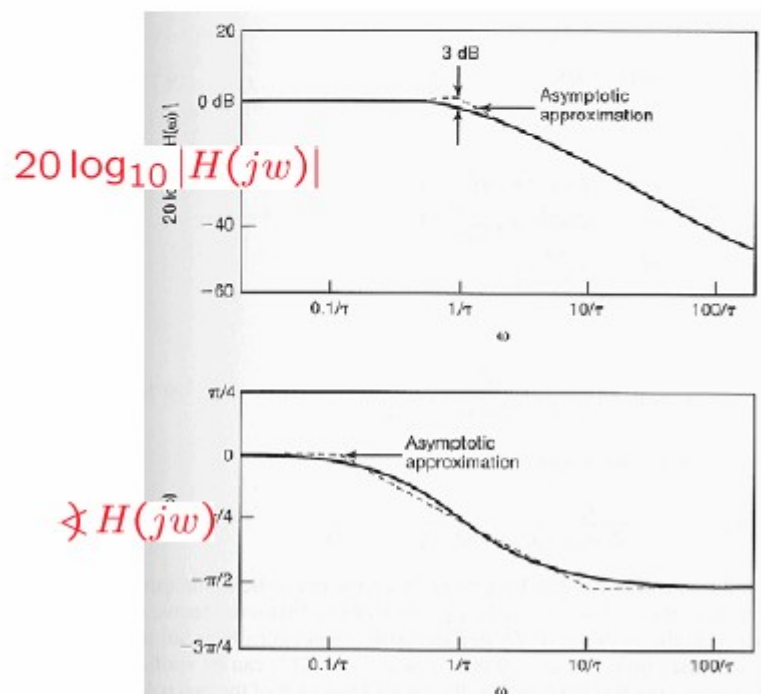
	$\mathcal{R}e\{s\} < -2$	$-2 < \mathcal{R}e\{s\} < -1$	$-1 < \mathcal{R}e\{s\}$
$\frac{1}{(s+1)} + \frac{1}{(s+2)}$	$- e^{-t}u(-t)$ $+$ $- e^{-2t}u(-t)$	$- e^{-t}u(-t)$ $+$ $e^{-2t}u(t)$	$e^{-t}u(t)$ $+$ $e^{-2t}u(t)$

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- First-Order Systems:

$$h(t) = \frac{1}{\tau} e^{-t/\tau} u(t) \xleftrightarrow{\mathcal{L}} H(s) = \frac{\frac{1}{\tau}}{s + \frac{1}{\tau}}, \quad \text{Re}\{s\} > -\frac{1}{\tau}$$

$$h(t) = \frac{1}{\tau} e^{-t/\tau} u(t) \xleftrightarrow{\mathcal{F}} H(j\omega) = \frac{\frac{1}{\tau}}{j\omega + \frac{1}{\tau}}$$



▪ Second-Order Systems:

$$h(t) = M [e^{c_1 t} - e^{c_2 t}] u(t)$$

$$c_1 = -\zeta w_n + w_n \sqrt{\zeta^2 - 1}$$

$$c_2 = -\zeta w_n - w_n \sqrt{\zeta^2 - 1}$$

$$M = \frac{w_n}{2\sqrt{\zeta^2 - 1}}$$

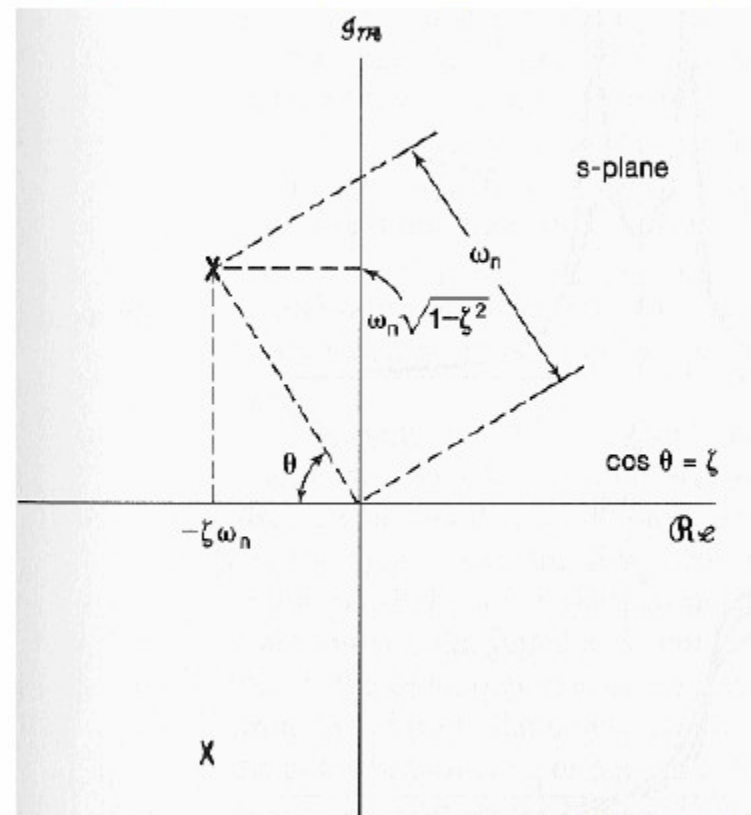
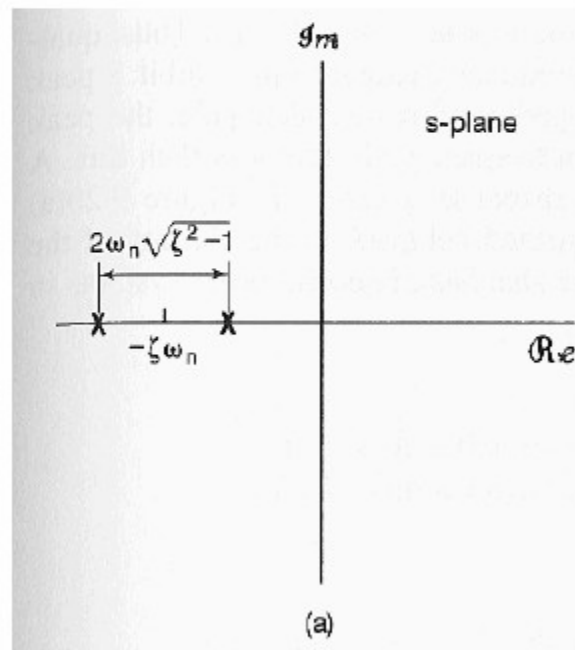
$$\Rightarrow H(jw) = \frac{w_n^2}{(jw)^2 + 2\zeta w_n(jw) + w_n^2}$$

$$\Rightarrow H(s) = \frac{w_n^2}{s^2 + 2\zeta w_n s + w_n^2} = \frac{w_n^2}{(s - c_1)(s - c_2)}$$

- $\zeta > 1$: c_1 & c_2 are real
- $0 < \zeta < 1$: c_1 & c_2 are complex

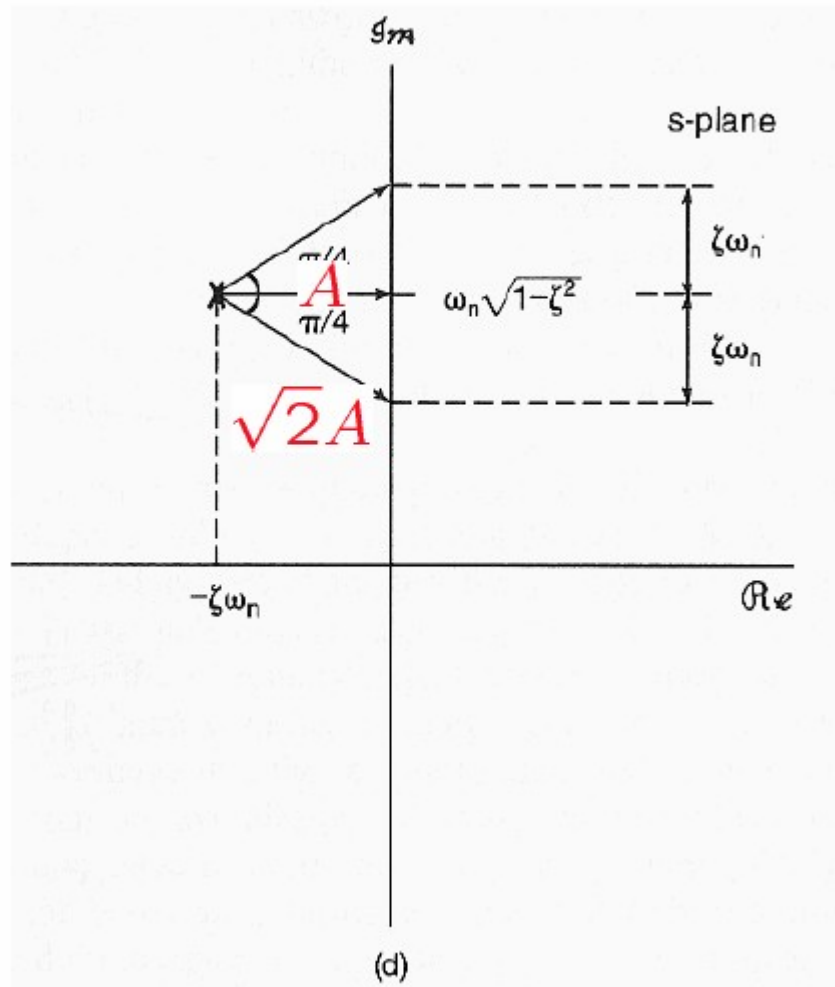
▪ Pole Locations:

- $\zeta > 1$: c_1 & c_2 are real
- $0 < \zeta < 1$: c_1 & c_2 are complex



minimum length at $w = w_n\sqrt{1-\zeta^2}!!!$

■ Relative Bandwidth B:



$$|H(j\omega)|_{\omega=\omega_n\sqrt{1-\zeta^2}}$$

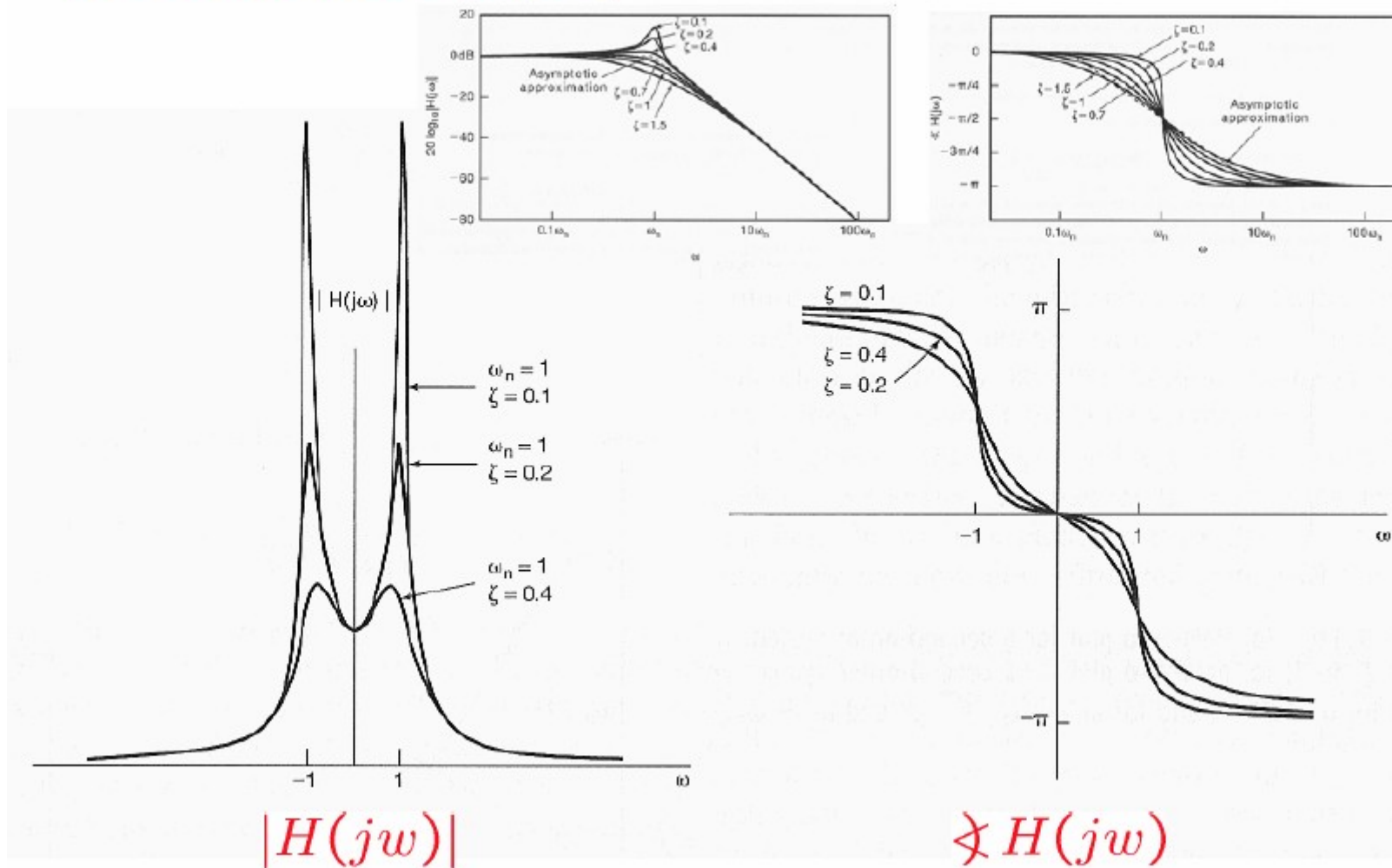
$$|H(j\omega)|_{\omega=\omega_n\sqrt{1-\zeta^2}\pm\zeta\omega_n}$$

$$\approx \text{ or } \leq \frac{1}{\sqrt{2}}$$

$$\Rightarrow B = 2\zeta$$

$$\Rightarrow \Delta \angle H(j\omega) = \frac{\pi}{2}$$

■ Frequency Response:



- The Nth-Order Systems:

$$H(jw) = \left(\frac{\frac{1}{\tau}}{jw + \frac{1}{\tau}} \right) \prod_i \left(\frac{w_{ni}^2}{(jw)^2 + 2\zeta_i w_{ni}(jw) + w_{ni}^2} \right)$$

$$H(s) = \left(\frac{b}{s - a} \right) \left(\prod_i \frac{w_{ni}^2}{s^2 + 2\zeta_i w_{ni}s + w_{ni}^2} \right)$$

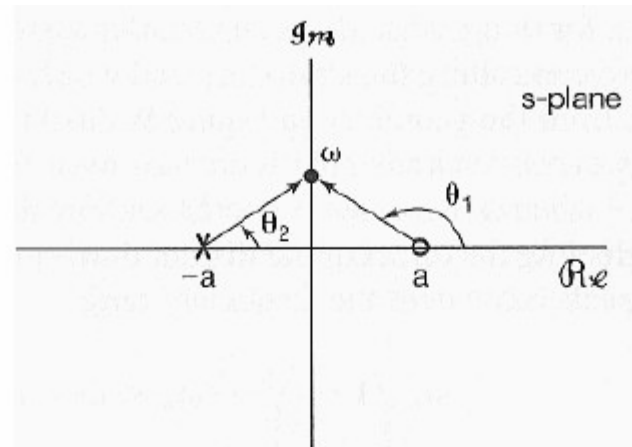
$$|H(jw)| = \prod_i |H_i(jw)|$$

$$\angle H(jw) = \sum_i \angle H_i(jw)$$

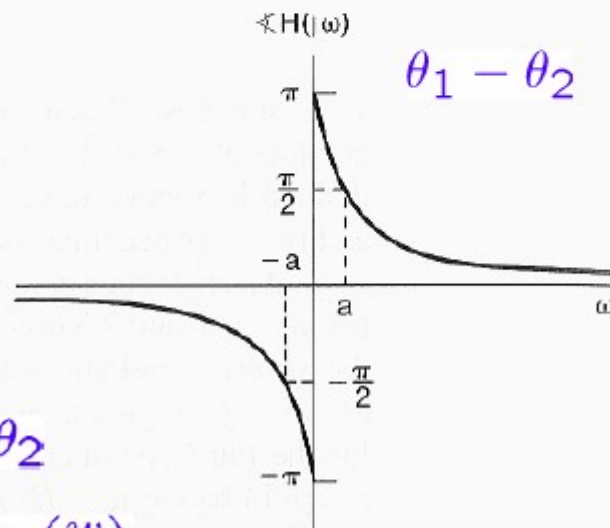
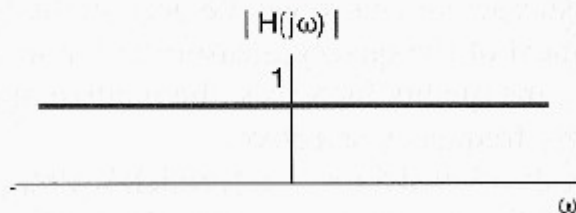
■ All-Pass Systems:

$$H(s) = \frac{s - a}{s + a},$$

$$H(j\omega) = \frac{j\omega - a}{j\omega + a},$$



$$|H(j\omega)| = 1$$



$$\begin{aligned} \angle H(j\omega) &= \theta_1 - \theta_2 = (\pi - \theta_2) - \theta_2 \\ &= \pi - 2\theta_2 = \pi - 2 \tan^{-1} \left(\frac{\omega}{a} \right) \end{aligned}$$

- The Laplace Transform
- The Region of Convergence for Laplace Transforms
- The Inverse Laplace Transform
- Geometric Evaluation of the Fourier Transform
- Properties of the Laplace Transform
- Some Laplace Transform Pairs
- Analysis & Characterization of LTI Systems Using the Laplace Transform
- System Function Algebra and Block Diagram Representations
- The Unilateral Laplace Transform

Section	Property
9.5.1	Linearity
9.5.2	Time Shifting
9.5.3	Shifting in the s-Domain
9.5.4	Time Scaling
9.5.5	Conjugation
9.5.6	Convolution
9.5.7	Differentiation in the Time Domain
9.5.8	Differentiation in the s-Domain
9.5.9	Integration in the Time Domain
9.5.10	Initial- and Final-Value Theorems

■ Linearity of the Laplace Transform:

$$X(s) = \int_{-\infty}^{\infty} x(t) e^{-st} dt$$

$$x(t) = \frac{1}{2\pi j} \int_{\sigma-j\infty}^{\sigma+j\infty} X(s) e^{st} ds$$

$$x_1(t) \xleftrightarrow{\mathcal{L}} X_1(s), \quad ROC = R_1$$

$$x_2(t) \xleftrightarrow{\mathcal{L}} X_2(s), \quad ROC = R_2$$

$$a x_1(t) + b x_2(t) \xleftrightarrow{\mathcal{L}} a X_1(s) + b X_2(s),$$

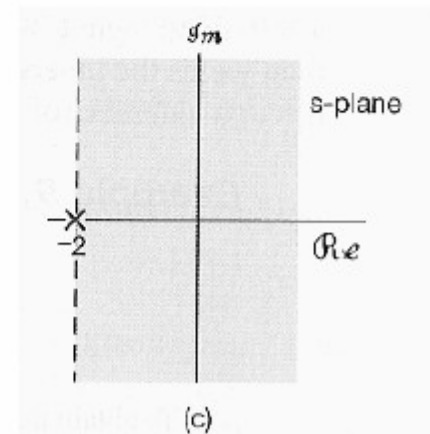
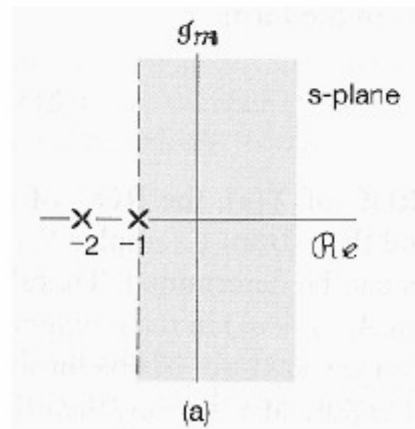
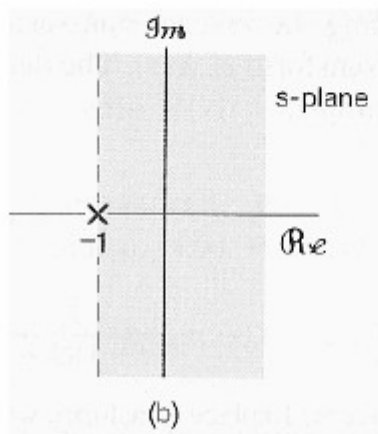
with ROC containing $R_1 \cap R_2$

- Example 9.13: $x(t) = x_1(t) - x_2(t)$

$$X_1(s) = \frac{1}{(s+1)}, \quad \mathcal{R}e\{s\} > -1$$

$$X_2(s) = \frac{1}{(s+1)(s+2)}, \quad \mathcal{R}e\{s\} > -1$$

$$X(s) = \frac{1}{(s+1)} - \frac{1}{(s+1)(s+2)} = \frac{1}{(s+2)}$$



■ Time Shifting:

$$x(t) \xleftrightarrow{\mathcal{L}} X(s), \quad ROC = R$$

$$X(s) = \int_{-\infty}^{\infty} x(t) e^{-st} dt$$

$$x(t) = \frac{1}{2\pi j} \int_{\sigma-j\infty}^{\sigma+j\infty} X(s) e^{st} ds$$

$$x(t-t_0) \xleftrightarrow{\mathcal{L}} e^{-st_0} X(s), \quad ROC = R$$

$$X_0(s) = \int_{-\infty}^{\infty} x(t-t_0) e^{-st} dt$$

▪ Shifting in the s-Domain:

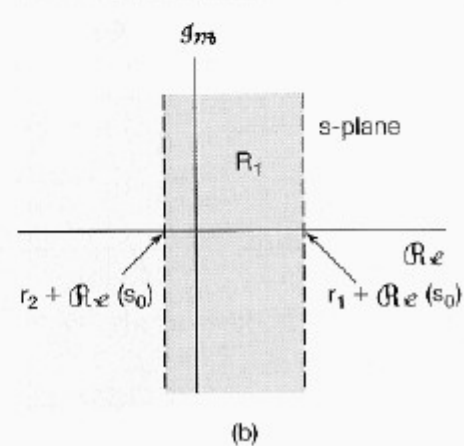
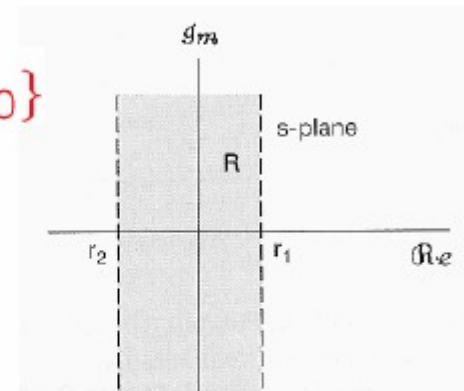
$$x(t) \xleftrightarrow{\mathcal{L}} X(s), \quad \text{ROC} = R$$

$$X(s) = \int_{-\infty}^{\infty} x(t) e^{-st} dt$$

$$x(t) = \frac{1}{2\pi j} \int_{\sigma-j\infty}^{\sigma+j\infty} X(s) e^{st} ds$$

$$e^{s_0 t} x(t) \xleftrightarrow{\mathcal{L}} X(s - s_0), \quad \text{ROC} = R + \mathcal{R}e\{s_0\}$$

$$X(s - s_0) = \int_{-\infty}^{\infty} x(t) e^{-(s - s_0)t} dt$$

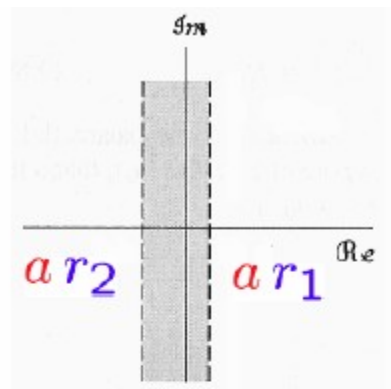
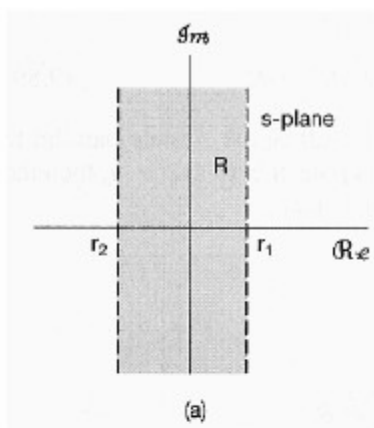


▪ Time Scaling:

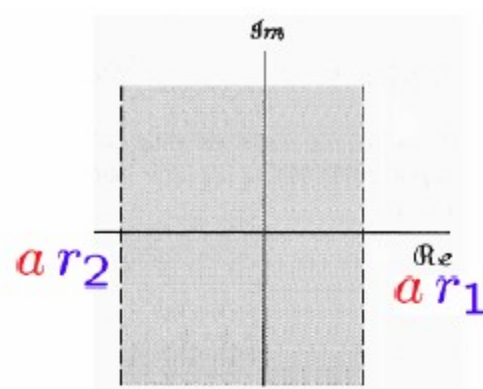
$$x(t) \xleftrightarrow{\mathcal{L}} X(s), \quad ROC = R$$

$$x(at) \xleftrightarrow{\mathcal{L}} \frac{1}{|a|} X\left(\frac{s}{a}\right), \quad ROC = aR$$

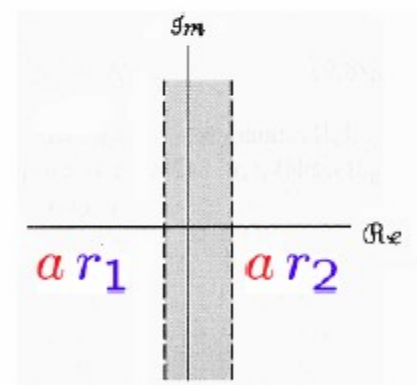
$$X_a(s) = \int_{-\infty}^{\infty} x(at) e^{-st} dt$$



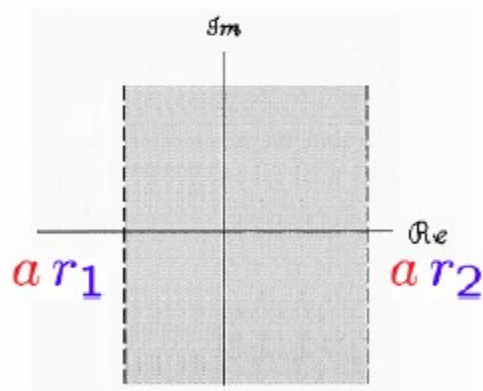
$$0 < a < 1$$



$$1 < a$$



$$-1 < a < 0$$



$$a < -1$$

$$x(-t) \xleftrightarrow{\mathcal{L}} X(-s), \quad ROC = -R$$

- Conjugation:

$$x(t) \xleftrightarrow{\mathcal{L}} X(s), \quad ROC = R$$

$$x^*(t) \xleftrightarrow{\mathcal{L}} X^*(s^*), \quad ROC = R$$

$$X(s) = \int_{-\infty}^{\infty} x(t) e^{-st} dt$$

$$X(s) = \int_{-\infty}^{\infty} x(t) e^{-st} dt$$

$$x(t) = \frac{1}{2\pi j} \int_{\sigma-j\infty}^{\sigma+j\infty} X(s) e^{st} ds$$

■ Convolution Property:

$$X(s) = \int_{-\infty}^{\infty} x(t) e^{-st} dt$$

$$x_1(t) \xleftrightarrow{\mathcal{L}} X_1(s), \quad ROC = R_1$$

$$x_2(t) \xleftrightarrow{\mathcal{L}} X_2(s), \quad ROC = R_2$$

$$x_1(t) * x_2(t) \xleftrightarrow{\mathcal{L}} X_1(s) X_2(s), \quad ROC \text{ containing } R_1 \cap R_2$$

$$\int_{-\infty}^{\infty} x_1(\tau) x_2(t - \tau) d\tau$$

- Differentiation in the Time & s-Domain:

$$x(t) \xleftrightarrow{\mathcal{L}} X(s), \quad ROC = R \qquad X(s) = \int_{-\infty}^{\infty} x(t) e^{-st} dt$$

$$\frac{d}{dt}x(t) \xleftrightarrow{\mathcal{L}} sX(s), \quad ROC \text{ containing } R \qquad x(t) = \frac{1}{2\pi j} \int_{\sigma-j\infty}^{\sigma+j\infty} X(s) e^{st} ds$$

$$-tx(t) \xleftrightarrow{\mathcal{L}} \frac{d}{ds}X(s), \quad ROC = R$$

$$x(t) = \frac{1}{2\pi j} \int_{\sigma-j\infty}^{\sigma+j\infty} X(s) e^{st} ds$$

$$X(s) = \int_{-\infty}^{\infty} x(t) e^{-st} dt$$

- Integration in the Time Domain:

$$x(t) \xleftrightarrow{\mathcal{L}} X(s), \quad ROC = R$$

$$\int_{-\infty}^t x(\tau) d\tau \xleftrightarrow{\mathcal{L}} \frac{1}{s} X(s), \quad ROC \text{ containing } R \cap \{\operatorname{Re}\{s\} > 0\}$$

$$\int_{-\infty}^t x(\tau) d\tau = x(t) * u(t)$$

- The Initial-Value Theorem:

$$\text{If } x(t) = 0 \text{ for } t < 0 \quad \Rightarrow \quad x(0^+) = \lim_{s \rightarrow \infty} sX(s)$$

- The Final-Value Theorem:

$$\text{If } x(t) = 0 \text{ for } t < 0 \text{ and } x(t) \text{ has a finite limit as } t \rightarrow \infty,$$

$$\Rightarrow \lim_{t \rightarrow \infty} x(t) = \lim_{s \rightarrow 0} sX(s)$$

$$\mathcal{L}\left\{\frac{d}{dt}x(t)\right\} = \int_0^\infty \frac{d}{dt}x(t) e^{-st} dt = sX(s) - x(0^+)$$

$$\lim_{s \rightarrow \infty} \int_0^\infty \frac{d}{dt}x(t) e^{-st} dt = \lim_{s \rightarrow \infty} \{sX(s) - x(0^+)\}$$

$$\lim_{s \rightarrow 0} \int_0^\infty \frac{d}{dt}x(t) e^{-st} dt = \lim_{s \rightarrow 0} \{sX(s) - x(0^+)\}$$

TABLE 9.1 PROPERTIES OF THE LAPLACE TRANSFORM

Section	Property	Signal	Laplace Transform	ROC
		$x(t)$	$X(s)$	R
		$x_1(t)$	$X_1(s)$	R_1
		$x_2(t)$	$X_2(s)$	R_2
9.5.1	Linearity	$ax_1(t) + bx_2(t)$	$aX_1(s) + bX_2(s)$	At least $R_1 \cap R_2$
9.5.2	Time shifting	$x(t - t_0)$	$e^{-st_0} X(s)$	R
9.5.3	Shifting in the s -Domain	$e^{t_0 t} x(t)$	$X(s - s_0)$	Shifted version of R (i.e., s is in the ROC if $s - s_0$ is in R)
9.5.4	Time scaling	$x(at)$	$\frac{1}{ a } X\left(\frac{s}{a}\right)$	Scaled ROC (i.e., s is in the ROC if s/a is in R)
9.5.5	Conjugation	$x^*(t)$	$X^*(s^*)$	R
9.5.6	Convolution	$x_1(t) * x_2(t)$	$X_1(s)X_2(s)$	At least $R_1 \cap R_2$
9.5.7	Differentiation in the Time Domain	$\frac{d}{dt} x(t)$	$sX(s)$	At least R
9.5.8	Differentiation in the s -Domain	$-tx(t)$	$\frac{d}{ds} X(s)$	R
9.5.9	Integration in the Time Domain	$\int_{-\infty}^t x(\tau) d\tau$	$\frac{1}{s} X(s)$	At least $R \cap \{\operatorname{Re}\{s\} > 0\}$
Initial- and Final-Value Theorems				
9.5.10	If $x(t) = 0$ for $t < 0$ and $x(t)$ contains no impulses or higher-order singularities at $t = 0$, then			
			$x(0^+) = \lim_{s \rightarrow \infty} sX(s)$	
	If $x(t) = 0$ for $t < 0$ and $x(t)$ has a finite limit as $t \rightarrow \infty$, then			
			$\lim_{t \rightarrow \infty} x(t) = \lim_{s \rightarrow 0} sX(s)$	

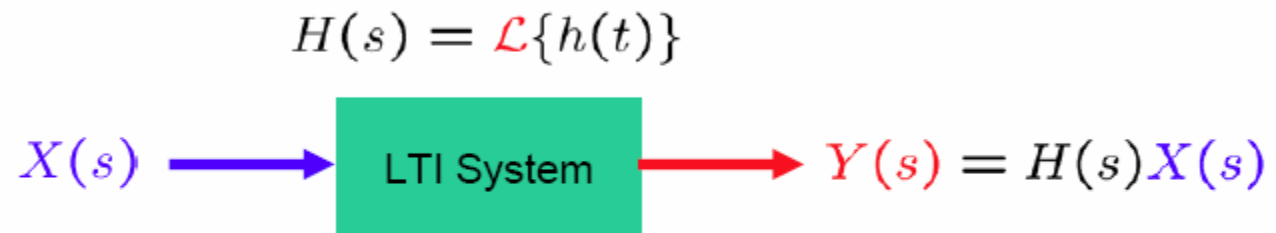
- The Laplace Transform
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TABLE 9.2 LAPLACE TRANSFORMS OF ELEMENTARY FUNCTIONS

Transform pair	Signal	Transform	ROC
1	$\delta(t)$	1	All s
2	$u(t)$	$\frac{1}{s}$	$\Re\{s\} > 0$
3	$-u(-t)$	$\frac{1}{s}$	$\Re\{s\} < 0$
4	$\frac{t^{n-1}}{(n-1)!} u(t)$	$\frac{1}{s^n}$	$\Re\{s\} > 0$
5	$-\frac{t^{n-1}}{(n-1)!} u(-t)$	$\frac{1}{s^n}$	$\Re\{s\} < 0$
6	$e^{-\alpha t} u(t)$	$\frac{1}{s + \alpha}$	$\Re\{s\} > -\alpha$
7	$-e^{-\alpha t} u(-t)$	$\frac{1}{s + \alpha}$	$\Re\{s\} < -\alpha$
8	$\frac{t^{n-1}}{(n-1)!} e^{-\alpha t} u(t)$	$\frac{1}{(s + \alpha)^n}$	$\Re\{s\} > -\alpha$
9	$-\frac{t^{n-1}}{(n-1)!} e^{-\alpha t} u(-t)$	$\frac{1}{(s + \alpha)^n}$	$\Re\{s\} < -\alpha$
10	$\delta(t - T)$	e^{-sT}	All s
11	$[\cos \omega_0 t] u(t)$	$\frac{s}{s^2 + \omega_0^2}$	$\Re\{s\} > 0$
12	$[\sin \omega_0 t] u(t)$	$\frac{\omega_0}{s^2 + \omega_0^2}$	$\Re\{s\} > 0$
13	$[e^{-\alpha t} \cos \omega_0 t] u(t)$	$\frac{s + \alpha}{(s + \alpha)^2 + \omega_0^2}$	$\Re\{s\} > -\alpha$
14	$[e^{-\alpha t} \sin \omega_0 t] u(t)$	$\frac{\omega_0}{(s + \alpha)^2 + \omega_0^2}$	$\Re\{s\} > -\alpha$
15	$u_n(t) = \frac{d^n \delta(t)}{dt^n}$	s^n	All s
16	$u_{-n}(t) = \underbrace{u(t) * \cdots * u(t)}_{n \text{ times}}$	$\frac{1}{s^n}$	$\Re\{s\} > 0$

- The Laplace Transform
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▪ Analysis & Characterization of LTI Systems:

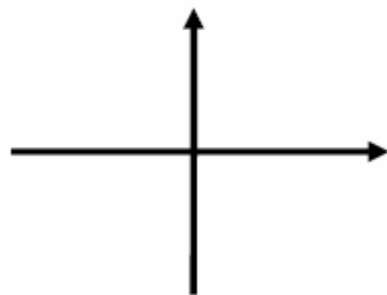


$H(s)$: system function
or transfer function

- Causality
- Stability

■ Causality:

- For a **causal** LTI system,
 $h(t) = 0$ for $t < 0$, and thus is **right sided**
- The **ROC** associated with the system function for a **causal** system is a **right-half plane**
- For a system with a **rational** system function, **causality** of the system is equivalent to the **ROC** being the **right-half plane** to the **right** of the **rightmost pole**



▪ Examples 9.17, 9.18, 9.19:

$$h(t) = e^{-t}u(t) \quad \xleftrightarrow{\mathcal{L}} \quad H(s) = \frac{1}{s+1}, \quad -1 < \mathcal{R}e\{s\}$$

$$h(t) = e^{-|t|} \quad \xleftrightarrow{\mathcal{L}} \quad H(s) = \frac{-2}{s^2 - 1}, \quad -1 < \mathcal{R}e\{s\} < +1$$

$$h(t) = e^{-(t+1)}u(t+1) \quad \xleftrightarrow{\mathcal{L}} \quad H(s) = \frac{e^s}{s+1}, \quad -1 < \mathcal{R}e\{s\}$$

$$\begin{cases} h(t) : & \text{causal} \\ H(s) : & \text{rational} \\ ROC : & \text{right-sided} \end{cases}$$

$$\begin{cases} h(t) : & \text{not causal} \\ H(s) : & \text{rational} \\ ROC : & \text{not right-sided} \end{cases}$$

$$\begin{cases} h(t) : & \text{not causal} \\ H(s) : & \text{not rational} \\ ROC : & \text{right-sided} \end{cases}$$

▪ Anti-causality:

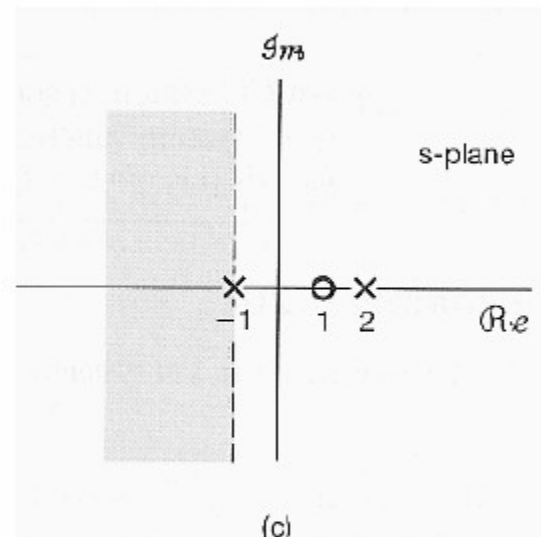
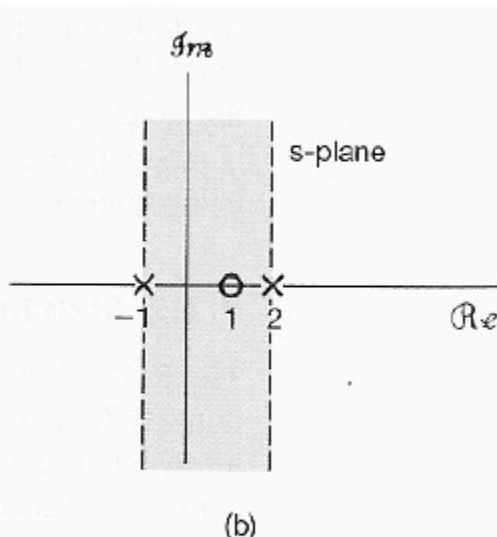
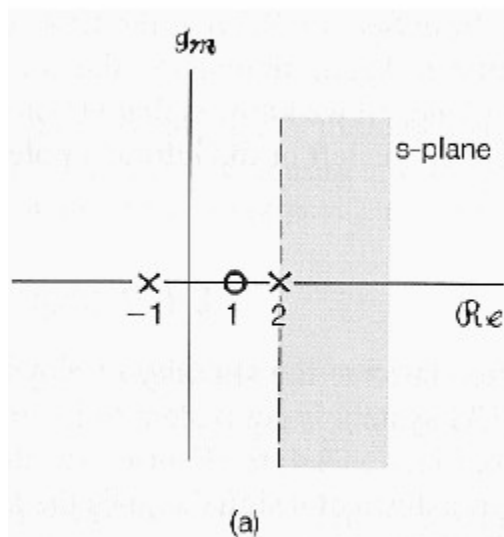
- For a **anti-causal** LTI system,
 $h(t) = 0$ for $t > 0$, and thus is **left sided**
- The **ROC** associated with the system function for a **anti-causal** system is a **left-half plane**
- For a system with a **rational** system function, **anti-causality** of the system is equivalent to the **ROC** being
the **left-half plane** to the **left** of the **leftmost pole**

- Stability:

- An LTI system is stable if and only if the ROC of its system function $H(s)$ includes the entire $j\omega$ -axis [i.e., $\text{Re}\{s\} = 0$]

■ Example 9.20:

$$H(s) = \frac{s - 1}{(s + 1)(s - 2)}$$



$$v(t) = \left(\frac{2}{3}e^{-t} + \frac{1}{3}e^{2t} \right) u(t)$$

$$h(t) = \frac{2}{3}e^{-t}u(t) - \frac{1}{3}e^{2t}u(-t)$$

$$h(t) = - \left(\frac{2}{3}e^{-t} + \frac{1}{3}e^{2t} \right) u(-t)$$

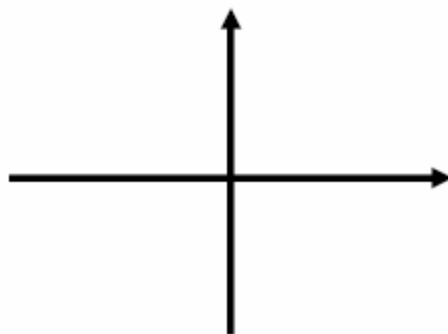
causal, (unstable)

stable, (not causal)

unstable, anticausal

- Stability:

- A **causal** system with **rational** system function $H(s)$ is **stable** if and only if
all of the **poles** of $H(s)$ lie in the **left-half** of s-plane,
i.e., all of the poles have **negative real parts**



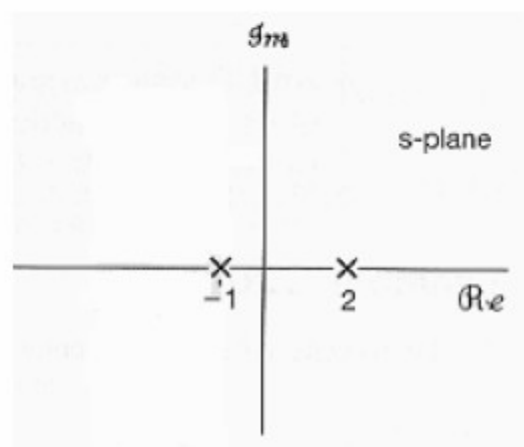
■ Examples 9.17, 9.21:

$$h(t) = e^{-t}u(t) \xleftrightarrow{\mathcal{L}} H(s) = \frac{1}{s+1}, \quad -1 < \operatorname{Re}\{s\}$$

$$h(t) = e^{2t}u(t) \xleftrightarrow{\mathcal{L}} H(s) = \frac{1}{s-2}, \quad 2 < \operatorname{Re}\{s\}$$

$$\begin{cases} h(t) : & \text{causal} \\ H(s) : & \text{stable, rational} \end{cases}$$

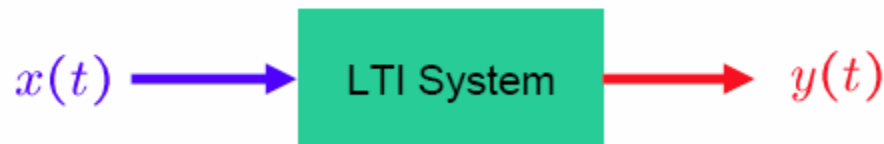
$$\begin{cases} h(t) : & \text{causal} \\ H(s) : & \text{unstable, rational} \end{cases}$$



▪ LTI Systems by Linear Constant-Coef Differential Equations:

$$\begin{aligned} a_N \frac{d^N y(t)}{dt^N} + a_{N-1} \frac{d^{N-1} y(t)}{dt^{N-1}} + \dots + a_1 \frac{dy(t)}{dt} + a_0 y(t) \\ = b_M \frac{d^M x(t)}{dt^M} + b_{M-1} \frac{d^{M-1} x(t)}{dt^{M-1}} + \dots + b_1 \frac{dx(t)}{dt} + b_0 x(t) \end{aligned}$$

$$\sum_{k=0}^N a_k \frac{d^k y(t)}{dt^k} = \sum_{k=0}^M b_k \frac{d^k x(t)}{dt^k}$$



$$Y(s) = X(s)H(s) \qquad H(s) = \frac{Y(s)}{X(s)}$$

$$\mathcal{L} \left\{ \sum_{k=0}^N a_k \frac{d^k y(t)}{dt^k} \right\} = \mathcal{L} \left\{ \sum_{k=0}^M b_k \frac{d^k x(t)}{dt^k} \right\}$$

$$\sum_{k=0}^N a_k \mathcal{L} \left\{ \frac{d^k y(t)}{dt^k} \right\} = \sum_{k=0}^M b_k \mathcal{L} \left\{ \frac{d^k x(t)}{dt^k} \right\}$$

$$\sum_{k=0}^N a_k s^k Y(s) = \sum_{k=0}^M b_k s^k X(s)$$

$$Y(s) \left[\sum_{k=0}^N a_k s^k \right] = X(s) \left[\sum_{k=0}^M b_k s^k \right]$$

$$\Rightarrow H(s) = \frac{Y(s)}{X(s)} = \frac{\sum_{k=0}^M b_k s^k}{\sum_{k=0}^N a_k s^k} = \frac{b_M s^M + \dots + b_1 s + b_0}{a_N s^N + \dots + a_1 s + a_0}$$

zeros

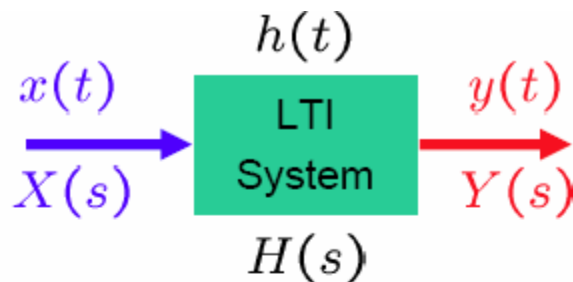
poles

■ Example 9.23:

$$\frac{dy(t)}{dt} + 3y(t) = x(t)$$

$$\Rightarrow sY(s) + 3Y(s) = X(s)$$

$$\Rightarrow (s + 3)Y(s) = X(s)$$

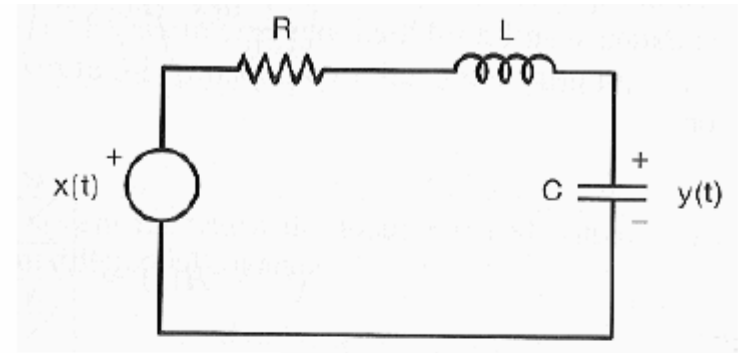


$$H(s) = \frac{Y(s)}{X(s)}$$

$$\Rightarrow H(s) = \frac{1}{s + 3}$$

- If **causal**, $\Rightarrow \mathcal{R}\{s\} > -3$, $\Rightarrow h(t) = e^{-3t}u(t)$
- If **anti-causal**, $\Rightarrow \mathcal{R}\{s\} < -3$, $\Rightarrow h(t) = -e^{-3t}u(-t)$

■ Example 9.24:



$$LC \frac{d^2 y(t)}{dt^2} + RC \frac{dy(t)}{dt} + y(t) = x(t)$$

$$\Rightarrow H(s) = \frac{\left(\frac{1}{LC}\right)}{s^2 + \left(\frac{R}{L}\right)s + \left(\frac{1}{LC}\right)} = \frac{\left(\frac{1}{LC}\right)}{(s-a)(s-b)}$$

- If $R, L, C > 0$, $\Rightarrow \mathcal{Re}\{a\}, \mathcal{Re}\{b\} < 0$

i.e., poles with negative real parts

▪ Example 9.25:

?



$$X(s) = \frac{1}{s+3}, \quad -3 < \mathcal{Re}\{s\}$$

$$Y(s) = \frac{1}{(s+1)(s+2)}, \quad -1 < \mathcal{Re}\{s\}$$

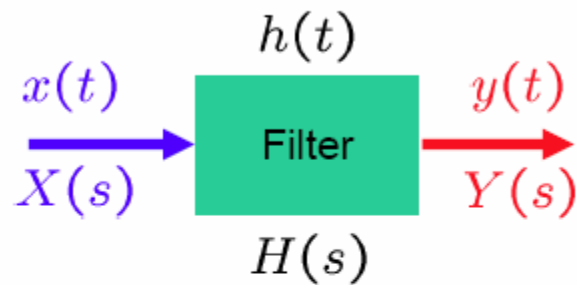
$$\Rightarrow H(s) = \frac{Y(s)}{X(s)} = \frac{s+3}{(s+1)(s+2)} = \frac{s+3}{s^2+3s+2}$$

$$\text{ROC} : -1 < \mathcal{Re}\{s\}$$

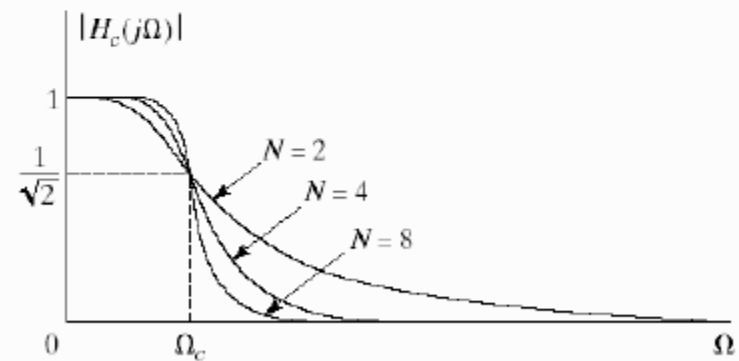
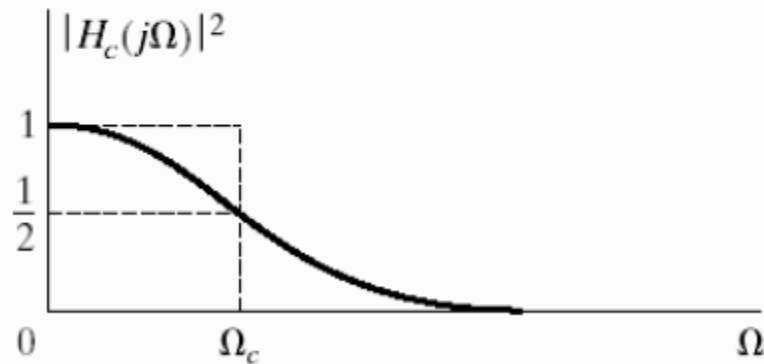
$\Rightarrow$ casual, stable

$$\Rightarrow \frac{d^2 y(t)}{dt^2} + 3 \frac{dy(t)}{dt} + 2 y(t) = \frac{dx(t)}{dt} + 3 x(t) \quad \begin{cases} y(0^-) = 0 \\ x(0^-) = 0 \end{cases}$$

■ Butterworth Lowpass Filters:



$$\left| H_c(j\Omega) \right|^2 = \frac{1}{1 + \left(\frac{j\Omega}{j\Omega_c} \right)^{2N}}$$



■ An Nth-Order Lowpass Butterworth Filters:

$$|B(jw)|^2 = \frac{1}{1 + (jw/jw_c)^{2N}}$$

- If impulse response is **real**, $\Rightarrow B^*(jw) = B(-jw)$

$$|B(jw)|^2 = B(jw)B^*(jw) = B(jw)B(-jw)$$

$$\Rightarrow B(jw) B(-jw) = \frac{1}{1 + (jw/jw_c)^{2N}}$$

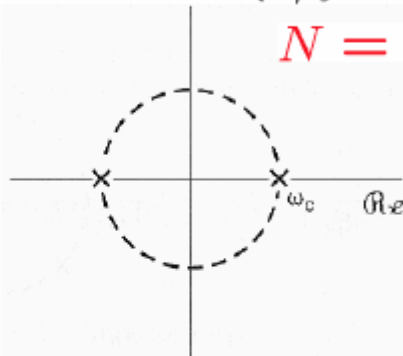
$$\Rightarrow B(s) B(-s) = \frac{1}{1 + (s/jw_c)^{2N}} \quad \text{at } s_p = (-1)^{1/2N}(jw_c)$$

$$\Rightarrow \begin{cases} |s_p| = w_c \\ \angle s_p = \frac{\pi(2k+1)}{2N} + \frac{\pi}{2} \end{cases} \Rightarrow s_p = w_c \exp \left(j \left[\frac{\pi(2k+1)}{2N} + \frac{\pi}{2} \right] \right)$$

- An Nth-Order Lowpass Butterworth Filters:

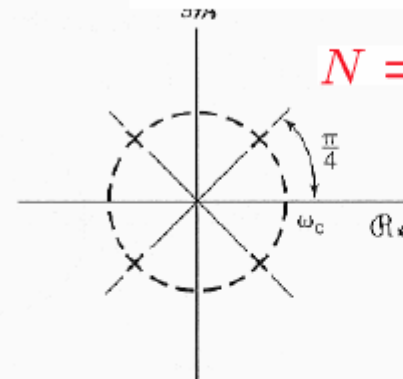
$$B(s) B(-s) = \frac{1}{1 + (s/j\omega_c)^{2N}}$$

$N = 1$

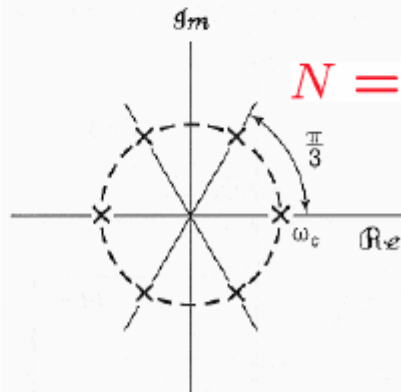


$$s_p = \omega_c \exp\left(j\left[\frac{\pi(2k+1)}{2N} + \frac{\pi}{2}\right]\right)$$

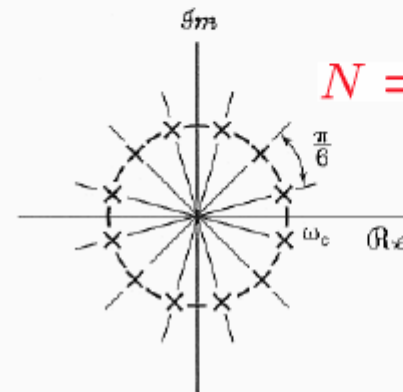
$N = 2$



$N = 3$

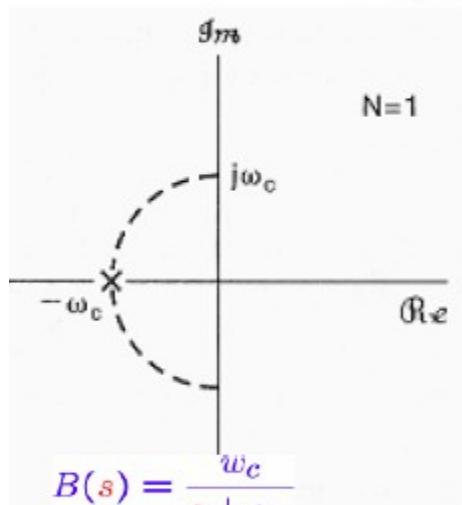


$N = 6$



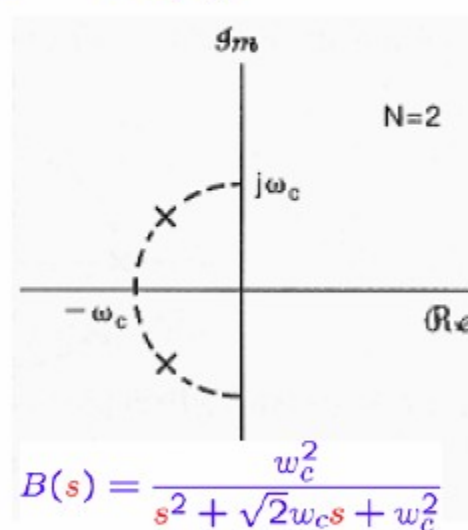
■ An Nth-Order Lowpass Butterworth Filters:

- Both $s = s_p$ and $s = -s_p$ are poles of $B(s)B(-s)$
- If the system is **stable** & **causal**
 $\Rightarrow$ **poles** of $H(s)$ are in the **left-hand** plane



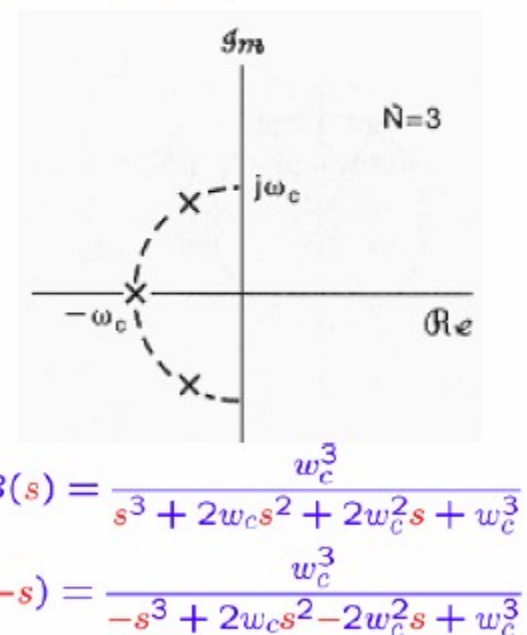
$$B(-s) = \frac{w_c}{-s + w_c}$$

$$\frac{dy(t)}{dt} + w_c y(t) = w_c x(t)$$



$$B(-s) = \frac{w_c^2}{s^2 - \sqrt{2}w_c s + w_c^2}$$

$$\frac{d^2 y(t)}{dt^2} + \sqrt{2}w_c \frac{dy(t)}{dt} + w_c^2 y(t) = w_c^2 x(t)$$



$$B(-s) = \frac{w_c^3}{-s^3 + 2w_c s^2 - 2w_c^2 s + w_c^3}$$

■ An Nth-Order Lowpass Butterworth Filters:

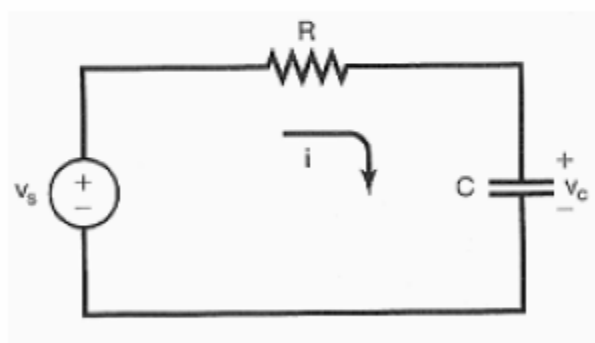
$$B(s) = \frac{w_c}{s + w_c}$$

$$B(s) = \frac{w_c^2}{s^2 + \sqrt{2}w_cs + w_c^2}$$

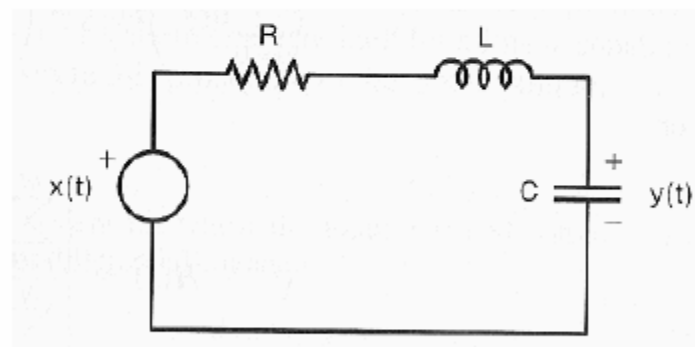
$$B(s) = \frac{w_c^3}{s^3 + 2w_cs^2 + 2w_c^2s + w_c^3}$$

$$\frac{dy(t)}{dt} + w_c y(t) = w_c x(t)$$

$$\frac{d^2y(t)}{dt^2} + \sqrt{2}w_c \frac{dy(t)}{dt} + w_c^2 y(t) = w_c^2 x(t)$$

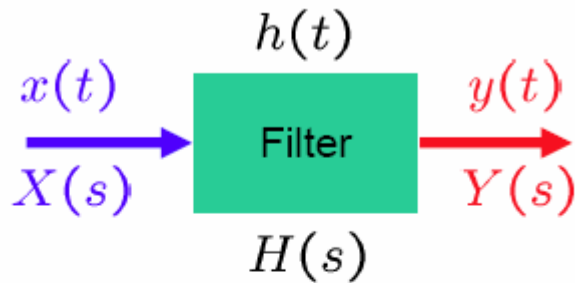


$$\Rightarrow \frac{dv_c(t)}{dt} + \frac{1}{RC}v_c(t) = \frac{1}{RC}v_s(t)$$



$$LC \frac{d^2y(t)}{dt^2} + RC \frac{dy(t)}{dt} + y(t) = x(t)$$

■ Chebyshev Filters:



$$V_N(x) = \cos(N \cos^{-1} x)$$

Nth-order Chebyshev polynomial

$$\left| H_c(j\Omega) \right|^2 = \frac{1}{1 + \epsilon^2 V_N^2\left(\frac{\Omega}{\Omega_c}\right)}$$

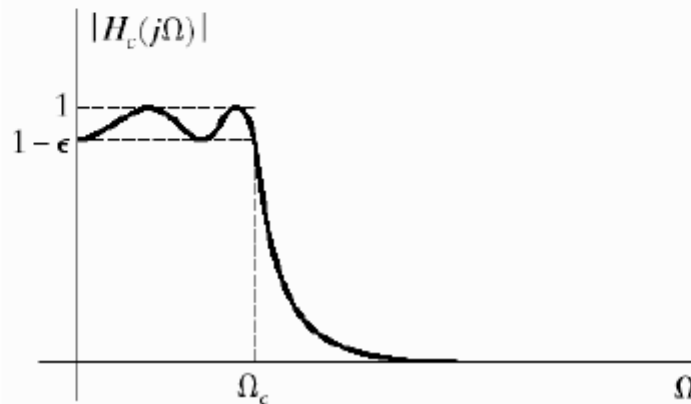
$$V_0(x) = 1$$

$$V_1(x) = x$$

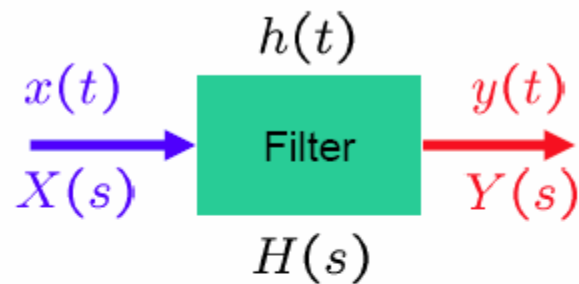
$$V_2(x) = 2x^2 - 1$$

...

...

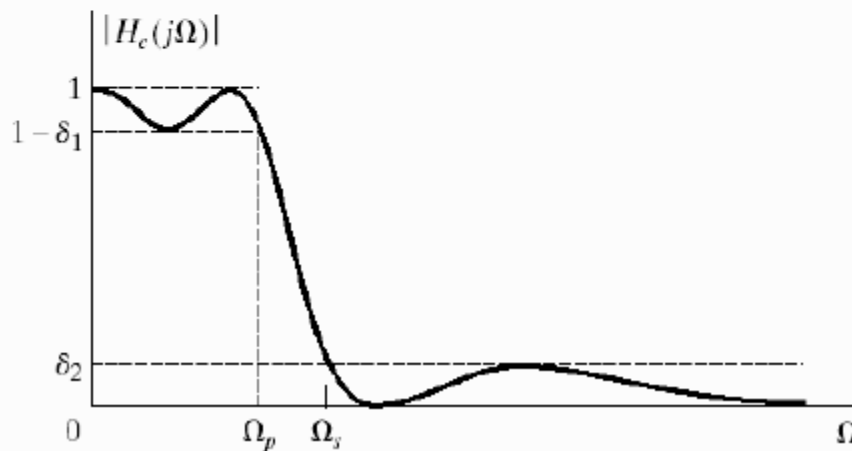


- Elliptic Filters:



$$\left| H_c(j\Omega) \right|^2 = \frac{1}{1 + \epsilon^2 U_N^2(\Omega)}$$

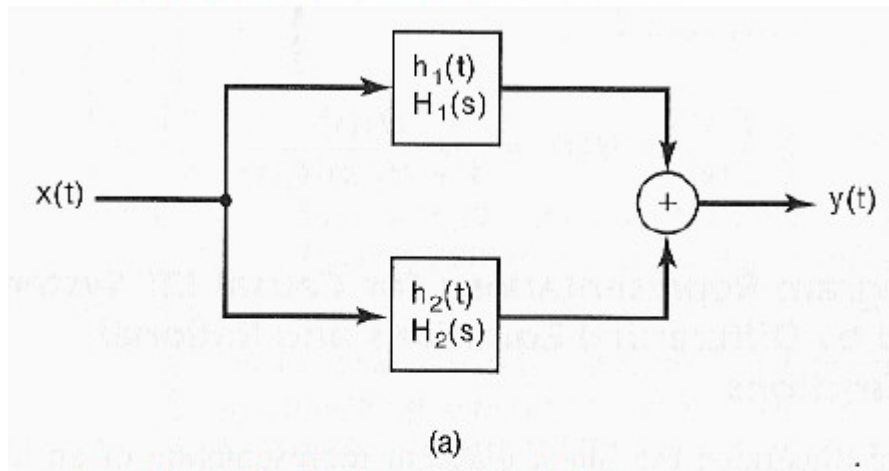
$U_N(x)$: Jacobian elliptic function



- The Laplace Transform
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■ System Function Blocks:

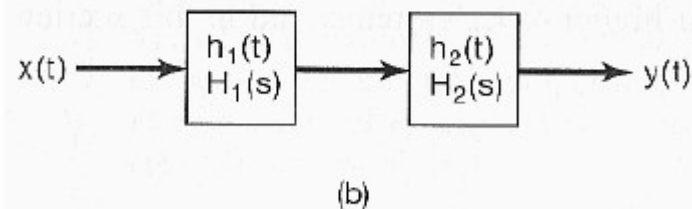
- parallel interconnection



$$h(t) = h_1(t) + h_2(t)$$

$$H(s) = H_1(s) + H_2(s)$$

- series interconnection

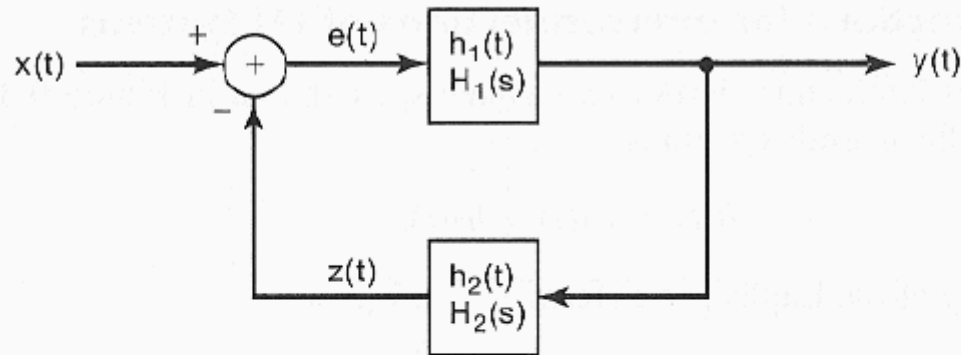


$$h(t) = h_1(t) * h_2(t)$$

$$H(s) = H_1(s) H_2(s)$$

■ System Function Blocks:

- feedback interconnection



$$Y = H_1 E$$

$$Z = H_2 Y$$

$$E = X - Z$$

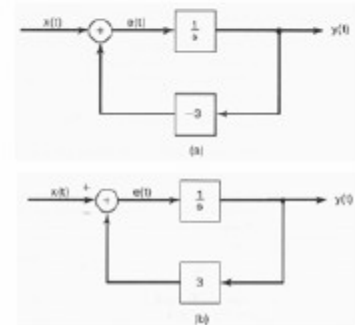
$$H(s) = \frac{H_1(s)}{1 + H_1(s)H_2(s)}$$

▪ Example 9.28:

- Consider a **causal LTI** system with system function

$$H(s) = \frac{1}{s+3} \Rightarrow Y(s) = \frac{1}{s+3}X(s) \Rightarrow \frac{d}{dt}y(t) + 3y(t) = x(t)$$

$$\Rightarrow \frac{d}{dt}y(t) = x(t) - 3y(t)$$

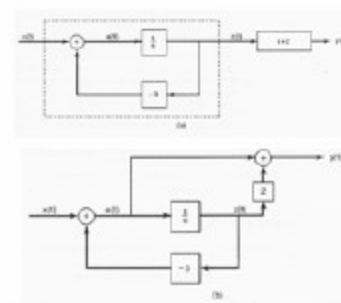
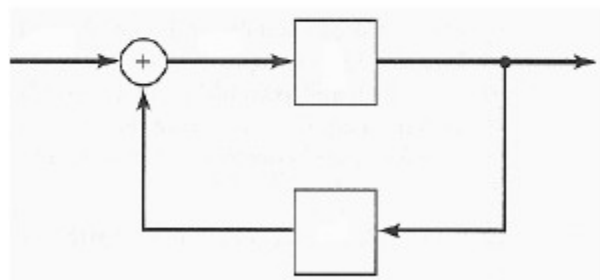


■ Example 9.29:

- Consider a **causal LTI** system with system function

$$H(s) = \frac{s+2}{s+3} = \left(\frac{1}{s+3} \right) (s+2)$$

$$\Rightarrow Z(s) \triangleq \frac{1}{s+3} X(s) \quad \& \quad Y(s) = (s+2) Z(s)$$



■ Example 9.30:

- Consider a **causal LTI** system with system function

$$H(s) = \frac{1}{(s+1)(s+2)} = \frac{1}{s^2 + 3s + 2}$$

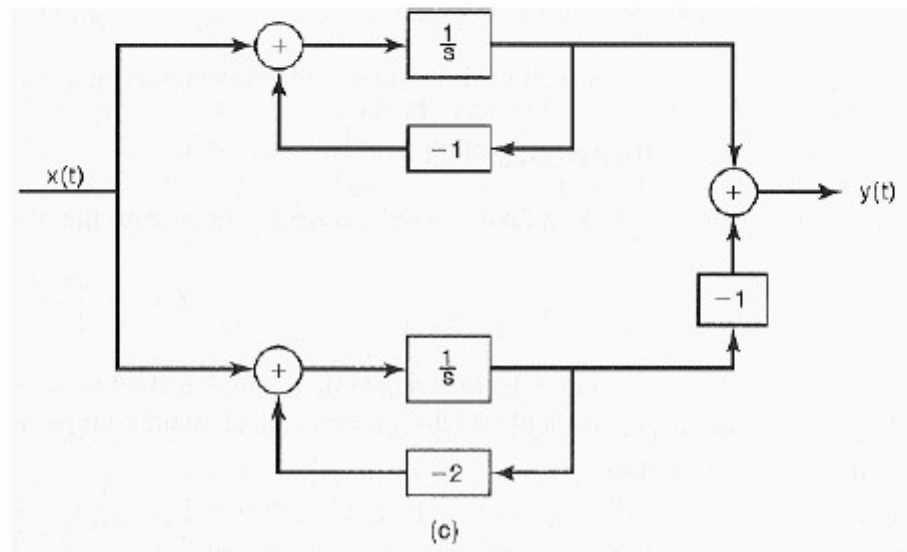
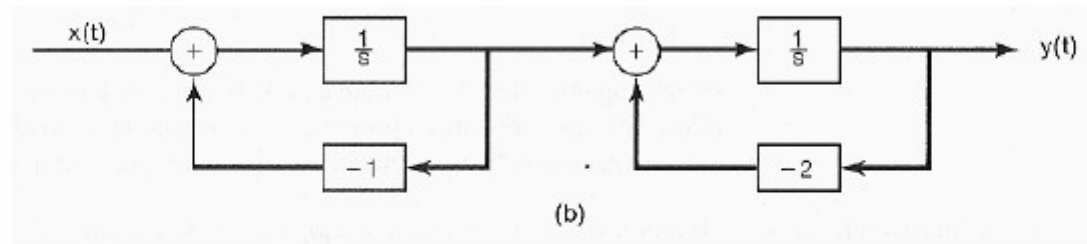
$$\Rightarrow s^2 Y + 3sY + 2Y = X$$

$$\Rightarrow \begin{cases} F = sY \\ E = sF = s^2 Y \end{cases} \Rightarrow E = s^2 Y = -3F - 2Y + X$$



■ Example 9.30:

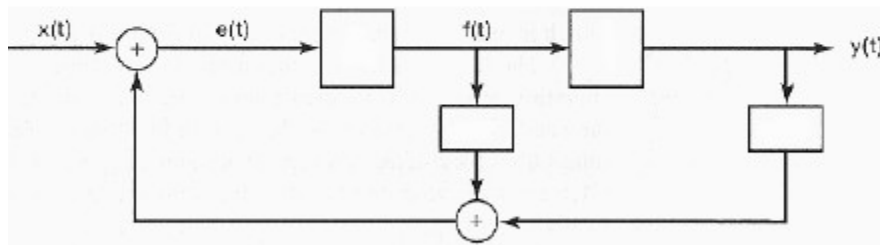
$$H(s) = \frac{1}{(s+1)(s+2)} = \left(\frac{1}{s+1}\right) \left(\frac{-1}{s+2}\right) = \left(\frac{1}{s+1}\right) + \left(\frac{-1}{s+2}\right)$$



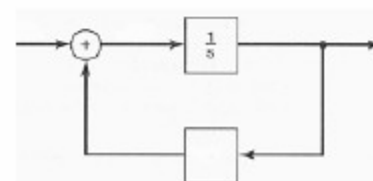
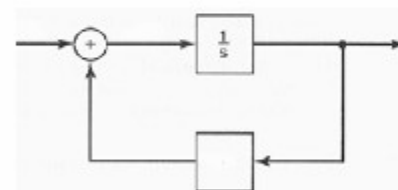
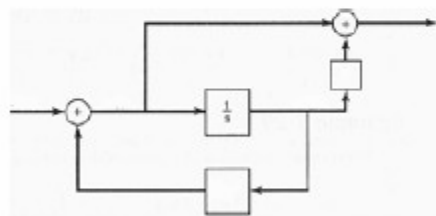
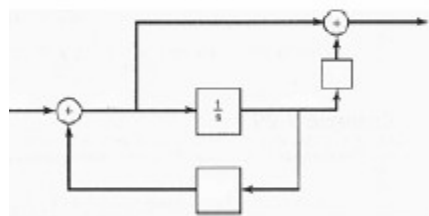
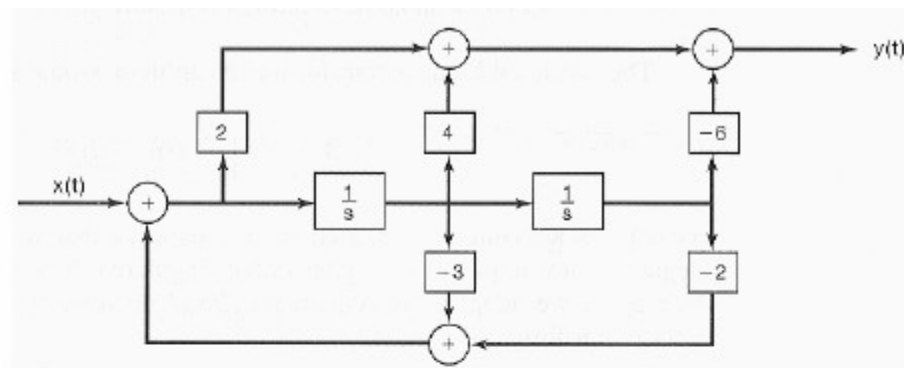
■ Example 9.31:

$$H(s) = \frac{2s^2 + 4s - 6}{s^2 + 3s + 2} = \left(\frac{1}{s^2 + 3s + 2} \right) (2s^2 + 4s - 6)$$

$$\Rightarrow Z(s) \triangleq \frac{1}{s^2 + 3s + 2} X(s) \quad \& \quad Y(s) = (2s^2 + 4s - 6) Z(s)$$



$$H(s) = \begin{cases} \frac{2s^2+4s-6}{s^2+3s+2} \\ \left(\frac{2(s-1)}{s+2}\right)\left(\frac{s+3}{s+1}\right) \\ \left(\frac{2(s-1)}{s+1}\right)\left(\frac{s+3}{s+2}\right) \\ 2 + \frac{6}{s+2} + \frac{-8}{s+1} \end{cases}$$



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- The Unilateral Laplace Transform of $x(t)$:

bilateral LT

$$X(s) \triangleq \int_{-\infty}^{\infty} x(t)e^{-st} dt$$

$$x(t) \xleftrightarrow{\mathcal{L}} X(s)$$

$$X(s) = \mathcal{L}\{x(t)\}$$

$$x(t) = \mathcal{L}^{-1}\{X(s)\}$$

unilateral LT

for causal system &
with nonzero init. cond.

$$\mathcal{X}(s) \triangleq \int_{0^-}^{\infty} x(t)e^{-st} dt$$

$$x(t) \xleftrightarrow{\mathcal{UL}} \mathcal{X}(s)$$

$$\mathcal{X}(s) = \mathcal{UL}\{x(t)\}$$

$$x(t) = \mathcal{UL}^{-1}\{\mathcal{X}(s)\}$$

ROC : a right-half plane

TABLE 9.3 PROPERTIES OF THE UNILATERAL LAPLACE TRANSFORM

Property	Signal	Unilateral Laplace Transform
	$x(t)$ $x_1(t)$ $x_2(t)$	$\mathfrak{X}(s)$ $\mathfrak{X}_1(s)$ $\mathfrak{X}_2(s)$
Linearity	$ax_1(t) + bx_2(t)$	$a\mathfrak{X}_1(s) + b\mathfrak{X}_2(s)$
Shifting in the s -domain	$e^{s_0 t} x(t)$	$\mathfrak{X}(s - s_0)$
Time scaling	$x(at), \quad a > 0$	$\frac{1}{a} \mathfrak{X}\left(\frac{s}{a}\right)$
Conjugation	$x^*(t)$	$x^*(s)$
Convolution (assuming that $x_1(t)$ and $x_2(t)$ are identically zero for $t < 0$)	$x_1(t) * x_2(t)$	$\mathfrak{X}_1(s)\mathfrak{X}_2(s)$
Differentiation in the time domain	$\frac{d}{dt} x(t)$	$s\mathfrak{X}(s) - x(0^-)$
Differentiation in the s -domain	$-tx(t)$	$\frac{d}{ds} \mathfrak{X}(s)$
Integration in the time domain	$\int_{0^-}^t x(\tau) d\tau$	$\frac{1}{s} \mathfrak{X}(s)$

Initial- and Final-Value Theorems

If $x(t)$ contains no impulses or higher-order singularities at $t = 0$, then

$$x(0^+) = \lim_{s \rightarrow \infty} s\mathfrak{X}(s)$$

$$\lim_{t \rightarrow \infty} x(t) = \lim_{s \rightarrow 0} s\mathfrak{X}(s)$$

$$x_1(t) = x_2(t) \equiv 0 \\ t < 0$$

$$s\mathfrak{X}(s) - x(0^-)$$

- Differentiation Property:

$$\begin{aligned}\mathcal{U}\mathcal{L}\left\{\frac{dx(t)}{dt}\right\} &= \int_{0^-}^{\infty} \frac{dx(t)}{dt} e^{-st} dt = x(t)e^{-st} \Big|_{0^-}^{\infty} + s \int_{0^-}^{\infty} x(t) e^{-st} dt \\ &= s\mathcal{X}(s) - x(0^-)\end{aligned}$$

$$\mathcal{U}\mathcal{L}\left\{\frac{d^2x(t)}{dt^2}\right\} = \int_{0^-}^{\infty} \frac{d^2x(t)}{dt^2} e^{-st} dt = s^2\mathcal{X}(s) - sx(0^-) - x'(0^-)$$

■ Example 9.38:

$$\frac{d^2 y(t)}{dt^2} + 3 \frac{dy(t)}{dt} + 2 y(t) = x(t) \quad \begin{cases} y(0^-) = \beta \\ y'(0^-) = \gamma \end{cases}$$

$$x(t) = \alpha u(t)$$

$$\Rightarrow \left[s^2 \mathcal{Y}(s) - \beta s - \gamma \right] + 3 \left[s \mathcal{Y}(s) - \beta \right] + 2 \left[\mathcal{Y}(s) \right] = \frac{\alpha}{s}$$

$$\Rightarrow \mathcal{Y}(s) = \underbrace{\frac{\beta(s+3)}{(s+1)(s+2)} + \frac{\gamma}{(s+1)(s+2)}}_{\text{zero-input response}} + \underbrace{\frac{\alpha}{s(s+1)(s+2)}}_{\text{zero-state response}}$$

■ Example 9.38:

- If $\alpha = 2$, $\beta = 3$, $\gamma = -5$

$$\Rightarrow \mathcal{Y}(s) = \frac{1}{s} - \frac{1}{s+1} + \frac{3}{s+2}$$

$$\Rightarrow y(t) = \left[1 - e^{-t} + 3e^{-2t} \right] u(t), \quad \text{for } t > 0$$

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 - Integration in the Time Domain
 - Time Shifting
 - Conjugation
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- The Unilateral LT

Introduction [\(Chap 1\)](#)

LTI & Convolution [\(Chap 2\)](#)

Bounded/Convergent

Periodic

FS

[\(Chap 3\)](#)

- CT
- DT

Aperiodic

FT

- CT [\(Chap 4\)](#)
- DT [\(Chap 5\)](#)

Unbounded/Non-convergent

LT

- CT [\(Chap 9\)](#)

zT

- DT [\(Chap 10\)](#)

Time-Frequency [\(Chap 6\)](#)

Communication [\(Chap 8\)](#)

CT-DT [\(Chap 7\)](#)

Control [\(Chap 11\)](#)