

Chapter X

Z Transform

- The z-Transform
- The Region of Convergence for z-Transforms
- The Inverse z-Transform
- Geometric Evaluation of the Fourier Transform
- Properties of the z-Transform
- Some Common z-Transform Pairs
- Analysis & Characterization of LTI Systems Using the z-Transforms
- System Function Algebra and Block Diagram Representations
- The Unilateral z-Transform

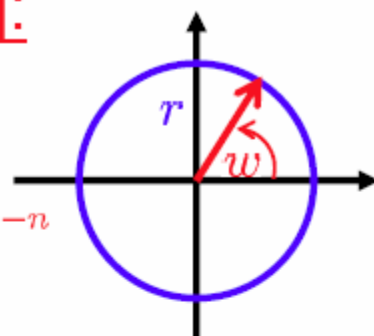
▪ The z-Transform of a General Signal $x[n]$:

$$z = e^{j\omega}$$

$$X(e^{j\omega}) \triangleq \sum_{n=-\infty}^{+\infty} x[n] e^{-j\omega n}$$

$$z = r e^{j\omega}$$

$$X(z) \triangleq \sum_{n=-\infty}^{+\infty} x[n] z^{-n}$$



$$x[n] \xleftrightarrow{\mathcal{F}} X(e^{j\omega})$$

$$x[n] \xleftrightarrow{\mathcal{Z}} X(z)$$

$$X(e^{j\omega}) = \mathcal{F} \{ x[n] \}$$

$$X(z) = \mathcal{Z} \{ x[n] \}$$

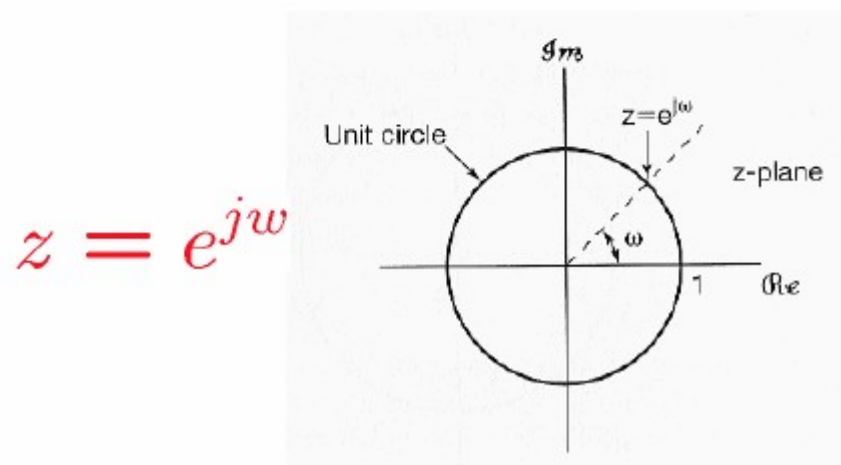
$$x[n] = \mathcal{F}^{-1} \{ X(e^{j\omega}) \}$$

$$x[n] = \mathcal{Z}^{-1} \{ X(z) \}$$

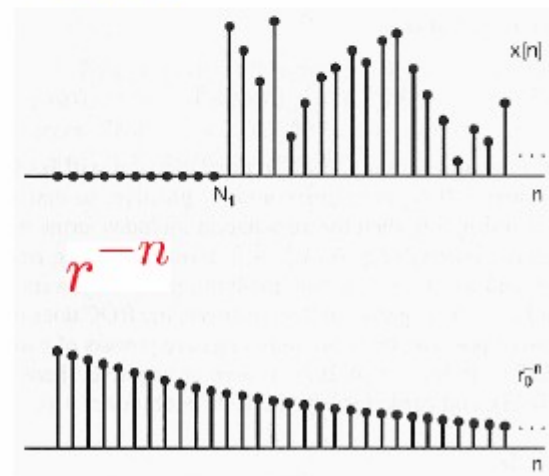
$$X(z) \Big|_{z=e^{j\omega}} = \mathcal{Z} \{ x[n] \} \Big|_{z=e^{j\omega}} = \mathcal{F} \{ x[n] \} = X(e^{j\omega})$$

▪ z-Transform & Fourier Transform:

$$\begin{aligned}
 X(re^{j\omega}) &= \sum_{n=-\infty}^{+\infty} x[n](re^{j\omega})^{-n} \\
 &= \sum_{n=-\infty}^{+\infty} \{x[n]r^{-n}\}e^{-j\omega n} \\
 &= \mathcal{F}\{x[n]r^{-n}\}
 \end{aligned}$$



$x[n]$

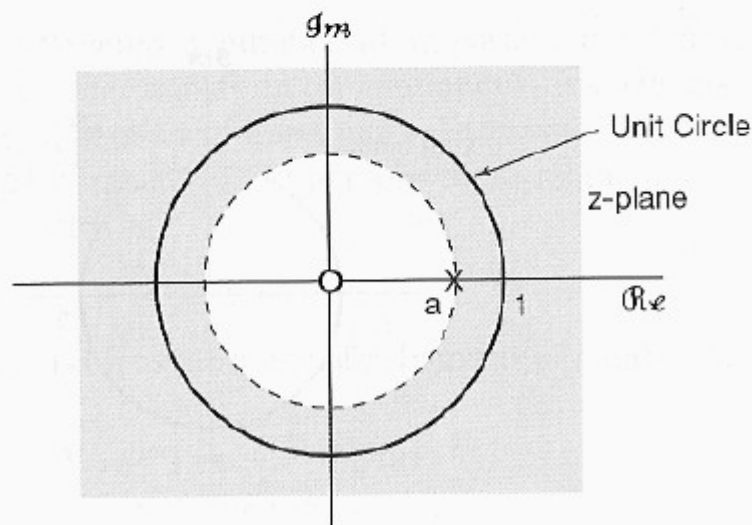


▪ Example 10.1: $x[n] = a^n u[n]$

$$\Rightarrow X(e^{j\omega}) = \sum_{n=-\infty}^{+\infty} a^n u[n] e^{-j\omega n} = \sum_{n=0}^{\infty} (ae^{-j\omega})^n = \frac{1}{1 - ae^{-j\omega}}$$

$$\Rightarrow X(z) = \sum_{n=-\infty}^{+\infty} a^n u[n] z^{-n} = \sum_{n=0}^{\infty} (az^{-1})^n \quad |az^{-1}| < 1$$

$$= \frac{1}{1 - az^{-1}} = \frac{z}{z - a}, \quad |z| > |a|$$



- For $|a| > 1$,

ROC does not include the unit circle,
 $\mathcal{F}\{a^n u[n]\}$ does not converge

- Example 10.2: $x[n] = -a^n u[-n - 1]$

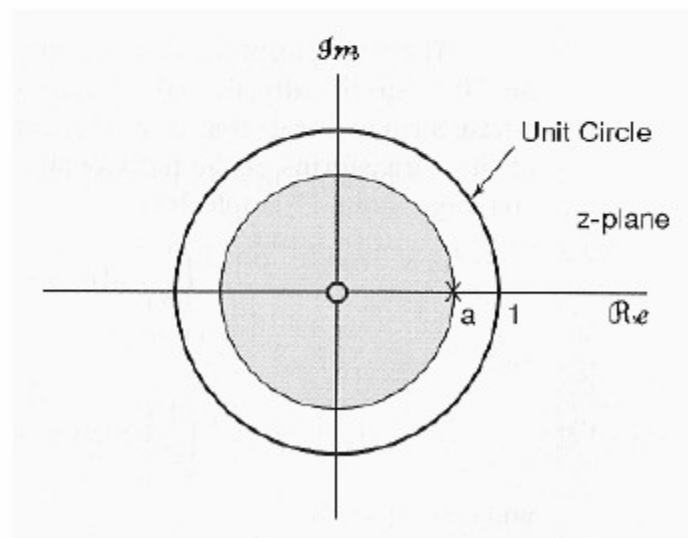
$$\Rightarrow X(z) = - \sum_{n=-\infty}^{+\infty} a^n u[-n - 1] z^{-n}$$

$$= - \sum_{n=-\infty}^{-1} a^n z^{-n} = - \sum_{n=1}^{\infty} a^{-n} z^n = 1 - \sum_{n=0}^{\infty} (a^{-1} z)^n$$

$$|a^{-1} z| < 1 \quad = 1 - \frac{1}{1 - a^{-1} z}$$

$$= \frac{1}{1 - az^{-1}} = \frac{z}{z - a}$$

$$|z| < |a|$$

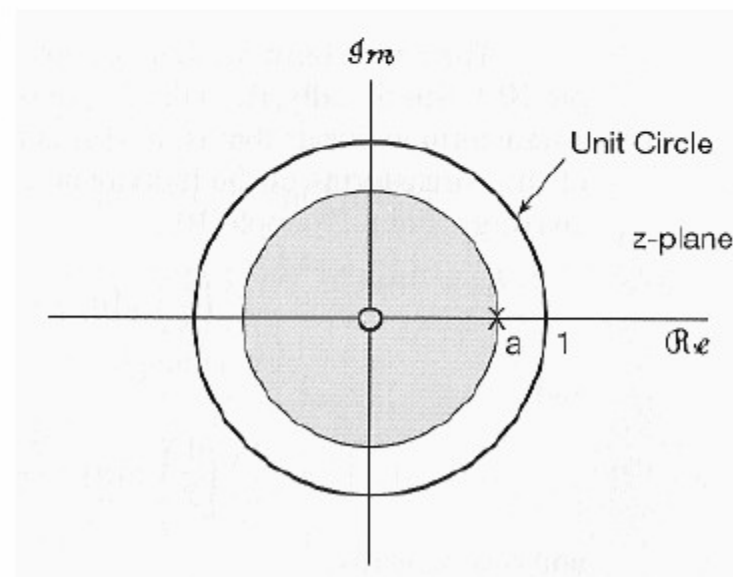
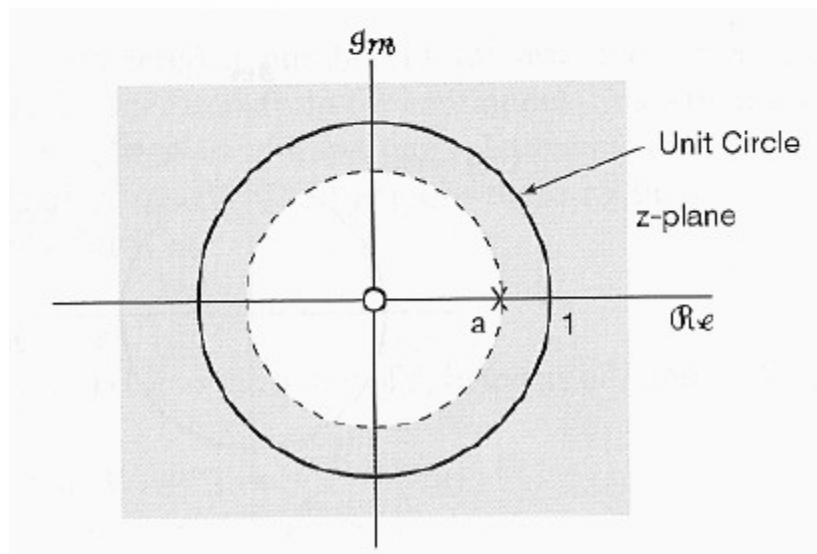


- Region of Convergence (ROC):

$$a^n u[n] \xleftrightarrow{z} \frac{z}{z-a}, \quad |z| > |a|$$

$$-a^n u[-n-1] \xleftrightarrow{z} \frac{z}{z-a}, \quad |z| < |a|$$

where Fourier transform of $x[n]r^{-n}$ converges



■ Example 10.3:

$$x[n] = 7 \left(\frac{1}{3}\right)^n u[n] - 6 \left(\frac{1}{2}\right)^n u[n]$$

$$X(z) = \sum_{n=-\infty}^{+\infty} \left\{ 7 \left(\frac{1}{3}\right)^n u[n] - 6 \left(\frac{1}{2}\right)^n u[n] \right\} z^{-n}$$

$$= 7 \sum_{n=-\infty}^{+\infty} \left(\frac{1}{3}\right)^n u[n] z^{-n} - 6 \sum_{n=-\infty}^{+\infty} \left(\frac{1}{2}\right)^n u[n] z^{-n}$$

$$= \frac{7}{1 - \frac{1}{3}z^{-1}} - \frac{6}{1 - \frac{1}{2}z^{-1}}$$

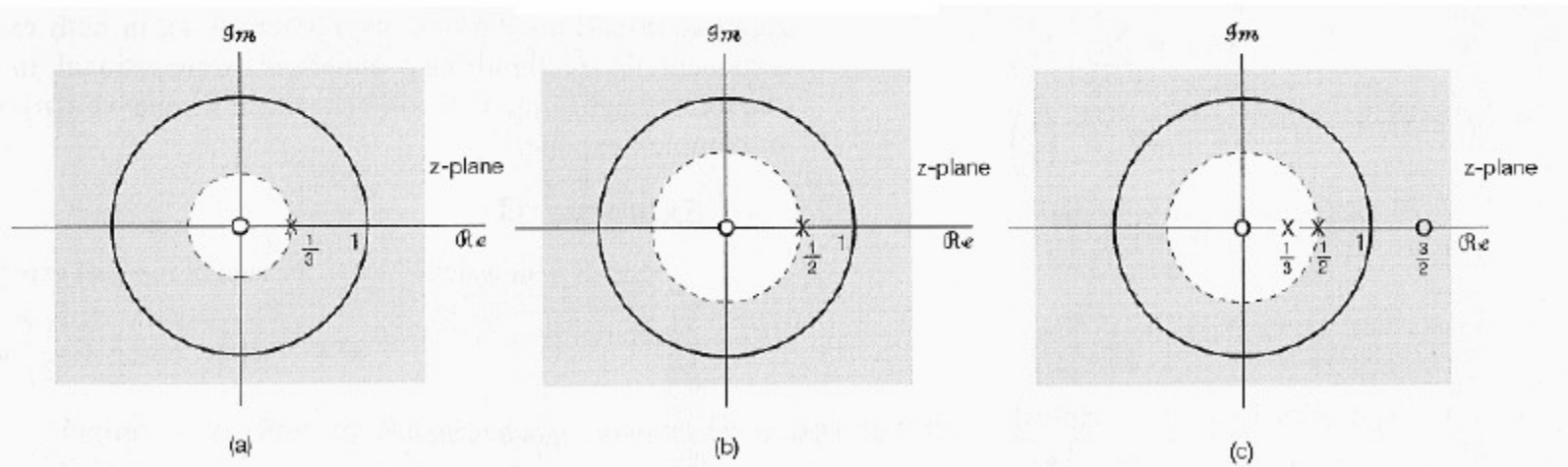
$$7 \cdot \left(\frac{1}{3}\right)^n u[n] \xleftrightarrow{z} 7 \cdot \frac{1}{1 - \frac{1}{3}z^{-1}}, \quad |z| > \frac{1}{3}$$

$$6 \cdot \left(\frac{1}{2}\right)^n u[n] \xleftrightarrow{z} 6 \cdot \frac{1}{1 - \frac{1}{2}z^{-1}}, \quad |z| > \frac{1}{2}$$

■ Example 10.3:

$$7 \left(\frac{1}{3}\right)^n u[n] - 6 \left(\frac{1}{2}\right)^n u[n] \xleftrightarrow{z} \frac{7}{1 - \frac{1}{3}z^{-1}} - \frac{6}{1 - \frac{1}{2}z^{-1}}, \quad |z| > \frac{1}{2}$$

$$\xleftrightarrow{z} \frac{z(z - \frac{3}{2})}{(z - \frac{1}{3})(z - \frac{1}{2})}, \quad |z| > \frac{1}{2}$$



■ Example 10.4:

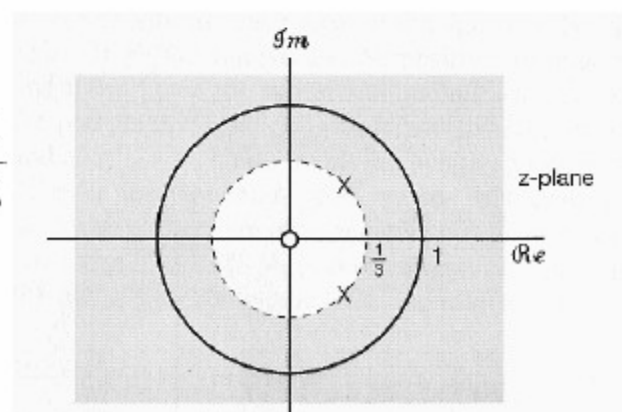
$$\left(\frac{1}{3}\right)^n \sin\left(\frac{\pi}{4}n\right) u[n] \xleftrightarrow{\mathcal{Z}} \frac{\frac{1}{3\sqrt{2}}z}{\left(z - \frac{1}{3}e^{j\pi/4}\right)\left(z - \frac{1}{3}e^{-j\pi/4}\right)}, \quad |z| > \frac{1}{3}$$

$$\sin\left(\frac{\pi}{4}n\right) = \frac{1}{2j} \left(e^{j\pi/4n} - e^{-j\pi/4n} \right) = \frac{1}{2j} \left(\left(e^{j\pi/4} \right)^n - \left(e^{-j\pi/4} \right)^n \right)$$

$$\left(\frac{1}{3}\right)^n \sin\left(\frac{\pi}{4}n\right) = \frac{1}{2j} \left(\left(\frac{1}{3} e^{j\pi/4} \right)^n - \left(\frac{1}{3} e^{-j\pi/4} \right)^n \right)$$

$$\left(\frac{1}{3} e^{j\pi/4} \right)^n u[n] \xleftrightarrow{\mathcal{Z}} \frac{z}{z - \frac{1}{3}e^{j\pi/4}}, \quad |z| > \frac{1}{3}$$

$$a^n u[n] \xleftrightarrow{\mathcal{Z}} \frac{z}{z - a}, \quad |z| > |a|$$

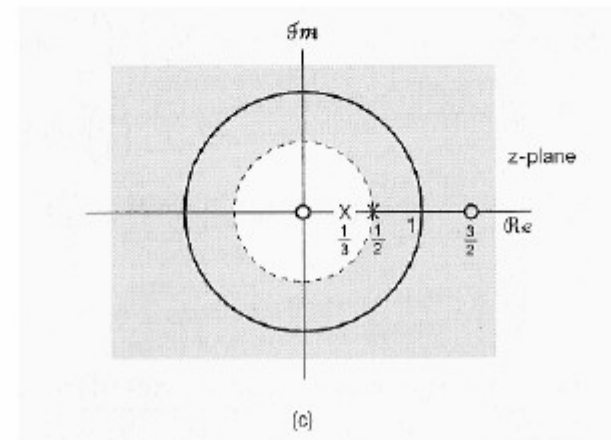
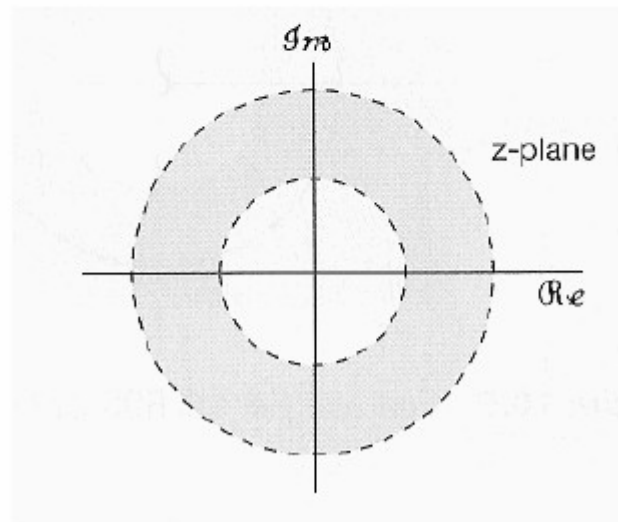


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■ Properties of ROC:

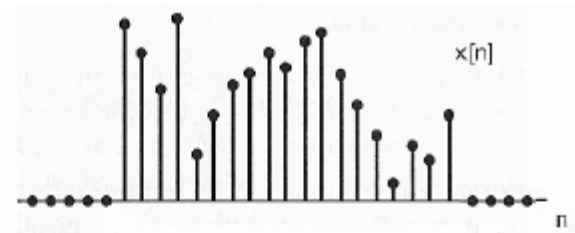
1. The ROC of $X(z)$ consists of a ring in the z-plane centered about the origin
2. The ROC does not contain any poles

$$\frac{z(z - \frac{3}{2})}{(z - \frac{1}{3})(z - \frac{1}{2})}, \quad |z| > \frac{1}{2}$$



■ Properties of ROC:

3. If $x[n]$ is of finite duration,
then the ROC is the entire z -plane,
except possibly $z = 0$ and/or $z = \infty$



$$X(z) \triangleq \sum_{n=-\infty}^{+\infty} x[n]z^{-n} = \sum_{n=N_1}^{N_2} x[n]z^{-n} \quad \text{is bounded}$$

- However,

$$|z| \rightarrow 0 \quad \Rightarrow \quad |z|^N \rightarrow \infty \quad \text{if } N \text{ is negative}$$

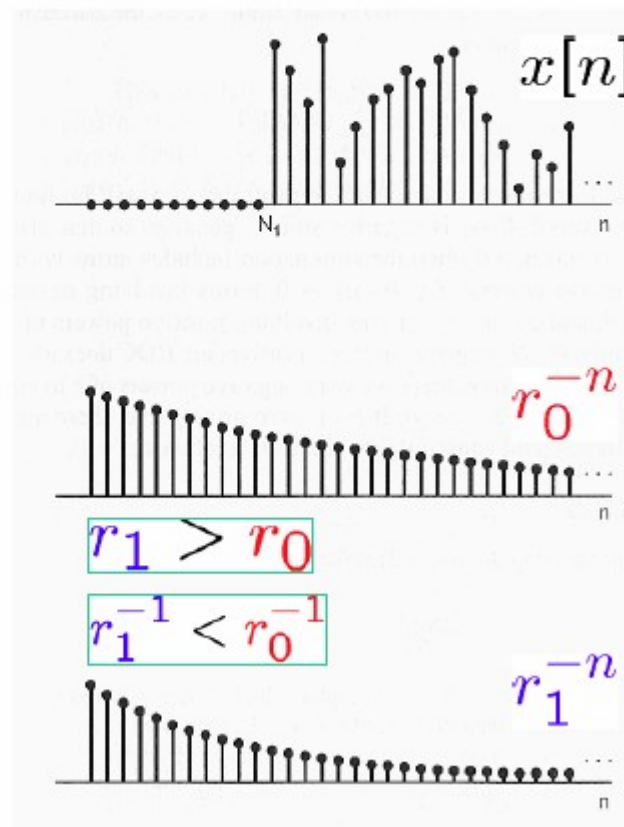
$$|z| \rightarrow \infty \quad \Rightarrow \quad |z|^N \rightarrow \infty \quad \text{if } N \text{ is positive}$$

- Properties of ROC:

4. If $x[n]$ is right-sided sequence, and if the circle $|z| = r_0$ is in the ROC, then all finite values of z for which $|z| > r_0$ will also be in the ROC

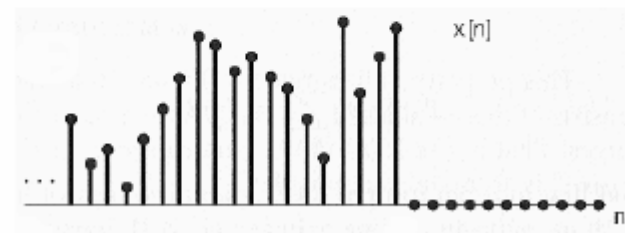
$$X(r_0 e^{j\omega}) = \sum_{n=N_1}^{\infty} \left\{ x[n] r_0^{-n} \right\} e^{-j\omega n} < \infty$$

$$\begin{aligned} X(r_1 e^{j\omega}) &= \sum_{n=N_1}^{\infty} \left\{ x[n] r_1^{-n} \right\} e^{-j\omega n} \\ &< \sum_{n=N_1}^{\infty} \left\{ x[n] r_0^{-n} \right\} e^{-j\omega n} < \infty \end{aligned}$$



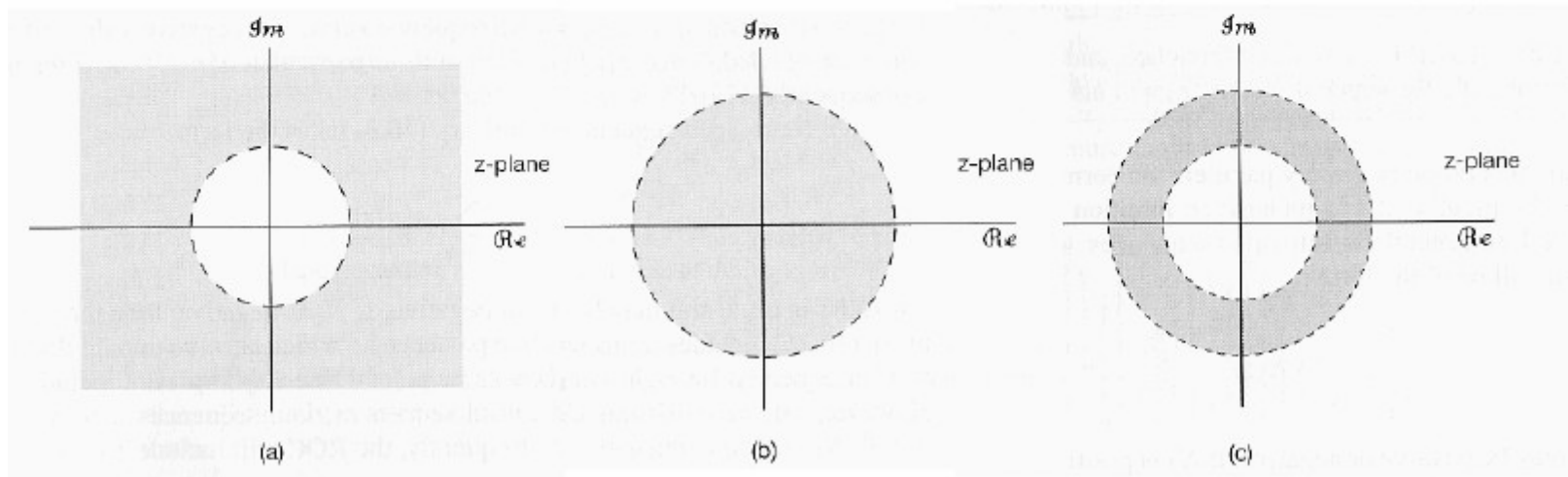
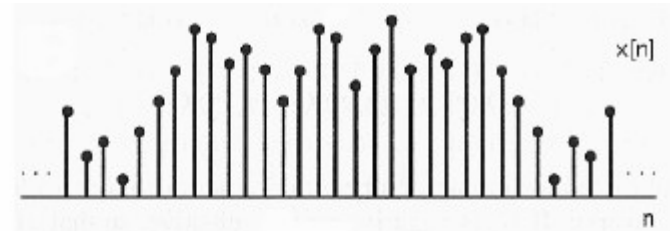
- Properties of ROC:

5. If $x[n]$ is left-sided sequence, and if the circle $|z| = r_0$ is in the ROC, then all values of z for which $0 < |z| < r_0$ will also be in the ROC

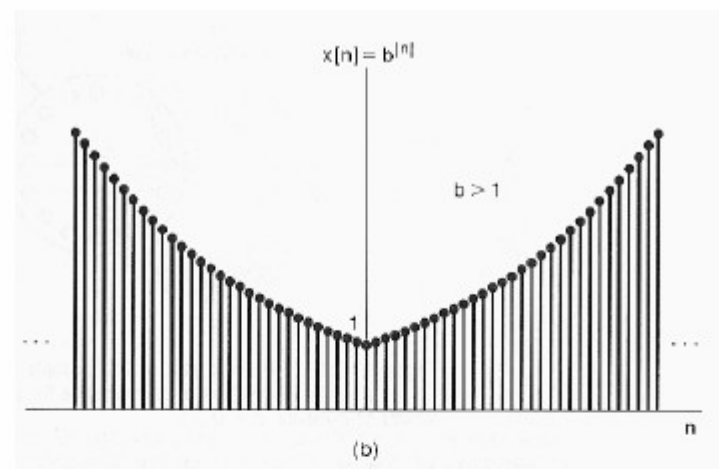
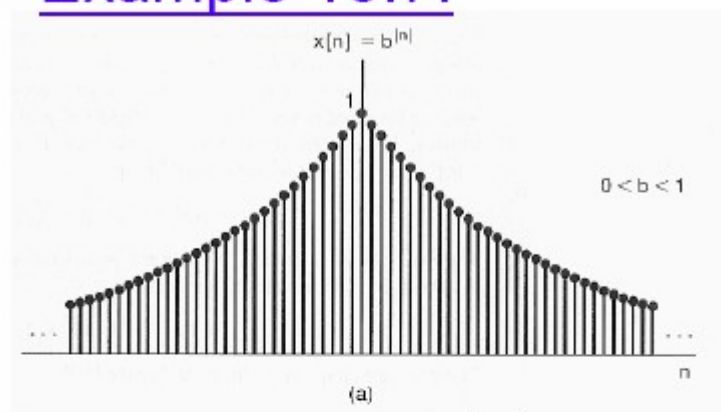


- Properties of ROC:

6. If $x[n]$ is two-sided, and if the circle $|z| = r_0$ is in the ROC, then the ROC will consist of a ring in the z-plane that includes the circle $|z| = r_0$



■ Example 10.7:



$$x[n] = b^{|n|}, \quad b > 0$$

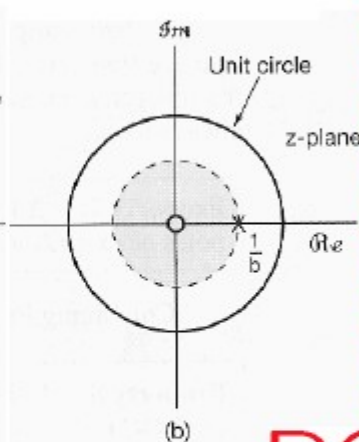
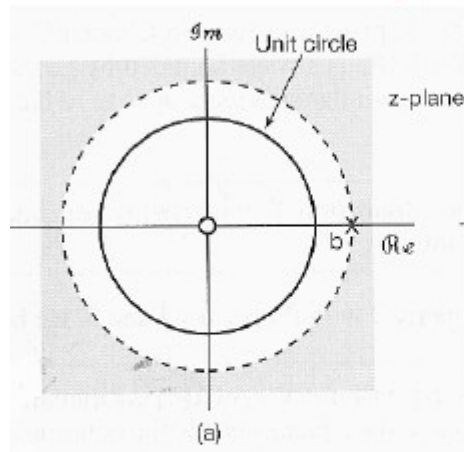
$$= b^n u[n] + b^{-n} u[-n-1]$$

$$X(z) = \frac{1}{1 - bz^{-1}} - \frac{1}{1 - b^{-1}z^{-1}}, \quad b < |z| < \frac{1}{b}$$

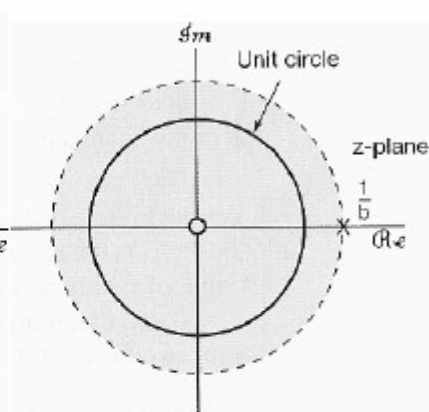
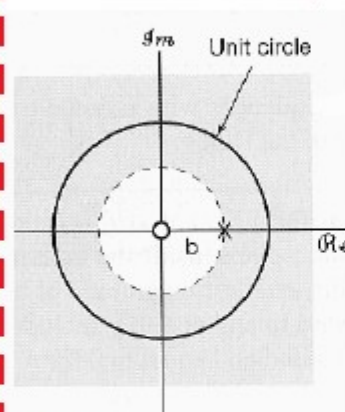
$$= \left(\frac{b^2 - 1}{b} \right) \frac{z}{(z - b)(z - b^{-1})}, \quad b < |z| < \frac{1}{b}$$

- Example 10.7:** $X(z) = \left(\frac{b^2 - 1}{b}\right) \frac{z}{(z - b)(z - b^{-1})}, \quad b < |z| < \frac{1}{b}$

$b > 1$

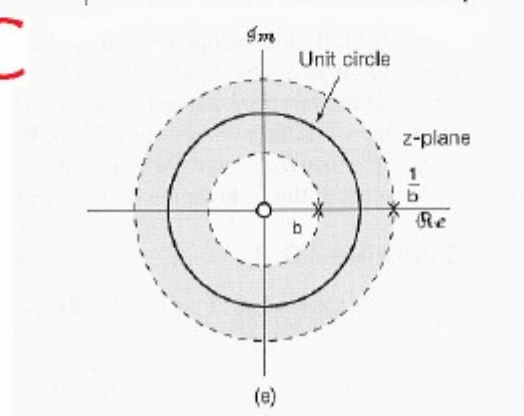


$0 < b < 1$



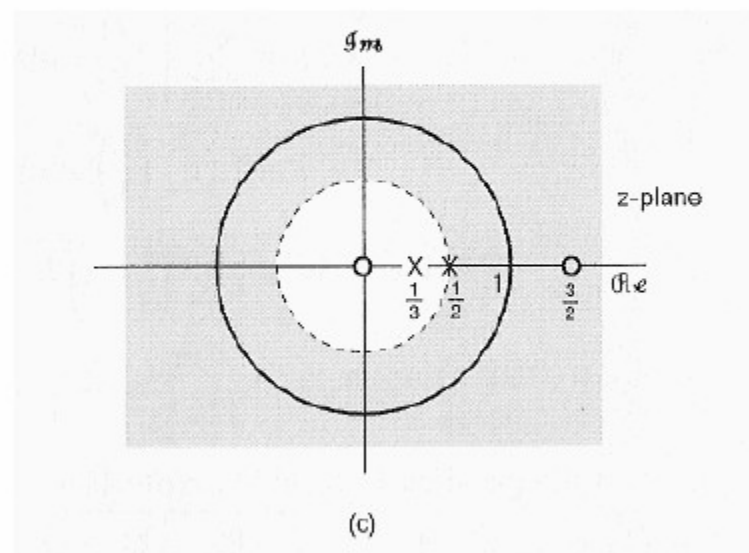
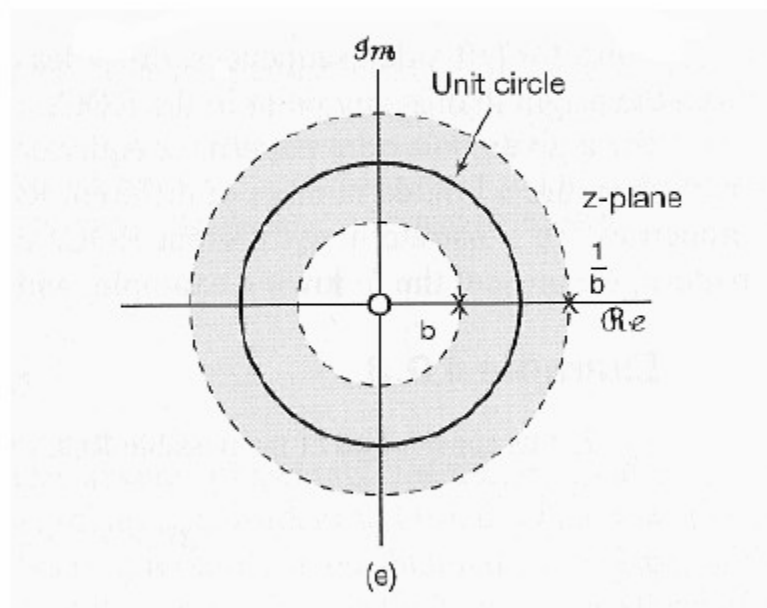
ROC

X



- Properties of ROC:

7. If the z-transform $X(z)$ of $x[n]$ is rational, then its ROC is bounded by poles or extends to ∞



$$X(z) = \left(\frac{b^2 - 1}{b} \right) \frac{z}{(z - b)(z - b^{-1})}, \quad b < |z| < \frac{1}{b}$$

$$\frac{z(z - \frac{3}{2})}{(z - \frac{1}{3})(z - \frac{1}{2})}, \quad |z| > \frac{1}{2}$$

▪ Properties of ROC:

8. If the z-transform $X(z)$ of $x[n]$ is rational

- If $x[n]$ is right sided,
then the ROC is the region in the z-plane
outside the outermost pole ---
i.e. outside the circle of radius equal to
the largest magnitude of the poles of $X(z)$

- Furthermore, if $x[n]$ is causal,
then the ROC also includes $z = \infty$
- $$X(z) = \sum_{n=-\infty}^{+\infty} x[n] z^{-n}$$
- $$= \sum_{n=0}^{+\infty} x[n] \left(\frac{1}{z}\right)^n$$

■ Properties of ROC:

9. If the z-transform $X(z)$ of $x[n]$ is rational and
If $x[n]$ is left sided,

then the ROC is the region in the z-plane
inside the innermost pole ---

i.e. inside the circle of radius equal to
the smallest magnitude of the poles of $X(z)$
other than any at $z = 0$
and extending inward and possibly including $z = 0$

In particular, if $x[n]$ is anti-causal,

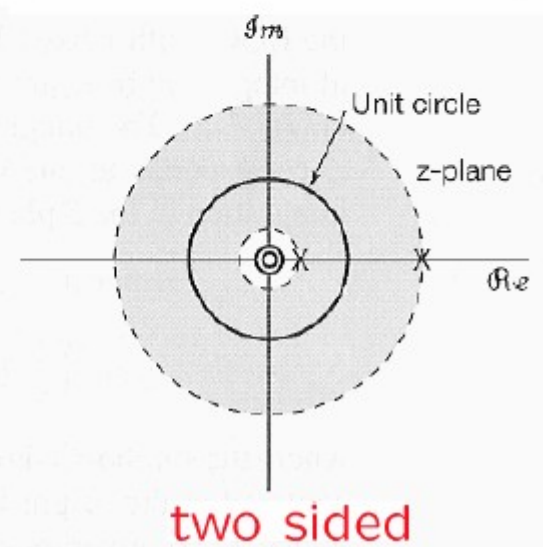
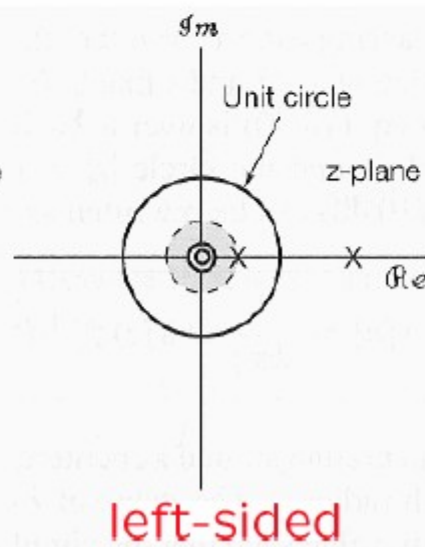
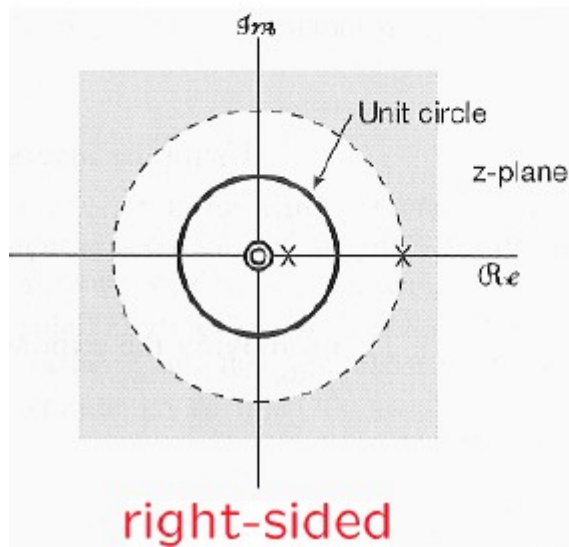
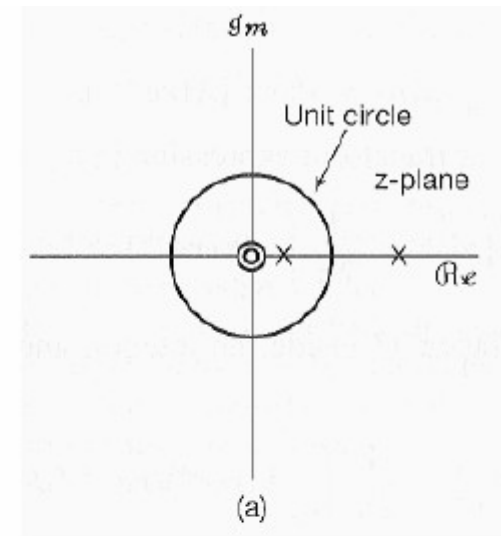
(i.e., if it is left sided and $= 0$ for $n > 0$),

then the ROC also includes $z = 0$

$$\begin{aligned} X(z) &= \sum_{n=-\infty}^{+\infty} x[n] z^{-n} \\ &= \sum_{n=-\infty}^0 x[n] z^{-n} \end{aligned}$$

■ Example 10.8:

$$X(z) = \frac{1}{(1 - \frac{1}{3}z^{-1})(1 - 2z^{-1})}$$



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■ The Inverse z-Transform:

$$X(z) = \sum_{n=-\infty}^{+\infty} x[n]z^{-n}$$

- By the use of contour integration

$$X(re^{jw}) = \mathcal{F} \left\{ x[n]r^{-n} \right\} \quad \forall z = re^{jw} \text{ in the ROC}$$

$$x[n]r^{-n} = \mathcal{F}^{-1} \left\{ X(re^{jw}) \right\}$$

$$x[n] = r^n \mathcal{F}^{-1} \left\{ X(re^{jw}) \right\}$$

$$= r^n \frac{1}{2\pi} \int_{2\pi} X(re^{jw}) e^{jwn} dw$$

$$\begin{aligned} z &= re^{jw} \\ dz &= jre^{jw} dw = jz dw \end{aligned} \quad = \frac{1}{2\pi} \int_{2\pi} X(re^{jw}) (re^{jw})^n dw$$

$$\Rightarrow x[n] = \frac{1}{2\pi j} \oint X(z) z^{n-1} dz$$

▪ The Inverse z-Transform:

- By the technique of partial fraction expansion

$$X(z) = \frac{A_1}{1 - a_1 z^{-1}} + \frac{A_2}{1 - a_2 z^{-1}} + \cdots + \frac{A_m}{1 - a_m z^{-1}}$$

$$x[n] = A_1 a_1^n u[n] - A_2 a_2^n u[-n - 1] + \cdots + x_m[n]$$

(if ROC outside $z = a_1$) (if ROC inside $z = a_2$)

■ Example 10.9:

$$X(z) = \frac{3 - \frac{5}{6}z^{-1}}{(1 - \frac{1}{4}z^{-1})(1 - \frac{1}{3}z^{-1})}, \quad |z| > \frac{1}{3}$$

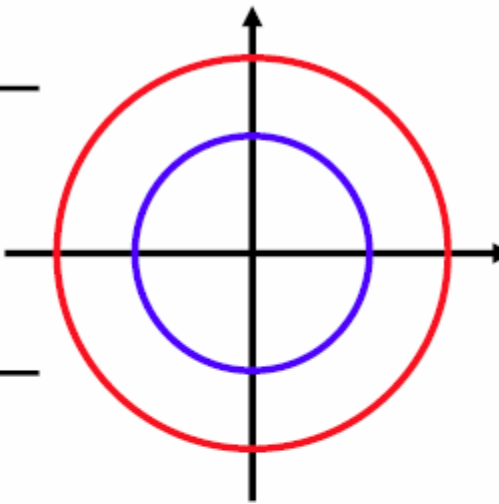
$$X(z) = \frac{1}{(1 - \frac{1}{4}z^{-1})} + \frac{2}{(1 - \frac{1}{3}z^{-1})}$$

$$\left(\frac{1}{4}\right)^n u[n] \xleftrightarrow{\mathcal{Z}} \frac{1}{(1 - \frac{1}{4}z^{-1})}, \quad |z| > \frac{1}{4}$$

$$2\left(\frac{1}{3}\right)^n u[n] \xleftrightarrow{\mathcal{Z}} \frac{2}{(1 - \frac{1}{3}z^{-1})}, \quad |z| > \frac{1}{3}$$

$$\Rightarrow x[n] = \left(\frac{1}{4}\right)^n u[n] + 2\left(\frac{1}{3}\right)^n u[n]$$

■ Examples 10.9, 10.10, 10.11:

	$ z < \frac{1}{4}$	$\frac{1}{4} < z $	
$\frac{1}{(1 - \frac{1}{4}z^{-1})}$	$-\left(\frac{1}{4}\right)^n u[-n-1]$	$\left(\frac{1}{4}\right)^n u[n]$	
	$ z < \frac{1}{3}$	$\frac{1}{3} < z $	
$\frac{1}{(1 - \frac{1}{3}z^{-1})}$	$-\left(\frac{1}{3}\right)^n u[-n-1]$	$\left(\frac{1}{3}\right)^n u[n]$	
	$ z < \frac{1}{4}$	$\frac{1}{4} < z < \frac{1}{3}$	$\frac{1}{3} < z $
$\frac{1}{(1 - \frac{1}{4}z^{-1})}$	$-\left(\frac{1}{4}\right)^n u[-n-1]$	$\left(\frac{1}{4}\right)^n u[n]$	$\left(\frac{1}{4}\right)^n u[n]$
$\frac{1}{(1 - \frac{1}{3}z^{-1})}$	$-\left(\frac{1}{3}\right)^n u[-n-1]$	$-\left(\frac{1}{3}\right)^n u[-n-1]$	$\left(\frac{1}{3}\right)^n u[n]$

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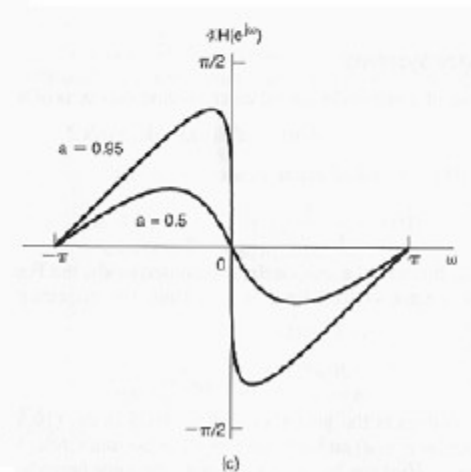
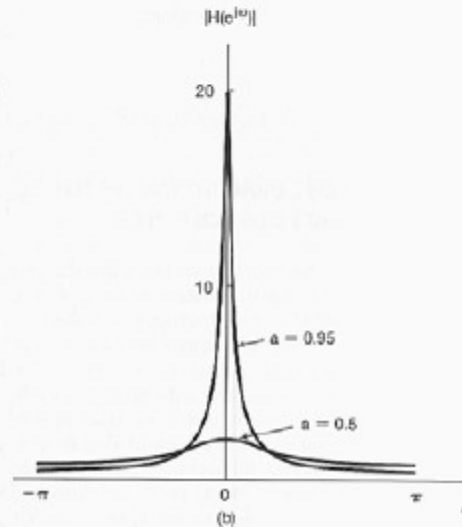
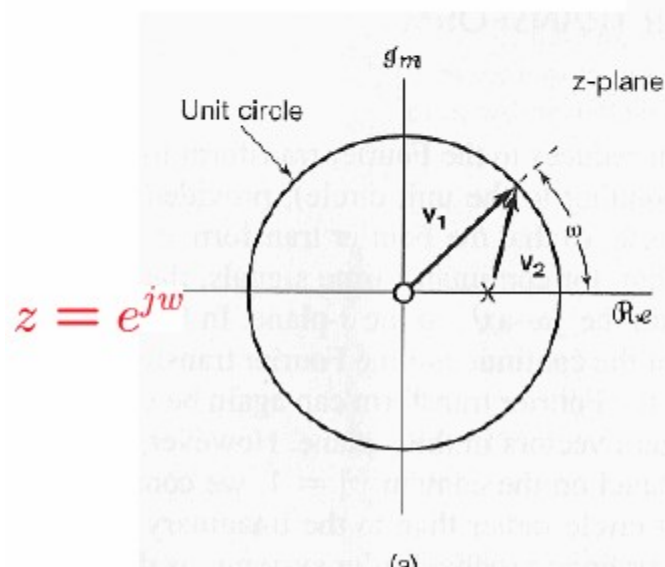
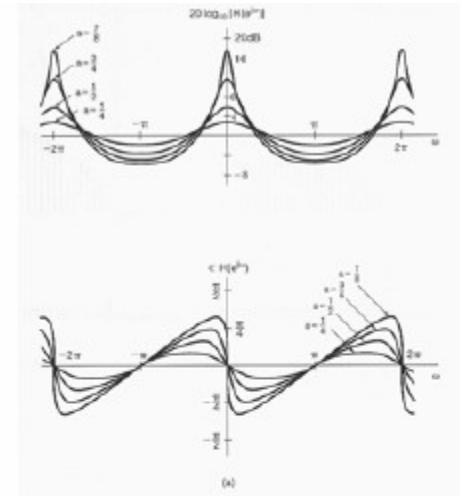
■ First-Order Systems:

$$h[n] = a^n u[n]$$

$$H(z) = \frac{1}{1 - az^{-1}} = \frac{z}{z - a}, \quad |z| > |a|$$

- For $|a| < 1$, ROC includes $|z| = 1$

$$\Rightarrow H(e^{j\omega}) = \frac{1}{1 - ae^{-j\omega}}$$

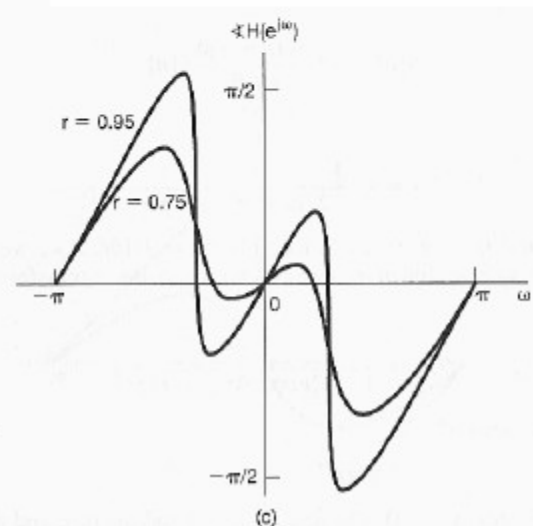
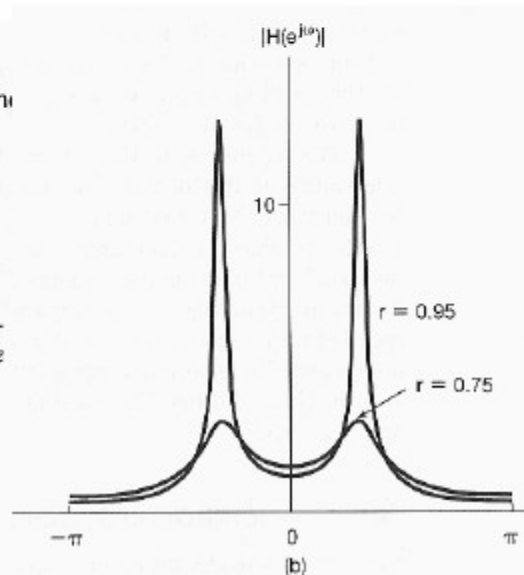
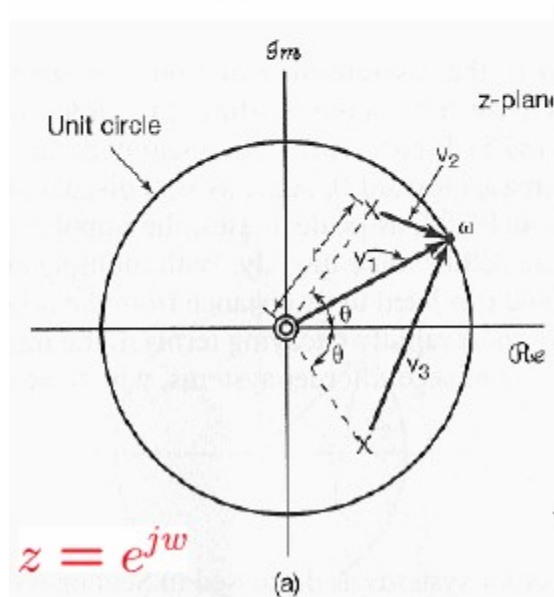
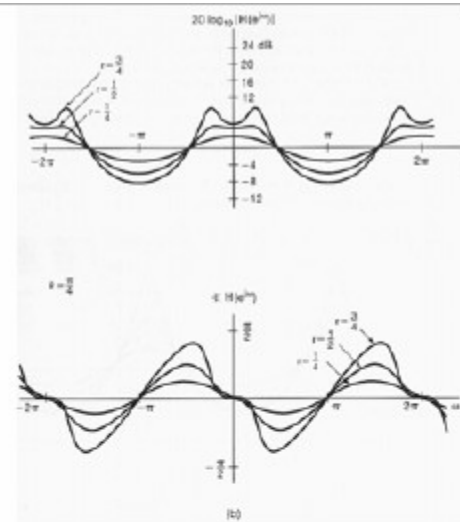


■ Second-Order Systems:

$$H(z) = \frac{1}{1 - 2r \cos \theta z^{-1} + r^2 z^{-2}}$$

$$= \frac{z^2}{(z - p_1)(z - p_2)}, \quad |z| > |a|$$

poles: $\begin{cases} p_1 = re^{j\theta} \\ p_2 = re^{-j\theta} \end{cases}$ zeros: $z_1 = z_2 = 0$



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- System Function Algebra and Block Diagram Representations
- The Unilateral z-Transform

Section	Property
10.5.1	Linearity
10.5.2	Time Shifting
10.5.3	Shifting in the z-Domain
10.5.4	Time Reversal
10.5.5	Time Expansion
10.5.6	Conjugation
10.5.7	Convolution
10.5.7	First Difference
10.5.7	Accumulation
10.5.8	Differentiation in the z-Domain
10.5.9	Initial-Value Theorems

- Linearity of the z-Transform:

$$X(z) = \sum_{n=-\infty}^{+\infty} x[n]z^{-n}$$

$$x[n] = \frac{1}{2\pi j} \oint X(z)z^{n-1}dz$$

$$x_1[n] \xleftrightarrow{\mathcal{Z}} X_1(z), \quad ROC = R_1$$

$$x_2[n] \xleftrightarrow{\mathcal{Z}} X_2(z), \quad ROC = R_2$$

$$ax_1[n] + bx_2[n] \xleftrightarrow{\mathcal{Z}} aX_1(z) + bX_2(z),$$

with ROC containing $R_1 \cap R_2$

- Time Shifting:

$$X(z) = \sum_{n=-\infty}^{+\infty} x[n]z^{-n}$$

$$x[n] = \frac{1}{2\pi j} \oint X(z)z^{n-1}dz$$

$$x[n] \xleftrightarrow{z} X(z), \quad ROC = R$$

$$x[n-n_0] \xleftrightarrow{z} z^{-n_0}X(z), \quad ROC = R$$

except for the possible
addition or deletion of
the origin or infinity

■ Scaling in the z-Domain:

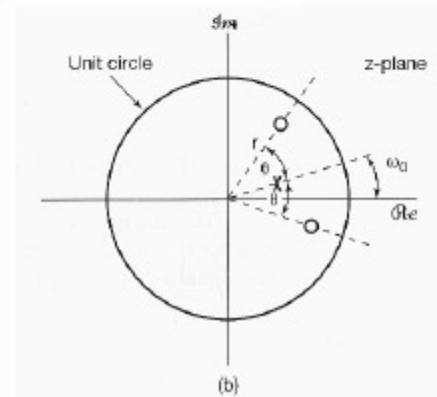
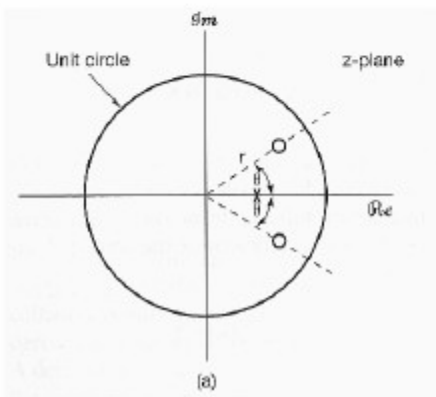
$$X(z) = \sum_{n=-\infty}^{+\infty} x[n] z^{-n}$$

$$x[n] = \frac{1}{2\pi j} \oint X(z) z^{n-1} dz$$

$$x[n] \xleftrightarrow{z} X(z), \quad ROC = R$$

$$z_0^n x[n] \xleftrightarrow{z} X\left(\frac{z}{z_0}\right), \quad ROC = |z_0| R$$

$$e^{j\omega_0 n} x[n] \xleftrightarrow{z} X(e^{-j\omega_0} z), \quad ROC = R$$



■ Time Reversal:

$$x[n] \xleftrightarrow{\mathcal{Z}} X(z), \quad ROC = R$$

$$X(z) = \sum_{n=-\infty}^{+\infty} x[n] z^{-n}$$

$$x[n] = \frac{1}{2\pi j} \oint X(z) z^{n-1} dz$$

$$x[-n] \xleftrightarrow{\mathcal{Z}} X\left(\frac{1}{z}\right), \quad ROC = \frac{1}{R}$$

■ Time Expansion:

$$X(z) = \sum_{n=-\infty}^{+\infty} x[n]z^{-n}$$

$$x[n] \xleftrightarrow{\mathcal{Z}} X(z), \quad ROC = R$$

$$x[n] = \frac{1}{2\pi j} \oint X(z) z^{n-1} dz$$

$$x_{(k)}[n] = \begin{cases} x[n/k], & \text{if } n = k \cdot m \\ 0, & \text{otherwise} \end{cases} \quad \begin{array}{l} k \text{ is a constant} \\ m \text{ is a new time variable} \end{array}$$

$$x_{(k)}[n] \xleftrightarrow{\mathcal{Z}} X(z^k), \quad ROC = R^{1/k}$$

■ Conjugation:

$$X(z) = \sum_{n=-\infty}^{+\infty} x[n] z^{-n}$$

$$x[n] \xleftrightarrow{Z} X(z), \quad ROC = R$$

$$x[n] = \frac{1}{2\pi j} \oint X(z) z^{n-1} dz$$

$$x^*[n] \xleftrightarrow{Z} X^*(z^*), \quad ROC = R$$

$$X(z) = \sum_{n=-\infty}^{+\infty} x[n] z^{-n}$$

■ Convolution Property:

$$X(z) = \sum_{n=-\infty}^{+\infty} x[n]z^{-n}$$

$$x[n] = \frac{1}{2\pi j} \oint X(z)z^{n-1}dz$$

$$x_1[n] \xleftrightarrow{\mathcal{Z}} X_1(z), \quad ROC = R_1$$

$$x_2[n] \xleftrightarrow{\mathcal{Z}} X_2(z), \quad ROC = R_2$$

$$x_1[n] * x_2[n] \xleftrightarrow{\mathcal{Z}} X_1(z)X_2(z), \quad \text{with } ROC \text{ containing } R_1 \cap R_2$$

$R_1 \cap R_2$ may be larger
if pole-zero cancellation
occurs in the product

- Differentiation in the z-Domain:

$$x[n] \xleftrightarrow{z} X(z), \quad ROC = R$$

$$nx[n] \xleftrightarrow{z} -z \frac{d}{dz} X(z), \quad ROC = R$$

$$X(z) = \sum_{n=-\infty}^{+\infty} x[n] z^{-n}$$

- The Initial-Value Theorem:

If $x[n] = 0$ for $n < 0$

$$X(z) = \sum_{n=-\infty}^{+\infty} x[n] z^{-n}$$

$$\Rightarrow x[0] = \lim_{z \rightarrow \infty} X(z) = x[0] + x[1] z^{-1} + x[2] z^{-2}$$

TABLE 10.1 PROPERTIES OF THE z -TRANSFORM

Section	Property	Signal	z -Transform	ROC
		$x[n]$	$X(z)$	R
		$x_1[n]$	$X_1(z)$	R_1
		$x_2[n]$	$X_2(z)$	R_2
10.5.1	Linearity	$ax_1[n] + bx_2[n]$	$aX_1(z) + bX_2(z)$	At least the intersection of R_1 and R_2
10.5.2	Time shifting	$x[n - n_0]$	$z^{-n_0}X(z)$	R , except for the possible addition or deletion of the origin
10.5.3	Scaling in the z -domain	$e^{j\omega_0 n}x[n]$	$X(e^{-j\omega_0}z)$	R
		$z_0^n x[n]$	$X\left(\frac{z}{z_0}\right)$	$z_0 R$
		$a^n x[n]$	$X(a^{-1}z)$	Scaled version of R (i.e., $ a R$ = the set of points $\{a z\}$ for z in R)
10.5.4	Time reversal	$x[-n]$	$X(z^{-1})$	Inverted R (i.e., R^{-1} = the set of points z^{-1} , where z is in R)
10.5.5	Time expansion	$x_{(k)}[n] = \begin{cases} x[r], & n = rk \\ 0, & n \neq rk \end{cases}$ for some integer r	$X(z^k)$	$R^{1/k}$ (i.e., the set of points $z^{1/k}$, where z is in R)
10.5.6	Conjugation	$x^*[n]$	$X^*(z^*)$	R
10.5.7	Convolution	$x_1[n] * x_2[n]$	$X_1(z)X_2(z)$	At least the intersection of R_1 and R_2
10.5.7	First difference	$x[n] - x[n - 1]$	$(1 - z^{-1})X(z)$	At least the intersection of R and $ z > 0$
10.5.7	Accumulation	$\sum_{k=-\infty}^n x[k]$	$\frac{1}{1 - z^{-1}}X(z)$	At least the intersection of R and $ z > 1$
10.5.8	Differentiation in the z -domain	$nx[n]$	$-z \frac{dX(z)}{dz}$	R
10.5.9	Initial Value Theorem If $x[n] = 0$ for $n < 0$, then $x[0] = \lim_{z \rightarrow \infty} X(z)$			

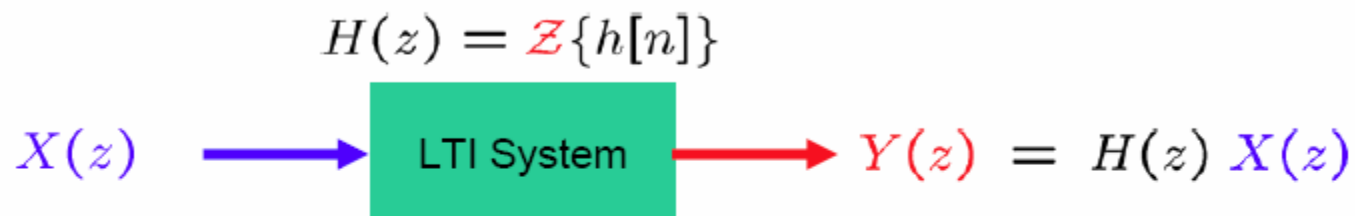
- The z-Transform
- The Region of Convergence for z-Transforms
- The Inverse z-Transform
- Geometric Evaluation of the Fourier Transform
- Properties of the z-Transform
- Some Common z-Transform Pairs
- Analysis & Characterization of LTI Systems Using the z-Transforms
- System Function Algebra and Block Diagram Representations
- The Unilateral z-Transform

TABLE 10.2 SOME COMMON z-TRANSFORM PAIRS

Signal	Transform	ROC
1. $\delta[n]$	1	All z
2. $u[n]$	$\frac{1}{1 - z^{-1}}$	$ z > 1$
3. $-u[-n - 1]$	$\frac{1}{1 - z^{-1}}$	$ z < 1$
4. $\delta[n - m]$	z^{-m}	All z , except 0 (if $m > 0$) or ∞ (if $m < 0$)
5. $\alpha^n u[n]$	$\frac{1}{1 - \alpha z^{-1}}$	$ z > \alpha $
6. $-\alpha^n u[-n - 1]$	$\frac{1}{1 - \alpha z^{-1}}$	$ z < \alpha $
7. $n\alpha^n u[n]$	$\frac{\alpha z^{-1}}{(1 - \alpha z^{-1})^2}$	$ z > \alpha $
8. $-n\alpha^n u[-n - 1]$	$\frac{\alpha z^{-1}}{(1 - \alpha z^{-1})^2}$	$ z < \alpha $
9. $[\cos \omega_0 n]u[n]$	$\frac{1 - [\cos \omega_0]z^{-1}}{1 - [2 \cos \omega_0]z^{-1} + z^{-2}}$	$ z > 1$
10. $[\sin \omega_0 n]u[n]$	$\frac{[\sin \omega_0]z^{-1}}{1 - [2 \cos \omega_0]z^{-1} + z^{-2}}$	$ z > 1$
11. $[r^n \cos \omega_0 n]u[n]$	$\frac{1 - [r \cos \omega_0]z^{-1}}{1 - [2r \cos \omega_0]z^{-1} + r^2 z^{-2}}$	$ z > r$
12. $[r^n \sin \omega_0 n]u[n]$	$\frac{[r \sin \omega_0]z^{-1}}{1 - [2r \cos \omega_0]z^{-1} + r^2 z^{-2}}$	$ z > r$

- The z-Transform
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▪ Analysis & Characterization of LTI Systems:



$H(z)$: system function
or transfer function

- Causality
- Stability

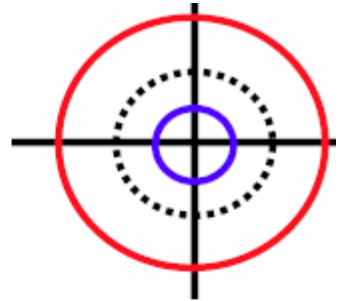
■ Causality:

- For a **causal** LTI system,
 $h[n] = 0$ for $n < 0$, and thus is **right-sided**
- A DT LTI system is **causal** if and only if
the **ROC** of the system function $H(z)$
is the **exterior of a circle** in the z -plane, **including infinity**
- A DT LTI system with a **rational** $H(z)$ is **causal** if and only if
 - (a) **ROC** is exterior of a circle **outside outermost pole**;
and **infinity** must be in the **ROC**; and
 - (b) **order of numerator** $\leq$ **order of denominator**

$$H(z) = \frac{b_0 z^M + b_1 z^{M-1} + \dots + b_M}{a_0 z^N + a_1 z^{N-1} + \dots + a_N} = \frac{Y(z)}{X(z)} \quad y[n+N] + \dots = x[n+M] + \dots$$

▪ Example 10.21:

$$H(z) = \frac{1}{1 - \frac{1}{2}z^{-1}} + \frac{1}{1 - 2z^{-1}}, \quad |z| > 2$$



$\Rightarrow$ *ROC*: the exterior of a circle of outside the outermost pole

$\Rightarrow$ the impulse response is right-sided

$$H(z) = \frac{2 - \frac{5}{2}z^{-1}}{(1 - \frac{1}{2}z^{-1})(1 - 2z^{-1})} = \frac{2z(z - \frac{5}{4})}{(z - \frac{1}{2})(z - 2)} = \frac{2z^2 - \frac{5}{2}z}{z^2 - \frac{5}{2}z + 1}$$

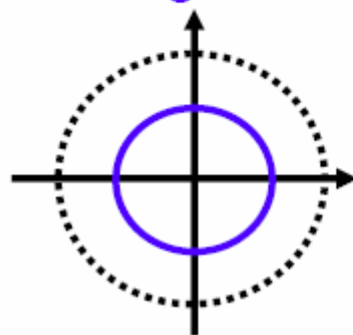
$\Rightarrow$ deg of num of $H(z)$ = deg of den of $H(z)$

$\Rightarrow$ the system is causal

$$\Rightarrow h[n] = \left[\left(\frac{1}{2}\right)^n + 2^n \right] u[n] \qquad \Rightarrow h[n] = 0, n < 0$$

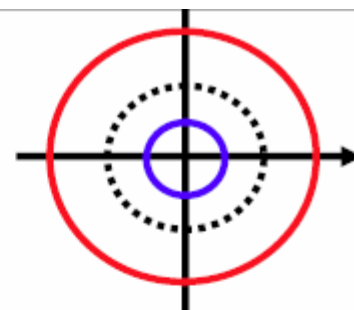
▪ Stability:

- An DT LTI system is **stable** if and only if the **ROC** of $H(z)$ includes the **unit circle** [i.e., $|z| = 1$]
- A **causal** LTI system with **rational** $H(z)$ is **stable** if and only if **all of the poles** of $H(z)$ lie in the **inside the unit circle**, i.e., all of the poles have **magnitude < 1**



▪ Example 10.22:

$$H(z) = \frac{1}{1 - \frac{1}{2}z^{-1}} + \frac{1}{1 - 2z^{-1}}, \quad |z| > 2$$



$\Rightarrow$ *ROC* does not include the unit circle $\Rightarrow$ **NOT stable**

$$\Rightarrow \text{i.e., } h[n] = \left[\left(\frac{1}{2} \right)^n + 2^n \right] u[n] \rightarrow \infty, \text{ as } n \rightarrow \infty$$

• If *ROC* = $1/2 < |z| < 2$ $\Rightarrow h[n] = \left(\frac{1}{2} \right)^n u[n] - 2^n u[-n - 1]$

$\Rightarrow$ the system is **NOT causal**, but **stable**

• If *ROC* = $|z| < 1/2$ $\Rightarrow h[n] = - \left[\left(\frac{1}{2} \right)^n + 2^n \right] u[-n - 1]$

$\Rightarrow$ the system is **neither causal nor stable**

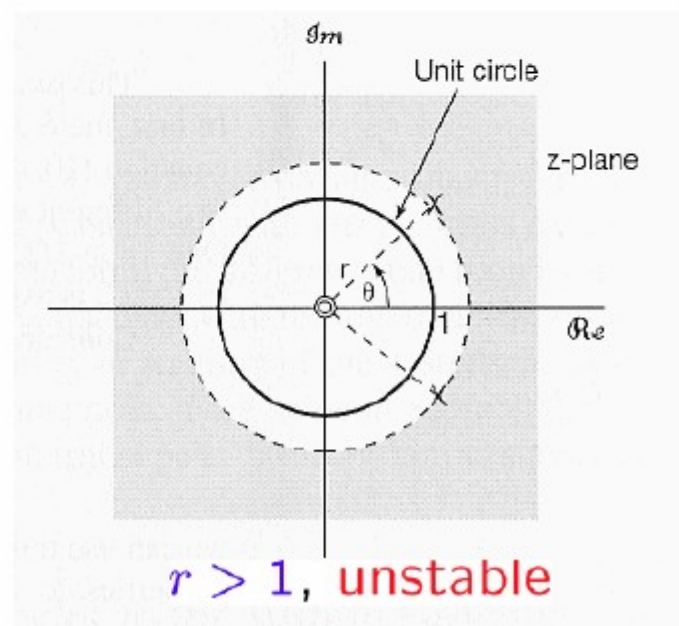
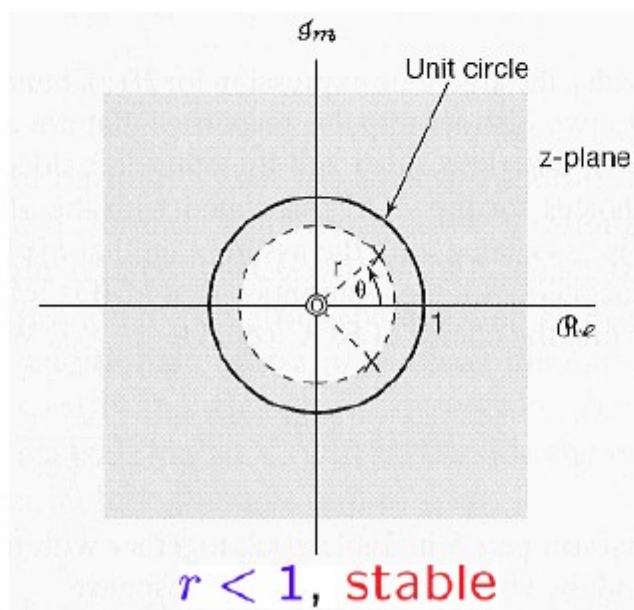
■ Example 10.24:

$$H(z) = \frac{1}{1 - (2r \cos \theta)z^{-1} + r^2 z^{-2}}$$

$$= \frac{z^2}{z^2 - (2r \cos \theta)z + r^2}$$

$$\Rightarrow z_1 = re^{j\theta}, \quad z_2 = re^{-j\theta}$$

If it is **causal**, $|z| > |r|$

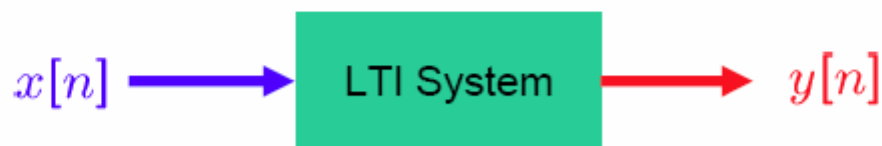


- LTI Systems by Linear Constant-Coef Difference Equations:

$$a_0y[n] + a_1y[n-1] + \cdots + a_{N-1}y[n-N+1] + a_Ny[n-N]$$

$$= b_0x[n] + b_1x[n-1] + \cdots + b_{M-1}x[n-M+1] + b_Mx[n-M]$$

$$\sum_{k=0}^N a_k y[n-k] = \sum_{k=0}^M b_k x[n-k]$$



$$Y(z) = X(z)H(z) \qquad H(z) = \frac{Y(z)}{X(z)}$$

$$x[n - n_0] \xleftrightarrow{\mathcal{Z}} z^{-n_0} X(z)$$

$$\mathcal{Z} \left\{ \sum_{k=0}^N a_k y[n - k] \right\} = \mathcal{Z} \left\{ \sum_{k=0}^M b_k x[n - k] \right\}$$

$$\sum_{k=0}^N a_k \mathcal{Z} \left\{ y[n - k] \right\} = \sum_{k=0}^M b_k \mathcal{Z} \left\{ x[n - k] \right\}$$

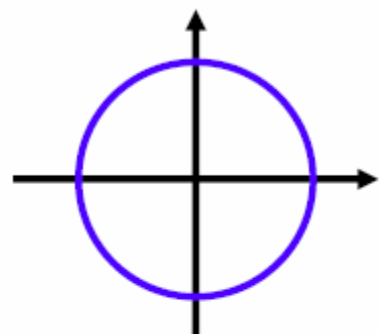
$$\sum_{k=0}^N a_k z^{-k} Y(z) = \sum_{k=0}^M b_k z^{-k} X(z)$$

$$\Rightarrow H(z) = \frac{Y(z)}{X(z)} = \frac{\sum_{k=0}^M b_k z^{-k}}{\sum_{k=0}^N a_k z^{-k}} = \frac{b_0 + b_1 z^{-1} + \dots + b_M z^{-M}}{a_0 + a_1 z^{-1} + \dots + a_N z^{-N}}$$

$$= \frac{b_0 z^N + b_1 z^{N-1} + \dots + b_M z^{N-M}}{a_0 z^N + a_1 z^{N-1} + \dots + a_N} \quad \begin{array}{l} \text{zeros} \\ \text{poles} \end{array}$$

▪ Example 10.25: $y[n] - \frac{1}{2}y[n-1] = x[n] + \frac{1}{3}x[n-1]$

$$\Rightarrow Y(z) - \frac{1}{2}z^{-1}Y(z) = X(z) + \frac{1}{3}z^{-1}X(z)$$



$$\Rightarrow H(z) = \frac{1 + \frac{1}{3}z^{-1}}{1 - \frac{1}{2}z^{-1}} = \frac{1}{1 - \frac{1}{2}z^{-1}} + \frac{\frac{1}{3}z^{-1}}{1 - \frac{1}{2}z^{-1}}$$

$$= \frac{z + \frac{1}{3}}{z - \frac{1}{2}} = \frac{z}{z - \frac{1}{2}} + \frac{\frac{1}{3}}{z - \frac{1}{2}}$$

- If $ROC = \{|z| > 1/2\}$, $\Rightarrow h[n]$ is right-sided

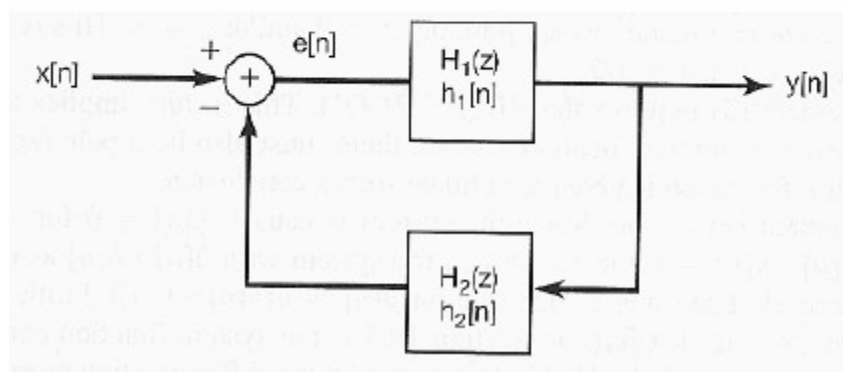
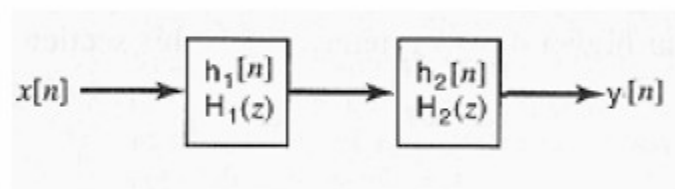
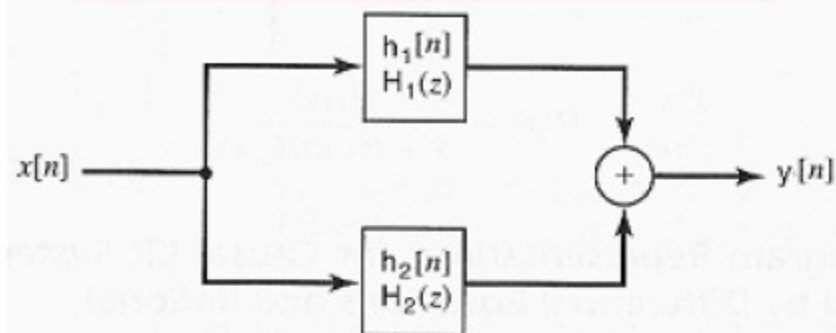
$$\Rightarrow h[n] = \left(\frac{1}{2}\right)^n u[n] + \frac{1}{3}\left(\frac{1}{2}\right)^{n-1} u[n-1]$$

- If $ROC = \{|z| < 1/2\}$, $\Rightarrow h[n]$ is left-sided

$$\Rightarrow h[n] = -\left(\frac{1}{2}\right)^n u[-n-1] - \frac{1}{3}\left(\frac{1}{2}\right)^{n-1} u[-n]$$

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■ System Function Blocks:



- parallel interconnection

$$h[n] = h_1[n] + h_2[n]$$

$$H(z) = H_1(z) + H_2(z)$$

- series interconnection

$$h[n] = h_1[n] * h_2[n]$$

$$H(z) = H_1(z) H_2(z)$$

- feedback interconnection

$$Y = H_1 E$$

$$Z = H_2 Y$$

$$E = X - Z$$

$$H(z) = \frac{H_1(z)}{1 + H_1(z)H_2(z)}$$

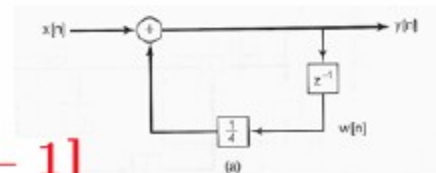
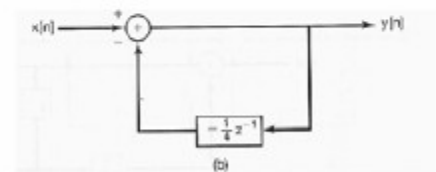
■ Example 10.28:

- Consider a **causal LTI** system with system function

$$H(z) = \frac{1}{1 - \frac{1}{4}z^{-1}}$$

$$\Rightarrow y[n] = \frac{1}{1 - \frac{1}{4}z^{-1}}x[n] \quad \Rightarrow y[n] - \frac{1}{4}y[n-1] = x[n]$$

$$\Rightarrow y[n] = \frac{1}{4}y[n-1] + x[n]$$

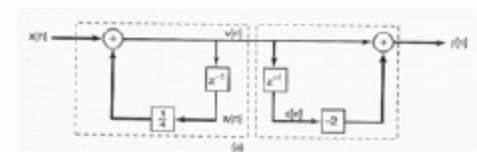
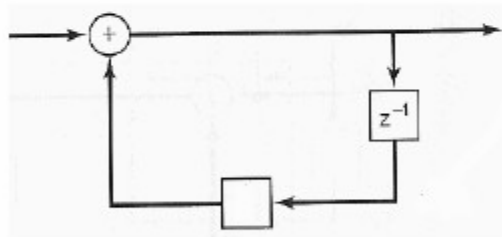


$$w[n] = y[n-1]$$

■ Example 10.29:

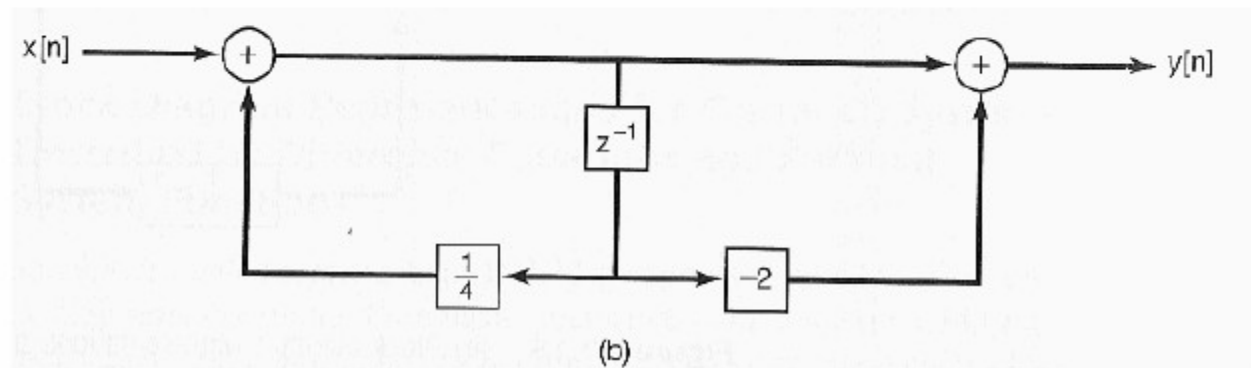
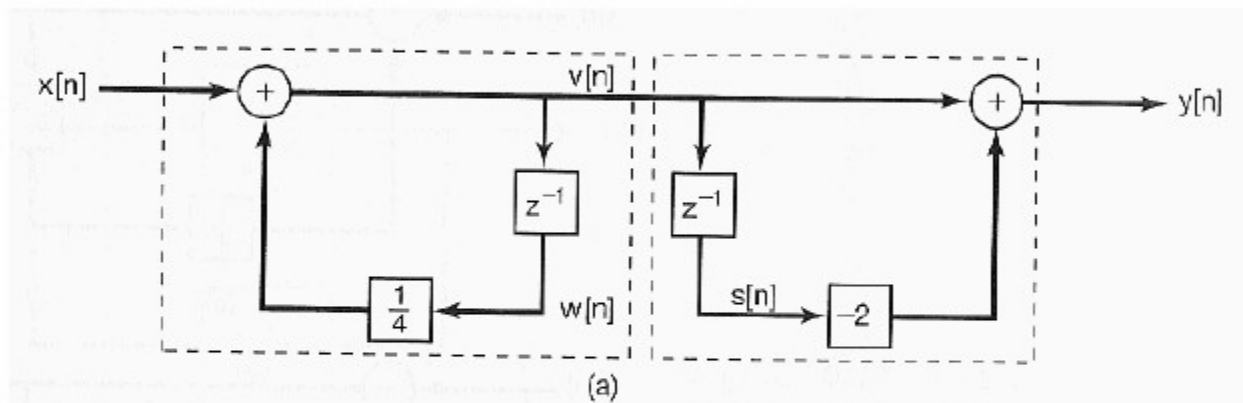
$$\frac{Y(z)}{X(z)} = H(z) = \frac{1 - 2z^{-1}}{1 - \frac{1}{4}z^{-1}} = \left(\frac{1}{1 - \frac{1}{4}z^{-1}} \right) (1 - 2z^{-1})$$

$$Y(z) = (1 - 2z^{-1})V(z) \qquad V(z) = \left(\frac{1}{1 - \frac{1}{4}z^{-1}} \right) X(z)$$



■ Example 10.29:

$$H(z) = \frac{1 - 2z^{-1}}{1 - \frac{1}{4}z^{-1}} = \left(\frac{1}{1 - \frac{1}{4}z^{-1}} \right) (1 - 2z^{-1})$$



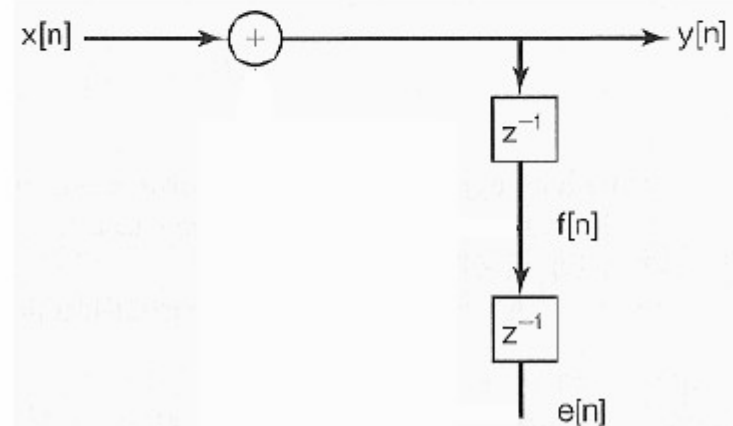
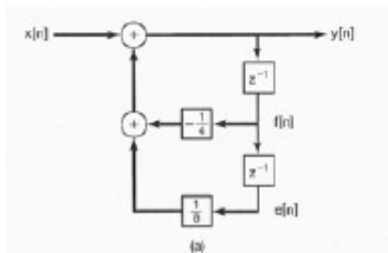
▪ Example 10.30:

$$H(z) = \frac{1}{(1 + \frac{1}{2}z^{-1})(1 - \frac{1}{4}z^{-1})} = \frac{1}{1 + \frac{1}{4}z^{-1} - \frac{1}{8}z^{-2}}$$

$$\Rightarrow y[n] + \frac{1}{4}y[n-1] - \frac{1}{8}y[n-2] = x[n]$$

$$\Rightarrow y[n] = -\frac{1}{4}y[n-1] + \frac{1}{8}y[n-2] + x[n]$$

$$\Rightarrow \begin{cases} f[n] = y[n-1] \\ e[n] = f[n-1] = y[n-2] \end{cases}$$

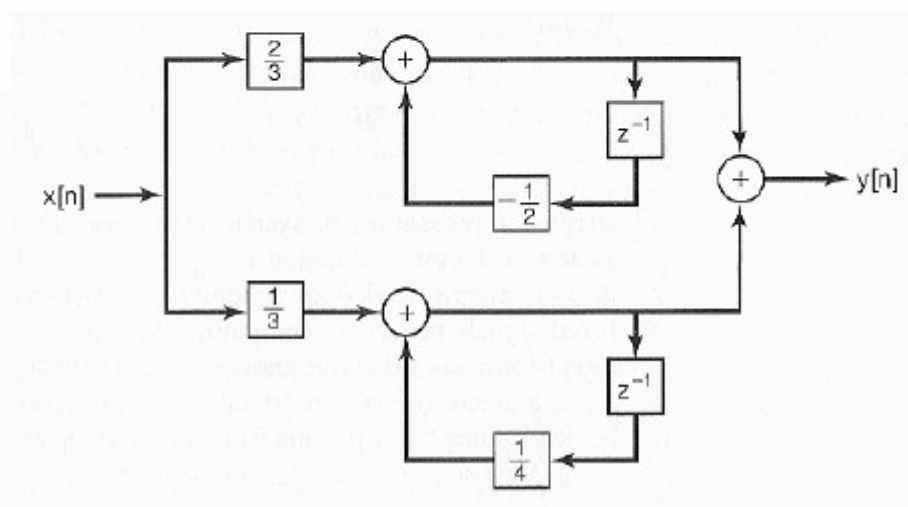
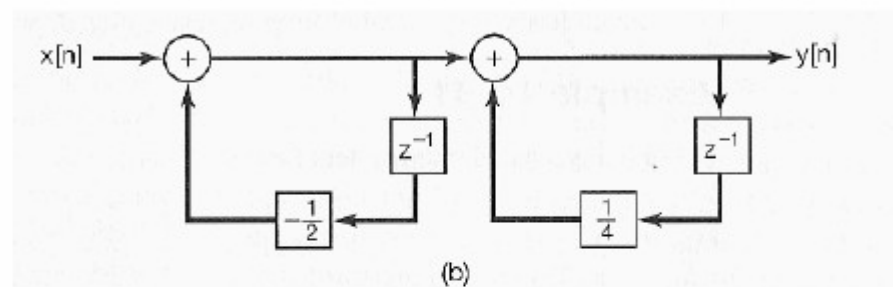
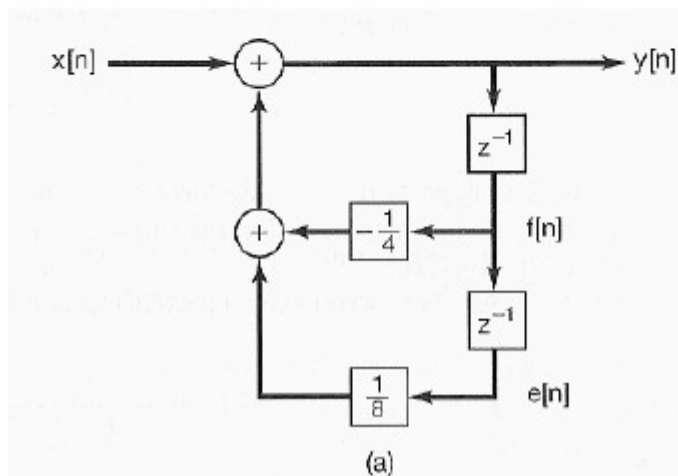


■ Example 10.30:

$$H(z) = \frac{1}{1 + \frac{1}{4}z^{-1} - \frac{1}{8}z^{-2}}$$

$$H(z) = \left(\frac{1}{1 + \frac{1}{2}z^{-1}} \right) \left(\frac{1}{1 - \frac{1}{4}z^{-1}} \right)$$

$$H(z) = \frac{\frac{2}{3}}{(1 + \frac{1}{2}z^{-1})} + \frac{\frac{1}{3}}{(1 - \frac{1}{4}z^{-1})}$$



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- The Unilateral z-Transform of x(t):

bilateral zT

$$X(z) \triangleq \sum_{n=-\infty}^{+\infty} x[n]z^{-n}$$

$$x[n] \xleftrightarrow{\mathcal{Z}} X(z)$$

$$X(z) = \mathcal{Z}\{x[n]\}$$

$$x[n] = \mathcal{Z}^{-1}\{X(z)\}$$

unilateral zT

for causal system &
with nonzero init. cond.

$$\mathcal{X}(z) \triangleq \sum_{n=0}^{+\infty} x[n]z^{-n}$$

$$x[n] \xleftrightarrow{\mathcal{UZ}} \mathcal{X}(z)$$

$$\mathcal{X}(z) = \mathcal{UZ}\{x[n]\}$$

$$x[n] = \mathcal{UZ}^{-1}\{\mathcal{X}(z)\}$$

ROC : exterior of a circle

■ Example 10.33: $x[n] = a^{n+1}u[n+1]$

- since $x[-1] = 1 \neq 0$, $X(z) \neq \mathcal{X}(z)$

$\Rightarrow$ bilateral transform : $X(z) = \frac{z}{1 - az^{-1}}, \quad |z| > |a|$

$\Rightarrow$ unilateral transform : $\mathcal{X}(z) = \sum_{n=0}^{+\infty} x[n]z^{-n}$

$$= \sum_{n=0}^{+\infty} a^{n+1}z^{-n}$$
$$= \frac{a}{1 - az^{-1}}, \quad |z| > |a|$$

TABLE 10.3 PROPERTIES OF THE UNILATERAL z -TRANSFORM

Property	Signal	Unilateral z -Transform
—	$x[n]$	$\mathcal{X}(z)$
—	$x_1[n]$	$\mathcal{X}_1(z)$
—	$x_2[n]$	$\mathcal{X}_2(z)$
Linearity	$ax_1[n] + bx_2[n]$	$a\mathcal{X}_1(z) + b\mathcal{X}_2(z)$
Time delay	$x[n-1]$	$z^{-1}\mathcal{X}(z) + x[-1]$
Time advance	$x[n+1]$	$z\mathcal{X}(z) - zx[0]$
Scaling in the z -domain	$e^{j\omega_0 n}x[n]$	$\mathcal{X}(e^{-j\omega_0}z)$
	$z_0^n x[n]$	$\mathcal{X}(z/z_0)$
	$a^n x[n]$	$\mathcal{X}(az)$
Time expansion	$x_k[n] = \begin{cases} x[m], & n = mk \\ 0, & n \neq mk \end{cases}$ for any m	$\mathcal{X}(z^k)$
Conjugation	$x^*[n]$	$\mathcal{X}^*(z^*)$
Convolution (assuming that $x_1[n]$ and $x_2[n]$ are identically zero for $n < 0$)	$x_1[n] * x_2[n]$	$\mathcal{X}_1(z)\mathcal{X}_2(z)$
First difference	$x[n] - x[n-1]$	$(1 - z^{-1})\mathcal{X}(z) - x[-1]$
Accumulation	$\sum_{k=0}^n x[k]$	$\frac{1}{1 - z^{-1}}\mathcal{X}(z)$
Differentiation in the z -domain	$nx[n]$	$-z \frac{d\mathcal{X}(z)}{dz}$
Initial Value Theorem		
$x[0] = \lim_{z \rightarrow \infty} \mathcal{X}(z)$		

- The **z**-Transform
- The **ROC** for **z**-T
- The **Inverse z**-T
- **Geometric Evaluation** of the **FT**
- **Properties of the z-T**
 - Linearity
 - Time Reversal
 - Convolution
 - Differentiation in the **z**-Domain
 - Time Shifting
 - Time Expansion
 - First Difference
 - Shifting in the **z**-Domain
 - Conjugation
 - Accumulation
 - Initial-Value Theorems
- **Some Common z-T Pairs**
- **Analysis & Charac. of LTI Systems Using the z-T**
- **System Function Algebra, Block Diagram Repre.**
- **The Unilateral z-T**

Introduction

(Chap 1)

LTI & Convolution

(Chap 2)

Bounded/Convergent

Periodic

FS

(Chap 3)

– CT
– DT

Aperiodic

FT

– CT (Chap 4)
– DT (Chap 5)

Unbounded/Non-convergent

LT

– CT (Chap 9)

zT

– DT (Chap 10)

Time-Frequency (Chap 6)

CT-DT (Chap 7)

Communication (Chap 8)

Control (Chap 11)