

EEM322 : Lecture 2

MODELING Of SYSTEMS
MODELING IN FREQUENCY DOMAIN

- Transfer function
- Laplace transformation
- Inverse Laplace transformation

Modeling

To study the **behaviour of a system** we must establish functional relationships between different variables. **These relationships are given by a set of mathematical equations, called the *mathematical model of the system*.**

Mathematical Model:

Mathematical model of a system is a tool used to study the behaviors of the system without having to do the experiments. OR Mathematical model: relationships

between measurable physical quantities (distances, current, flows, unemployment, etc.) in the system are described as mathematical Relations in the model.

The model-building process always involves approximations and simplifying assumptions, valid for a particular set of operating conditions. This is because the physical reality is always so complex that it is not possible to make a manageable model taking into account all the aspects with all their details.

The key factors to be examined in developing a mathematical model for a system are as follows:

- The structure
- The components and their inter-connections
- The **parameters**
- The **variables**
- The **input and outputs**
- The **initial conditions**

Modelling and analysis of dynamic systems:

An accurate mathematical model that describes a system completely must be determined in order to analyze a dynamic system.

Dynamic system can be completely described by

- known differential equations or by
- experimental test data

The ability to analyze the system and determine its performance depends on how well the characteristics can be expressed mathematically.

Modelling similarities (Analogous systems)

The analysis technique, once developed for a second order system, would be applicable to *any second order system*, irrespective of whether the system is electrical, mechanical, thermal or hydraulic in nature. This general approach of developing an analysis technique, based on the type of mathematical model of the system, is the key factor which permits a common systems approach to be used for studying problems from different branches of engineering and even non-engineering disciplines, like biological systems, economic systems, management systems, etc.

Mathematical Model:

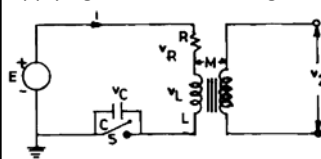
The mathematical model is a set of equations relating the different system variables.

This relationship is governed by the physical laws applicable to the particular system.

(example: Kirchhoff's laws.)

assume that the current through the coil and the voltage across the capacitor are zero, $t = 0$.

Applying Kirchhoff's voltage law



$$v_L + v_R = E \quad \text{or} \quad L \frac{di}{dt} + iR = E.$$

The output variable V_2 is related to the internal variable i by $v_2 = M \frac{di}{dt}$

There is another operating condition for which the above model does not provide the correct response. This condition occurs when the switch S is opened after remaining closed for some time. In this case the capacitor C will be in series with R and L . The circuit equation will be,

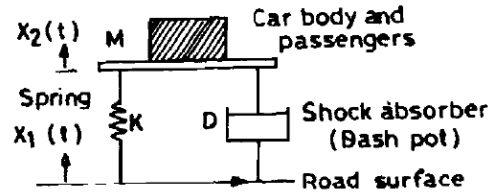
$$L \frac{di}{dt} + Ri + \frac{1}{C} \int i dt = E. \quad \longrightarrow \quad L \frac{d^2 i}{dt^2} + R \frac{di}{dt} + \frac{i}{C} = 0.$$

Assumptions: L is assumed to be independent of current i , that is, the inductance is assumed to be linear.

Automobile Suspension System:

The object of this system is to damp out vibrations and jerks produced in a vehicle due to uneven road surface. **The suspension system absorbs these jerks and prevents them from being transmitted to the passengers**, thus improving the riding quality of the vehicle.

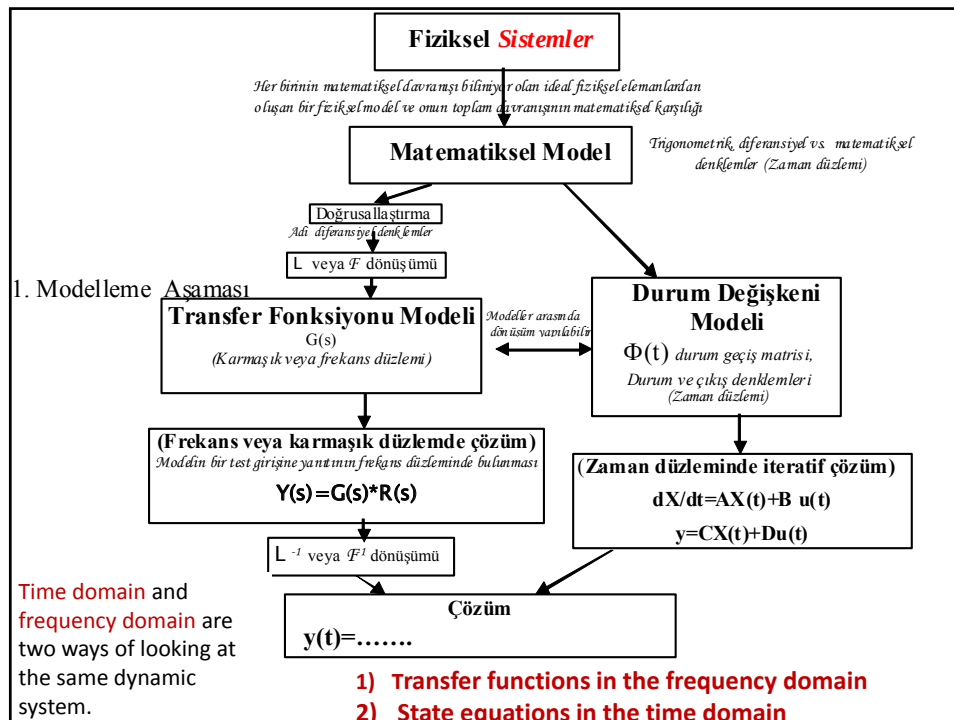
- The **mass of the car body** and passengers is represented by the **mass M**,
- suspension springs** by the ideal spring with spring constant **K**,
- shock absorber** by an ideal **dash-pot** with coefficient of viscous **friction D**.



The system **variables** are: **displacement x**, **velocity u** and **acceleration, a**.
the **input variable** is the vertical displacement $x_1(t)$ of the axle due to undulations in the road surface.

The output variable is the vertical **motion** $x_2(t)$ of the mass M . The mathematical model should relate these input and output variables.

$$M \frac{d^2 x}{dt^2} + D \frac{dx}{dt} + K x = 0.$$



Modelling in frequency domain

Transfer function describes relationship between two signals as a function of S . A frequency-domain transfer function is limited to describing elements that are **linear and time invariant**. The **Laplace Transform is used in system theory to simplify the problem of solving differential equations. (ODE's)**

There are **three steps in the solution process**.

1. The Laplace transform is used to convert the **differential equation into an algebraic equation**.
2. The algebraic equation is solved for the Laplace transform of the solution, and the concept of *partial fractions* is used to separate the solution into its elementary components.
3. Finally, the **inverse Laplace transform** is taken of each of the elementary components to find the solution in the time domain.

COMPLEX FUNCTIONS

Complex variable. A complex number has a real part and an imaginary part, both of which are constant. If the real part and/or imaginary part are variables, a complex number is called a *complex variable*. In the Laplace transformation we use the notation s as a complex variable; that is,

$$s = \sigma + j\omega \quad F(s) = F_x + jF_y \quad j = \sqrt{-1} \quad \text{The magnitude of } F(s) \text{ is } \sqrt{F_x^2 + F_y^2}$$

Laplace Transform: Use Laplace transformation - to obtain linear approximation of physical systems. The Laplace Transform is used in system theory to simplify the problem of solving differential equations.

$$L[f(t)] = F(s) = \int_0^{\infty} f(t)e^{-st} dt$$

Definition: Ratio of Laplace transformed output $C(s)$, divided by the Laplace transformed input $R(s)$ with assumed zero initial conditions.

Example: Find the Laplace transform

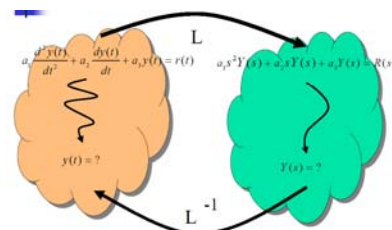
$$L[f(t)] = \int_0^{\infty} te^{-5t} dt$$

$\frac{1}{s^2}$

$\frac{1}{s+a}$

$$f(t) = te^{-5t} = \frac{1}{(s+5)^2}$$

$$s = \sigma + j\omega$$



Find the Laplace transform of $f(t) = Ae^{-at}u(t)$

Example:

$$= A \int_0^{\infty} e^{-(s+a)t} dt$$

$$L[f(t)] = \int_0^{\infty} Ae^{-at} e^{-st} dt$$

$$= A \left[\frac{1}{s+a} e^{-(s+a)t} \right]_0^{\infty}$$

$$= A \cdot \left(\frac{1}{s+a} \right) = \frac{A}{s+a}$$

	$f(t)$	$F(s)$
1	Unit impulse $\delta(t)$	1
2	Unit step $1(t)$	$\frac{1}{s}$
3	t	$\frac{1}{s^2}$
4	$\frac{t^{n-1}}{(n-1)!} \quad (n = 1, 2, 3, \dots)$	$\frac{1}{s^n}$
5	$t^n \quad (n = 1, 2, 3, \dots)$	$\frac{n!}{s^{n+1}}$
6	e^{-at}	$\frac{1}{s+a}$
7	te^{-at}	$\frac{1}{(s+a)^2}$
8	$\frac{1}{(n-1)!} t^{n-1} e^{-at} \quad (n = 1, 2, 3, \dots)$	$\frac{1}{(s+a)^n}$
9	$t^n e^{-at} \quad (n = 1, 2, 3, \dots)$	$\frac{n!}{(s+a)^{n+1}}$

10	$\sin \omega t$	$\frac{\omega}{s^2 + \omega^2}$
11	$\cos \omega t$	$\frac{s}{s^2 + \omega^2}$
12	$\sinh \omega t$	$\frac{\omega}{s^2 - \omega^2}$
13	$\cosh \omega t$	$\frac{s}{s^2 - \omega^2}$
14	$\frac{1}{a} (1 - e^{-at})$	$\frac{1}{s(s+a)}$
15	$\frac{1}{b-a} (e^{-at} - e^{-bt})$	$\frac{1}{(s+a)(s+b)}$
16	$\frac{1}{b-a} (be^{-bt} - ae^{-at})$	$\frac{s}{(s+a)(s+b)}$
17	$\frac{1}{ab} \left[1 + \frac{1}{a-b} (be^{-at} - ae^{-bt}) \right]$	$\frac{1}{s(s+a)(s+b)}$

18	$\frac{1}{a^2} (1 - e^{-at} - ate^{-at})$	$\frac{1}{s(s+a)^2}$
19	$\frac{1}{a^2} (at - 1 + e^{-at})$	$\frac{1}{s^2(s+a)}$
20	$e^{-at} \sin \omega t$	$\frac{\omega}{(s+a)^2 + \omega^2}$
21	$e^{-at} \cos \omega t$	$\frac{s+a}{(s+a)^2 + \omega^2}$
22	$\frac{\omega_n}{\sqrt{1-\xi^2}} e^{-\xi\omega_n t} \sin \omega_n \sqrt{1-\xi^2} t$	$\frac{\omega_n^2}{s^2 + 2\xi\omega_n s + \omega_n^2}$
23	$-\frac{1}{\sqrt{1-\xi^2}} e^{-\xi\omega_n t} \sin(\omega_n \sqrt{1-\xi^2} t - \phi)$ $\phi = \tan^{-1} \frac{\sqrt{1-\xi^2}}{\xi}$	$\frac{s}{s^2 + 2\xi\omega_n s + \omega_n^2}$
24	$1 - \frac{1}{\sqrt{1-\xi^2}} e^{-\xi\omega_n t} \sin(\omega_n \sqrt{1-\xi^2} t + \phi)$ $\phi = \tan^{-1} \frac{\sqrt{1-\xi^2}}{\xi}$	$\frac{\omega_n^2}{s(s^2 + 2\xi\omega_n s + \omega_n^2)}$

25	$1 - \cos \omega t$	$\frac{\omega^2}{s(s^2 + \omega^2)}$
26	$\omega t - \sin \omega t$	$\frac{\omega^3}{s^2(s^2 + \omega^2)}$
27	$\sin \omega t - \omega t \cos \omega t$	$\frac{2\omega^3}{(s^2 + \omega^2)^2}$
28	$\frac{1}{2\omega} t \sin \omega t$	$\frac{s}{(s^2 + \omega^2)^2}$
29	$t \cos \omega t$	$\frac{s^2 - \omega^2}{(s^2 + \omega^2)^2}$
30	$\frac{1}{\omega_2^2 - \omega_1^2} (\cos \omega_1 t - \cos \omega_2 t) \quad (\omega_1^2 \neq \omega_2^2)$	$\frac{s}{(s^2 + \omega_1^2)(s^2 + \omega_2^2)}$
31	$\frac{1}{2\omega} (\sin \omega t + \omega t \cos \omega t)$	$\frac{s^2}{(s^2 + \omega^2)^2}$

$$C(s) = \frac{(s+2)}{(s+5)} \cdot R(s) = \frac{(s+2)}{(s+5)} \cdot \frac{1}{s} = \frac{A}{s} + \frac{B}{(s+5)} \rightarrow c(t) = \frac{2}{5} + \frac{3}{5} e^{-5t}$$

Example for inverse Laplace

Obtain the inverse Laplace transform of **Inverse Laplace**

$$G(s) = \frac{s^3 + 5s^2 + 9s + 7}{(s+1)(s+2)} \quad G(s) = s + 2 + \frac{s+3}{(s+1)(s+2)}$$

Note that the Laplace transform of the unit-impulse function $\delta(t)$ is 1 and that the Laplace transform of $d\delta(t)/dt$ is s . The third term on the right-hand side of this last equation is $F(s)$ in Example

$$g(t) = \frac{d}{dt} \delta(t) + 2\delta(t) + 2e^{-t} - e^{-2t}, \quad \text{for } t \geq 0-$$

Find the inverse Laplace transform of $F(s) = \frac{2s+12}{s^2+2s+5}$

$$s^2 + 2s + 5 = (s+1+j2)(s+1-j2)$$

Noting that $s^2 + 2s + 5 = (s+1)^2 + 2^2$ and referring to the Laplace transforms of $e^{-\alpha t} \sin \omega t$ and $e^{-\alpha t} \cos \omega t$, rewritten thus,

$$\begin{aligned} \mathcal{L}[e^{-\alpha t} \sin \omega t] &= \frac{\omega}{(s+\alpha)^2 + \omega^2} \\ \mathcal{L}[e^{-\alpha t} \cos \omega t] &= \frac{s+\alpha}{(s+\alpha)^2 + \omega^2} \\ F(s) &= \frac{2s+12}{s^2+2s+5} = \frac{10+2(s+1)}{(s+1)^2+2^2} \\ &= 5 \frac{2}{(s+1)^2+2^2} + 2 \frac{s+1}{(s+1)^2+2^2} \\ &= 5 \mathcal{L}^{-1} \left[\frac{2}{(s+1)^2+2^2} \right] + 2 \mathcal{L}^{-1} \left[\frac{s+1}{(s+1)^2+2^2} \right] \\ &= 5e^{-t} \sin 2t + 2e^{-t} \cos 2t, \quad \text{for } t \geq 0 \end{aligned}$$

PROBLEM: Given the following differential equation, solve for $y(t)$ if all initial conditions are zero. Use the Laplace Transform

$$\frac{d^2 y}{dt^2} + 12 \frac{dy}{dt} + 32y = 32u(t)$$

SOLUTION:

$$s^2 Y(s) + 12sY(s) + 32Y(s) = \frac{32}{s}$$

$$Y(s) = \frac{32}{s(s^2 + 12s + 32)} = \frac{32}{s(s+4)(s+8)}$$

$$Y(s) = \frac{32}{s(s+4)(s+8)} = \frac{K_1}{s} + \frac{K_2}{(s+4)} + \frac{K_3}{(s+8)}$$

inverse Laplace transforms

$$y(t) = (1 - 2e^{-4t} + e^{-8t})u(t)$$

$$\left[\begin{aligned} K_1 &= \frac{32}{(s+4)(s+8)} \Big|_{s \rightarrow 0} = 1 \\ K_2 &= \frac{32}{s(s+8)} \Big|_{s \rightarrow -4} = -2 \\ K_3 &= \frac{32}{s(s+4)} \Big|_{s \rightarrow -8} = 1 \\ Y(s) &= \frac{1}{s} - \frac{2}{(s+4)} + \frac{1}{(s+8)} \end{aligned} \right.$$

Find: a) the partial fraction expansion of $X(s)$, and b) $x(t)$.

$$X(s) = \frac{5s+3}{(s+1)(s+2)(s+3)} \quad X(s) = \frac{K_1}{s+1} + \frac{K_2}{s+2} + \frac{K_3}{s+3}$$

$$K_1 = [(s+1)X(s)]_{s=-1} = \left(\frac{5s+3}{(s+2)(s+3)} \right)_{s=-1} = -1$$

$$K_2 = [(s+2)X(s)]_{s=-2} = \left(\frac{5s+3}{(s+1)(s+3)} \right)_{s=-2} = 7$$

$$K_3 = [(s+3)X(s)]_{s=-3} = \left(\frac{5s+3}{(s+1)(s+2)} \right)_{s=-3} = -6$$

$$X(s) = \left(\frac{-1}{s+1} \right) + \left(\frac{7}{s+2} \right) + \left(\frac{-6}{s+3} \right)$$

$$x(t) = -e^{-t} + 7e^{-2t} - 6e^{-3t}$$

$$X(s) = \frac{5s+15}{(s+2)(s^2+3s+9)} \quad \text{Find: a) the partial fraction expansion of } X(s), \text{ and b) } x(t).$$

$$X(s) = \left(\frac{K}{s+2} \right) + \left(\frac{As+B}{s^2+3s+9} \right)$$

$$K = [(s+2)X(s)]_{s=-2} = \left[\frac{5s+15}{s^2+3s+9} \right]_{s=-2} = \frac{5}{7}$$

$$\begin{aligned} 5s+15 &= K(s^2+3s+9) + (As+B)(s+2) \\ &= (K+A)s^2 + (2A+B+3K)s + (9K+2B) \end{aligned}$$

$$A+K=0 \quad (s^2) \quad A = -\frac{5}{7} \quad B = \frac{30}{7}$$

$$\begin{aligned} 2A+B+3K &= 5 \quad (s^1) \\ 9K+2B &= 15 \quad (s^0) \end{aligned} \quad X(s) = \frac{5}{7} \left(\frac{1}{s+2} \right) - \frac{5}{7} \left(\frac{s-6}{s^2+3s+9} \right) = \frac{5}{7} \left(\frac{1}{s+2} \right) - \frac{5}{7} \left(\frac{s+(-6)}{(s+\frac{3}{2})^2 + \frac{27}{4}} \right)$$

$$x(t) = \frac{5}{7} \left[e^{-2t} - 3.055 e^{-1.5t} \sin(2.5981t + \phi) \right]$$

Partial-fraction expansion when $F(s)$ involves multiple poles.

$$F(s) = \frac{s^2 + 2s + 3}{(s + 1)^3} \quad F(s) = \frac{B(s)}{A(s)} = \frac{b_1}{s + 1} + \frac{b_2}{(s + 1)^2} + \frac{b_3}{(s + 1)^3}$$

$$(s + 1)^3 \frac{B(s)}{A(s)} = b_1(s + 1)^2 + b_2(s + 1) + b_3$$

$$\begin{aligned} b_3 &= \left[(s + 1)^3 \frac{B(s)}{A(s)} \right]_{s=-1} = \left[(s^2 + 2s + 3) \right]_{s=-1} = 2 \\ b_2 &= \left\{ \frac{d}{ds} \left[(s + 1)^3 \frac{B(s)}{A(s)} \right] \right\}_{s=-1} = \left[\frac{d}{ds} (s^2 + 2s + 3) \right]_{s=-1} = (2s + 2)_{s=-1} = 0 \\ b_1 &= \frac{1}{2!} \left\{ \frac{d^2}{ds^2} \left[(s + 1)^3 \frac{B(s)}{A(s)} \right] \right\}_{s=-1} = \frac{1}{2!} \left[\frac{d^2}{ds^2} (s^2 + 2s + 3) \right]_{s=-1} = \frac{1}{2} (2) = 1 \end{aligned}$$

$$\begin{aligned} f(t) &= \mathcal{L}^{-1}[F(s)] \\ &= \mathcal{L}^{-1} \left[\frac{1}{s + 1} \right] + \mathcal{L}^{-1} \left[\frac{0}{(s + 1)^2} \right] + \mathcal{L}^{-1} \left[\frac{2}{(s + 1)^3} \right] \\ &= e^{-t} + 0 + t^2 e^{-t} \\ &= (1 + t^2) e^{-t}, \quad \text{for } t \geq 0 \end{aligned}$$

Example: Roots of the Denominator of $F(s)$ Are Real and Repeated An example of an $F(s)$ with real and repeated roots in the denominator is

$$F(s) = \frac{2}{(s + 1)(s + 2)^2}$$

Solution:

$$F(s) = \frac{2}{(s + 1)(s + 2)^2} = \frac{K_1}{(s + 1)} + \frac{K_2}{(s + 2)^2} + \frac{K_3}{(s + 2)}$$

then $K_1 = 2$, which can be found as previously described

K_2 can be isolated by multiplying by $(s + 2)^2$, yielding

$$\frac{2}{s + 1} = (s + 2)^2 \frac{K_1}{(s + 1)} + K_2 + (s + 2)K_3$$

Letting s approach -2 , $K_2 = -2$. To find

K_3 we see that if we differentiate with respect to s ,

$$\frac{-2}{(s + 1)^2} = \frac{(s + 2)s}{(s + 1)^2} K_1 + K_3$$

K_3 is isolated and can be found if we let s approach -2 . Hence, $K_3 = -2$.

$$f(t) = 2e^{-t} - 2te^{-2t} - 2e^{-2t}$$

The Transfer Function :

a general nth-order, linear, time-invariant differential equation,

$$a_n \frac{d^n c(t)}{dt^n} + a_{n-1} \frac{d^{n-1} c(t)}{dt^{n-1}} + \dots + a_0 c(t) = b_m \frac{d^m r(t)}{dt^m} + b_{m-1} \frac{d^{m-1} r(t)}{dt^{m-1}} + \dots + b_0 r(t)$$

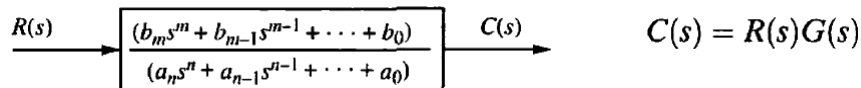
c(t) is the output, r(t) is the input

$$\begin{aligned} a_n s^n C(s) + a_{n-1} s^{n-1} C(s) + \dots + a_0 C(s) + \text{initial condition terms involving } c(t) \\ = b_m s^m R(s) + b_{m-1} s^{m-1} R(s) + \dots + b_0 R(s) + \text{initial condition terms involving } r(t) \end{aligned}$$

If we assume that all initial conditions are zero,

$$\frac{C(s)}{R(s)} = G(s) = \frac{(b_m s^m + b_{m-1} s^{m-1} + \dots + b_0)}{(a_n s^n + a_{n-1} s^{n-1} + \dots + a_0)}$$

The transfer function can be represented as a block diagram



PROBLEM: Find the transfer function represented by;

$$\frac{dc(t)}{dt} + 2c(t) = r(t)$$

SOLUTION: Taking the Laplace transform of both sides, assuming zero initial conditions, we have

$$sC(s) + 2C(s) = R(s) \qquad G(s) = \frac{C(s)}{R(s)} = \frac{1}{s+2}$$

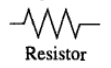
System Response from the Transfer Function:

If r(t) is a step function, find the system response $R(t) = u(t),$

$$C(s) = R(s)G(s) = \frac{1}{s(s+2)} \qquad C(s) = \frac{1/2}{s} - \frac{1/2}{s+2}$$

$$c(t) = \frac{1}{2} - \frac{1}{2} e^{-2t}$$

Electrical Network Transfer Functions: (Mesh Analysis/ Nodal Analysis)

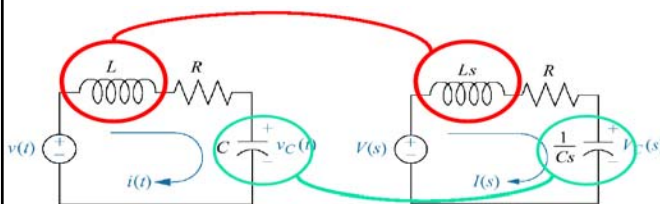
Component	Voltage-current	Current-voltage	Voltage-charge	Impedance $Z(s) = V(s)/I(s)$	Admittance $Y(s) = I(s)/V(s)$
Capacitor 	$v(t) = \frac{1}{C} \int_0^t i(\tau) d\tau$	$i(t) = C \frac{dv(t)}{dt}$	$v(t) = \frac{1}{C} q(t)$	$\frac{1}{Cs}$	Cs
Resistor 	$v(t) = Ri(t)$	$i(t) = \frac{1}{R} v(t)$	$v(t) = R \frac{dq(t)}{dt}$	R	$\frac{1}{R} = G$
Inductor 	$v(t) = L \frac{di(t)}{dt}$	$i(t) = \frac{1}{L} \int_0^t v(\tau) d\tau$	$v(t) = L \frac{d^2 q(t)}{dt^2}$	Ls	$\frac{1}{Ls}$

Note: In [electrical engineering](#), the **admittance** (Y) is a measure of how easily a circuit or device will allow a current to flow. It is defined as the [inverse](#) of the [impedance](#) (Z).

$$Y(s) = \frac{1}{Z(s)} = \frac{I(s)}{V(s)}$$

Note: These components are considered both **passive and linear**. Passive because they involve no internal source of energy (although inductors and capacitors can store energy).

Transfer Function—Single Loop via the Differential Equation: PROBLEM: Find the transfer function relating the capacitor voltage, $V_C(s)$, to the input voltage, $V(s)$



$$\begin{aligned} \left(Ls + R + \frac{1}{Cs} \right) I(s) &= V(s) \\ \frac{I(s)}{V(s)} &= \frac{1}{Ls + R + \frac{1}{Cs}} \\ V_C(s) &= I(s) \frac{1}{Cs} \end{aligned}$$

$$L \frac{d^2 q(t)}{dt^2} + R \frac{dq(t)}{dt} + \frac{1}{C} q(t) = v(t)$$

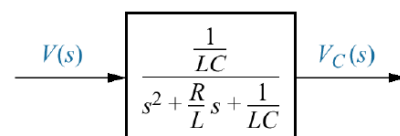
$$L \frac{di(t)}{dt} + Ri(t) + \frac{1}{C} \int_0^t i(\tau) d\tau = v(t)$$

$$q(t) = C v_C(t)$$

$$LC \frac{d^2 v_C(t)}{dt^2} + RC \frac{dv_C(t)}{dt} + v_C(t) = v(t)$$

$$(LCs^2 + RCs + 1)V_C(s) = V(s)$$

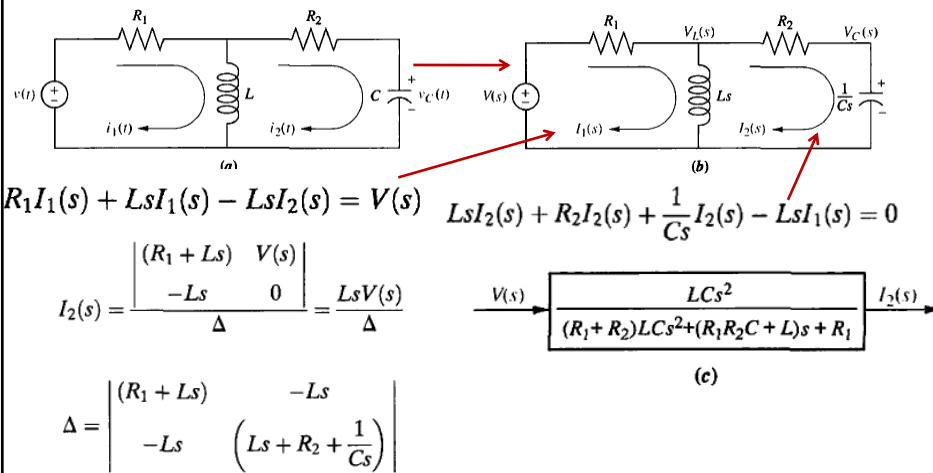
$$\frac{V_C(s)}{V(s)} = \frac{1/LC}{s^2 + \frac{R}{L}s + \frac{1}{LC}}$$



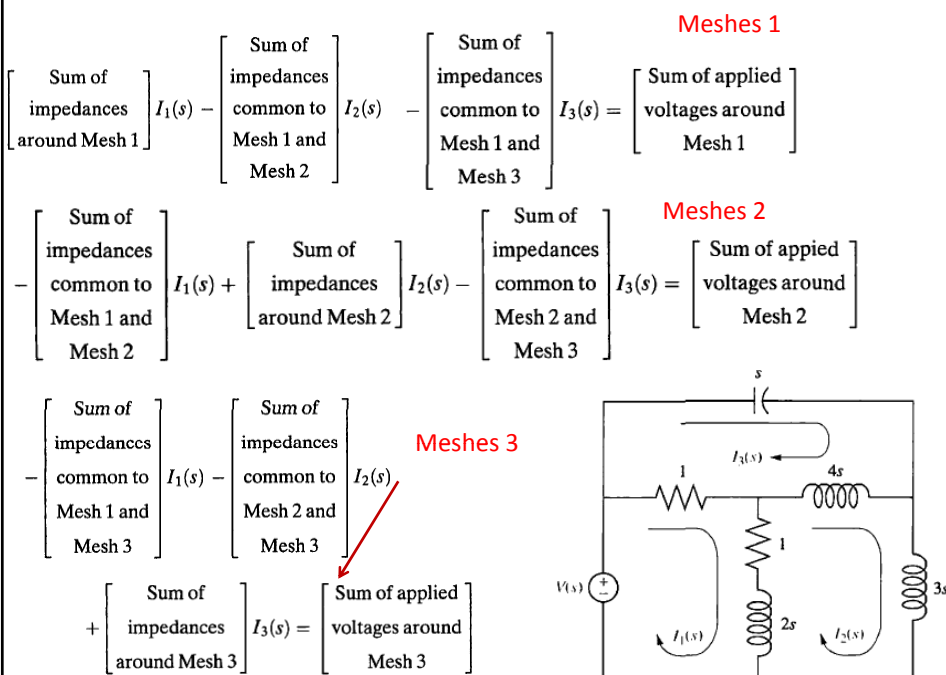
Complex Circuits via Mesh Analysis

1. Replace passive element values with their impedances.
2. Replace all sources and time variables with their Laplace transform.
3. Assume a transform current and a current direction in each mesh.
4. Write Kirchhoffs voltage law around each mesh.
5. Solve the simultaneous equations for the output.

Transfer Function—Multiple Loops: , find the transfer function, $I_2(s)/V(s)$.

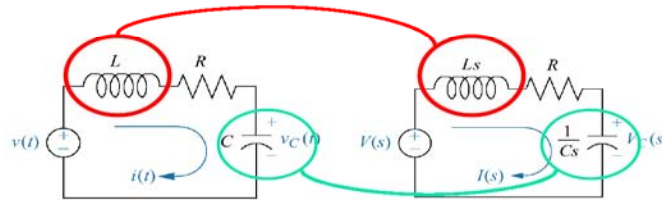


Example for Mesh Equations



Transfer Function (Nodal Analysis)—Single Node via Transform Methods:

SOLUTION: The transfer function can be obtained by **summing currents flowing out of the node whose voltage is $V_C(s)$ in Figure**. We assume that currents leaving the node are positive and currents entering the node are negative. The currents consist of the current through the capacitor and the current flowing through the series resistor and inductor.



$$\frac{V_C(s)}{I/Cs} + \frac{V_C(s) - V(s)}{R + Ls} = 0$$

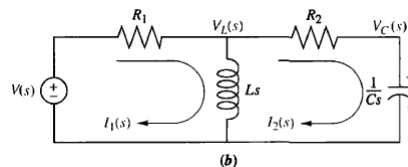
where $V_C(s)/(1/Cs)$ is the current flowing out of the node through the capacitor, and $[V_C(s) - V(s)]/(R + Ls)$ is the current flowing out of the node through the

Nodal Analysis: PROBLEM: Find the transfer function, $V_C(s)/V(s)$, for the circuit in Figure analysis.

Solution: sum currents at the nodes

$$\frac{V_L(s) - V(s)}{R_1} + \frac{V_L(s)}{Ls} + \frac{V_L(s) - V_C(s)}{R_2} = 0$$

$$CsV_C(s) + \frac{V_C(s) - V_L(s)}{R_2} = 0$$



Rearranging and expressing the resistances as conductances

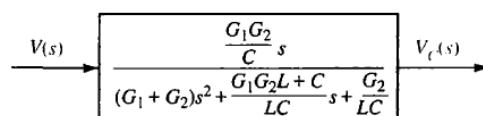
$$G_1 = 1/R_1$$

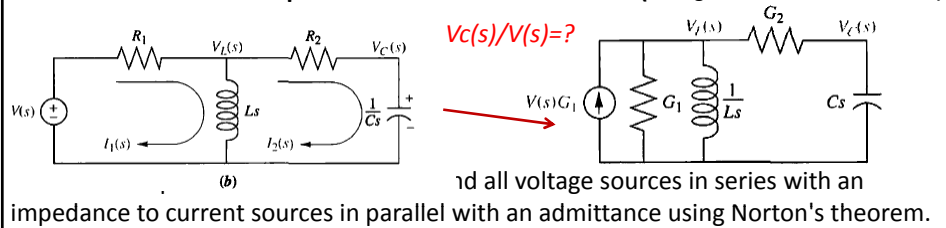
$$G_2 = 1/R_2$$

$$\left(G_1 + G_2 + \frac{1}{Ls}\right)V_L(s) - G_2V_C(s) = V(s)G_1$$

$$-G_2V_L(s) + (G_2 + Cs)V_C(s) = 0$$

$$\frac{V_C(s)}{V(s)} = \frac{\frac{G_1G_2}{C}s}{(G_1 + G_2)s^2 + \frac{G_1G_2L + C}{LC}s + \frac{G_2}{LC}}$$



Transfer Function—Multiple Nodes with Current Sources (Voltage source to Current source)


summing currents at the node $V_L(s)$,

$$G_1 V_L(s) + \frac{1}{Ls} V_L(s) + G_2 [V_L(s) - V_C(s)] = V(s) G_1$$

currents at the node $V_C(s)$

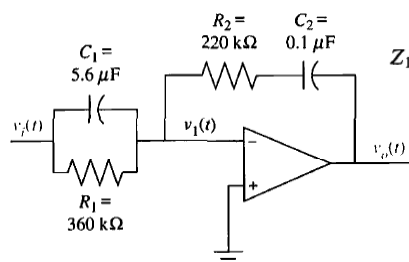
$$Cs V_C(s) + G_2 [V_C(s) - V_L(s)] = 0$$

$$\left[\begin{array}{c} \text{Sum of admittances} \\ \text{connected to Node 1} \end{array} \right] V_L(s) - \left[\begin{array}{c} \text{Sum of admittances} \\ \text{common to the two} \\ \text{nodes} \end{array} \right] V_C(s) = \left[\begin{array}{c} \text{Sum of applied} \\ \text{currents at Node 1} \end{array} \right]$$

$$- \left[\begin{array}{c} \text{Sum of admittances} \\ \text{common to the two} \\ \text{nodes} \end{array} \right] V_L(s) + \left[\begin{array}{c} \text{Sum of admittances} \\ \text{connected to Node 2} \end{array} \right] V_C(s) = \left[\begin{array}{c} \text{Sum of applied} \\ \text{currents at Node 2} \end{array} \right]$$

Transfer Function—Inverting Operational Amplifier Circuit:

Find the transfer function, $V_O(s)/V_i(s)$, for the circuit given in Figure



$$Z_1(s) = \frac{1}{C_1 s + \frac{1}{R_1}} = \frac{1}{5.6 \times 10^{-6} s + \frac{1}{360 \times 10^3}} = \frac{360 \times 10^3}{2.016s + 1}$$

$$Z_2(s) = R_2 + \frac{1}{C_2 s} = 220 \times 10^3 + \frac{10^7}{s}$$

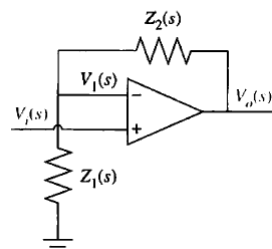
$$\left| \frac{V_O(s)}{V_i(s)} = -\frac{Z_2(s)}{Z_1(s)} \right. \quad \longrightarrow \quad \frac{V_O(s)}{V_i(s)} = -1.232 \frac{s^2 + 45.95s + 22.55}{s}$$

Transfer Function: Noninverting Operational Amplifier

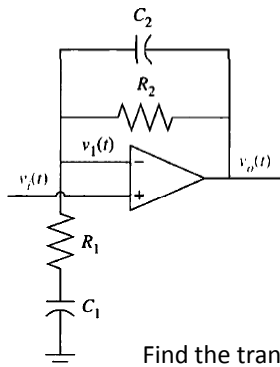
$$V_O(s) = A(V_i(s) - V_1(s)) \quad \frac{V_O(s)}{V_i(s)} = \frac{A}{1 + AZ_1(s)/(Z_1(s) + Z_2(s))}$$

$$V_1(s) = \frac{Z_1(s)}{Z_1(s) + Z_2(s)} V_O(s)$$

$$\frac{V_O(s)}{V_i(s)} = \frac{Z_1(s) + Z_2(s)}{Z_1(s)}$$

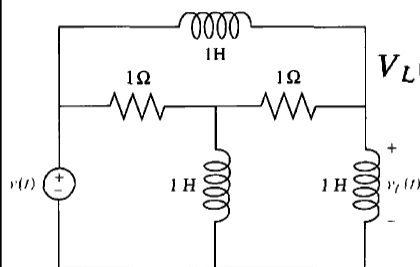


Transfer Function: Examples



Solution

$$\frac{V_o(s)}{V_i(s)} = \frac{C_2 C_1 R_2 R_1 s^2 + (C_2 R_2 + C_1 R_2 + C_1 R_1)s + 1}{C_2 C_1 R_2 R_1 s^2 + (C_2 R_2 + C_1 R_1)s + 1}$$

Find the transfer function, $G(s) = V_L(s)/V(s)$, for

Solution

$$V_L(s)/V(s) = (s^2 + 2s + 1)/(s^2 + 5s + 2)$$

Translational Mechanical System Transfer Functions:

Force-velocity, force-displacement, and impedance translational relationships for springs, viscous dampers, and mass

Component	Force-velocity	Force-displacement	Impedance $Z_M(s) = F(s)/X(s)$
Spring 	$f(t) = K \int_0^t v(\tau) d\tau$	$f(t) = Kx(t)$	K
Viscous damper 	$f(t) = f_v v(t)$	$f(t) = f_v \frac{dx(t)}{dt}$	$f_v s$
Mass 	$f(t) = M \frac{dv(t)}{dt}$	$f(t) = M \frac{d^2 x(t)}{dt^2}$	Ms^2

Mechanical systems parallel electrical networks to such an extent that there are analogies between electrical and mechanical components and variables. **Mechanical systems, like electrical networks, have three passive, linear components.** Two of them, the spring and the mass, are energy-storage elements; one of them, the **viscous damper, dissipates energy**.

We now create analogies between electrical and mechanical systems

Comparing the force-velocity column to the voltage-current column, we see that **mechanical force** is analogous to **electrical voltage** and **mechanical velocity** is analogous to **electrical current**.

Force-velocity	Voltage-current
----------------	-----------------

$$f(t) = K \int_0^t v(\tau) d\tau \quad v(t) = \frac{1}{C} \int_0^t i(\tau) d\tau$$

$$f(t) = f_v v(t) \quad v(t) = Ri(t)$$

$$f(t) = M \frac{dv(t)}{dt} \quad v(t) = L \frac{di(t)}{dt}$$

Comparing the force-displacement column the voltage-charge Column leads to the analogy between the **mechanical displacement** and **electrical charge**. We also see that the **spring** is analogous to the **capacitor**, The **viscous damper** is analogous to the **resistor**, and the **mass is analogous to the inductor**.

Force-displacement	Voltage-charge
--------------------	----------------

$$f(t) = Kx(t) \quad v(t) = \frac{1}{C} q(t)$$

$$f(t) = f_v \frac{dx(t)}{dt} \quad v(t) = R \frac{dq(t)}{dt}$$

$$f(t) = M \frac{d^2x(t)}{dt^2} \quad v(t) = L \frac{d^2q(t)}{dt^2}$$

summing forces written in terms of velocity is analogous to summing voltages written in terms of current, and the resulting mechanical differential equations are analogous to mesh equations.

Another analogy can be drawn by comparing the **force-velocity** column to the **current-voltage column in reverse order**. Here the analogy is between force and current and between velocity and voltage. Also, the **spring is analogous to the inductor**, the **viscous damper is analogous to the resistor**, and the **mass is analogous to the capacitor**. Thus, summing forces written in terms of velocity is analogous to summing currents written in terms of voltage and the resulting mechanical differential equations are analogous to **nodal equations**.

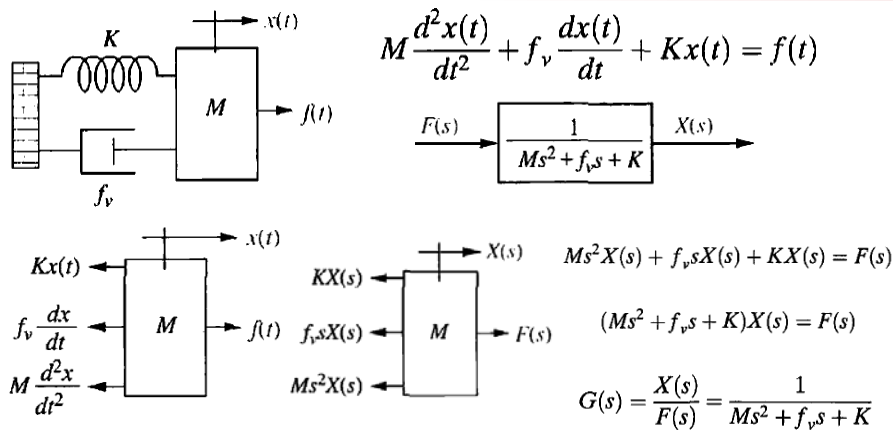
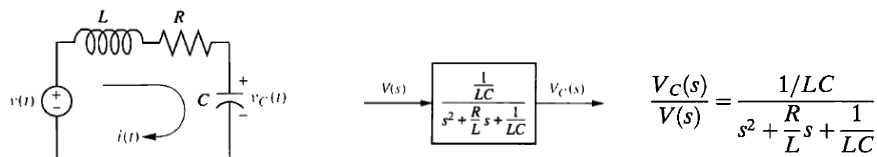
Force-velocity	Current-voltage
----------------	-----------------

$$f(t) = K \int_0^t v(\tau) d\tau \quad i(t) = C \frac{dv(t)}{dt}$$

$$f(t) = f_v v(t) \quad i(t) = \frac{1}{R} v(t)$$

$$f(t) = M \frac{dv(t)}{dt} \quad i(t) = \frac{1}{L} \int_0^t v(\tau) d\tau$$

Transfer functions for translational mechanical systems. The first example, is similar to the simple RLC network .see Figure and Find the transfer function, $X(s)/F(s)$,



Assume positive direction of motion, for example, to the right. **This assumed positive direction of motion is similar to assuming a current direction in an electrical loop.** Using our assumed direction of positive motion, we first draw a free-body diagram, placing on the body all forces that act on the body either in the direction of motion or opposite to it. Next we use Newton's law to form a differential equation of motion by summing the forces and setting the sum equal to zero. Finally, assuming zero initial conditions, we take the Laplace transform of the differential equation,

SOLUTION of example in previous page: Begin the solution by drawing the free-body diagram shown in Figure . Place on the mass all forces felt by the mass. We assume the mass is traveling toward the right. Thus, only the applied force points to the right; all other forces impede the motion and act to oppose it. Hence, the spring, viscous damper, and the force due to acceleration point to the left.

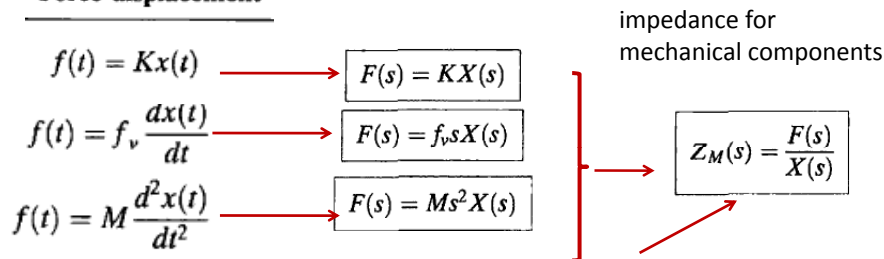
$$(Ms^2 + f_v s + K)X(s) = F(s)$$

[Sum of impedances] $X(s)$ = [Sum of applied forces]

$$G(s) = \frac{X(s)}{F(s)} = \frac{1}{Ms^2 + f_v s + K}$$

Taking the Laplace transform of the force-displacement column

Force-displacement



$$[\text{Sum of impedances}]X(s) = [\text{Sum of applied forces}]$$

Many mechanical systems are similar to multiple-loop and multiple-node electrical networks, **In mechanical systems, the number of equations of motion required is equal to the number of linearly independent motions. Linear independence implies that a point of motion in a system can still move if all other points of motion are held still.** Another name for the number of linearly independent motions is the number of **degrees of freedom**

Multiple-loop and multiple-node electrical networks,

Many mechanical systems are similar to multiple-loop and multiple-node electrical networks, where more than one simultaneous differential equation is required to describe the system. In mechanical systems, **the number of equations of motion required is equal to the number of linearly independent motions.** Linear independence implies that a point of motion in a system can still move if all other points of motion are held still. Another name for the number of **linearly independent motions** is **the number of degrees of freedom**. *This discussion is not meant to imply that these motions are not coupled to one another; in general, they are.* For example, in a two-loop electrical network, each loop current depends on the other loop current, but if we open-circuit just one of the loops, the other current can still exist if there is a voltage source in that loop. Similarly, in a mechanical system with two degrees of freedom, one point of motion can be held still while the other point of motion moves under the influence of an applied force.

In order to work such a problem, we draw the **free-body diagram** for each **point of motion** and then use **superposition**. For each free-body diagram we begin by **holding all other points of motion still and finding the forces acting on the body due only to its own motion.** Then we hold the body still and activate the other points of motion one at a time, placing on the original body the forces created by the adjacent motion.

Transfer Function—Two Degrees of Freedom

FIGURE 2.18 a. Forces on M1 due only to motion of M1; b. forces on M1 due only to motion of M2', c. all forces on M1

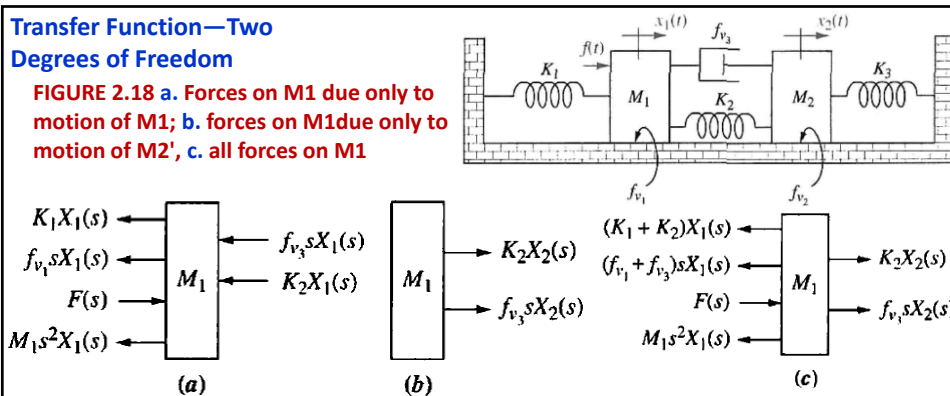
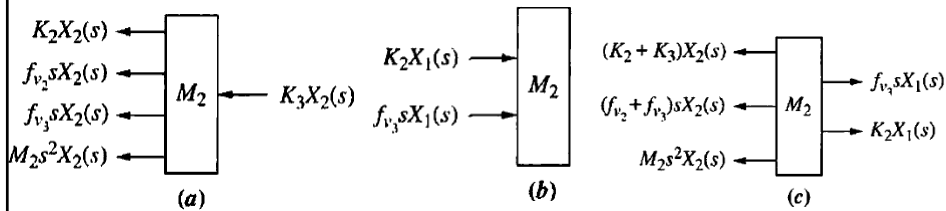


FIGURE 2.19 a. Forces on M2 due only to motion of M2, b. forces on M2 due only to motion of M1; c. all forces on M2



SOLUTION: The system has two degrees of freedom, since each mass can be moved in the horizontal direction while the other is held still. Superposition is used to draw the freebody diagrams. For example, the forces on M1 are due to (1) its own motion and (2) the motion of M2 transmitted to M1 through the system. We will consider these two sources separately. If we hold M2 still and move M1 to the right, we see the forces shown in Figure 2.18(A). If we hold M1 still and move M2 to the right, we see the forces shown in Figure 2.18(b). The total force on M1 is the superposition, or sum, of the forces just discussed. This result is shown in Figure 2.18(c). For M2, we proceed in a similar fashion: First we move M2 to the right while holding M1 still; then we move M1 to the right and hold M2 still. For each case we evaluate the forces on M2. The results appear in Figure 2.19. The Laplace transform of the equations of motion can now be written from Figures 2.18(c) and 2.19(c) as

$$[M_1 s^2 (f_{v_1} + f_{v_3})s + (K_1 + K_2)]X_1(s) - (f_{v_3}s + K_2)X_2(s) = F(s) \quad (2.118a)$$

$$-(f_{v_3}s + K_2)X_1(s) + [M_2 s^2 + (f_{v_2} + f_{v_3})s + (K_2 + K_3)]X_2(s) = 0 \quad (2.118b)$$

$$\frac{X_2(s)}{F(s)} = G(s) = \frac{(f_{v_3}s + K_2)}{\Delta}$$

$$\Delta = \begin{vmatrix} [M_1 s^2 + (f_{v_1} + f_{v_3})s + (K_1 + K_2)] & -(f_{v_3}s + K_2) \\ -(f_{v_3}s + K_2) & [M_2 s^2 + (f_{v_2} + f_{v_3})s + (K_2 + K_3)] \end{vmatrix}$$

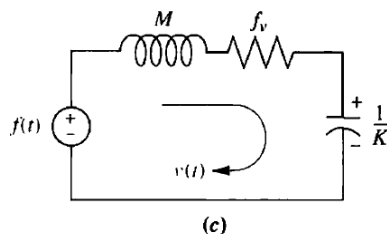
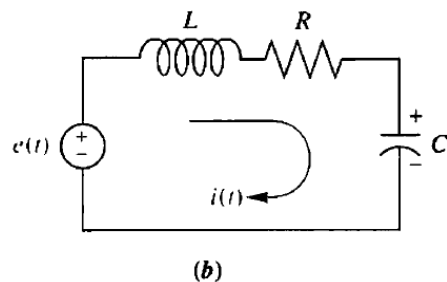
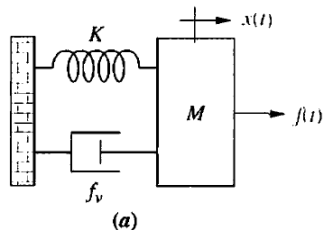
$$[M_1 s^2 (f_{v_1} + f_{v_3}) s + (K_1 + K_2)] X_1(s) - (f_{v_3} s + K_2) X_2(s) = F(s) \quad (2.118a)$$

$$-(f_{v_3} s + K_2) X_1(s) + [M_2 s^2 + (f_{v_2} + f_{v_3}) s + (K_2 + K_3)] X_2(s) = 0 \quad (2.118b)$$

similar to electrical mesh equations:

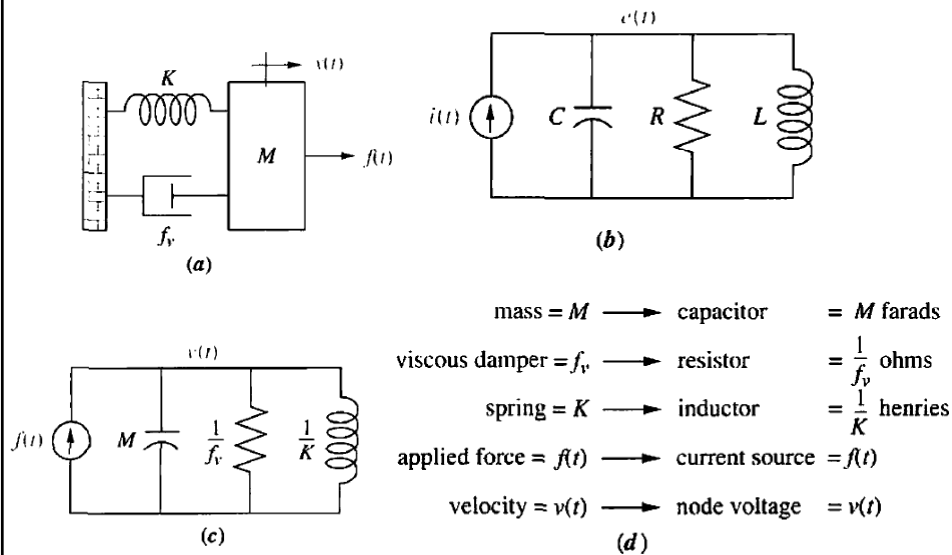
$$\begin{aligned} \left[\begin{array}{c} \text{Sum of} \\ \text{impedances} \\ \text{connected} \\ \text{to the motion} \\ \text{at } x_1 \end{array} \right] X_1(s) - \left[\begin{array}{c} \text{Sum of} \\ \text{impedances} \\ \text{between} \\ x_1 \text{ and } x_2 \end{array} \right] X_2(s) &= \left[\begin{array}{c} \text{Sum of} \\ \text{applied forces} \\ \text{at } x_1 \end{array} \right] \\ - \left[\begin{array}{c} \text{Sum of} \\ \text{impedances} \\ \text{between} \\ x_1 \text{ and } x_2 \end{array} \right] X_1(s) + \left[\begin{array}{c} \text{Sum of} \\ \text{impedances} \\ \text{connected} \\ \text{to the motion} \\ \text{at } x_2 \end{array} \right] X_2(s) &= \left[\begin{array}{c} \text{Sum of} \\ \text{applied forces} \\ \text{at } x_2 \end{array} \right] \end{aligned}$$

Electric Circuit Analogs: Series Analog

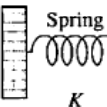
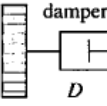
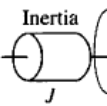


mass = M	→ inductor	= M henries
viscous damper = f_v	→ resistor	= f_v ohms
spring = K	→ capacitor	= $\frac{1}{K}$ farads
applied force = $f(t)$	→ voltage source	= $f(t)$
velocity = $v(t)$	→ mesh current	= $v(t)$

(d)

Electric Circuit Analogs: Parallel Analog**Rotational Mechanical System Transfer Functions:**

Rotational mechanical systems are handled the same way as translational mechanical systems, except that torque replaces force and angular displacement replaces translational displacement.

Component	Torque-angular velocity	Torque-angular displacement	Impedance $Z_M(s) = T(s)/\theta(s)$
 Spring K	$T(t) = K \int_0^t \omega(\tau) d\tau$	$T(t) = K\theta(t)$	K
 Viscous damper D	$T(t) = D\omega(t)$	$T(t) = D \frac{d\theta(t)}{dt}$	Ds
 Inertia J	$T(t) = J \frac{d\omega(t)}{dt}$	$T(t) = J \frac{d^2\theta(t)}{dt^2}$	Js^2

Torque-angular velocity, torque-angular displacement, and impedance rotational relationships for springs, viscous dampers, and inertia

Rotational mechanical systems are modelled in almost the same way as translational systems except that...

- We replace displacement, $x(t)$ with angular displacement $\theta(t)$; Angular velocity is $\omega(t)$
- We replace force with **torque**

For a force F acting on a body at point P , torque is defined as,

$$T = FR \sin \phi$$

where R is the distance from P to the body's axis of rotation and ϕ is the angle the force makes to the ray from the axis of rotation to P . Hence, if the force is perpendicular to the axis of rotation then,

$$T = FR$$

A rotating body can be considered a system of particles with masses $m_1, m_2, m_3, \dots$. The **moment of inertia** is defined as,

$$J = m_1 R_1^2 + m_2 R_2^2 + m_3 R_3^2 + \dots$$

The total kinetic energy is,

$$K = \frac{1}{2} J \omega^2$$

Recall that the kinetic energy for a translational system is $\frac{1}{2} m v^2$. So J is analogous to mass in translational motion. Also, similar to the equation $F = ma$ in translational systems, we can relate torque and angular acceleration,

$$T(t) = J \frac{d\omega}{dt} = J \frac{d^2\theta}{dt^2}$$

We define the components of our rotational system as springs, viscous dampers, and rotating masses.

Find the transfer function: $\theta_2(s)/T(s)$

(a) (b) Draw a free-body diagram of J_1 , using superposition

(a) J_1 rotating, J_2 held still (b) J_2 rotating, J_1 held still (c) All torques on J_1

a. Torques on J_2 due only to the motion of J_2 ; b. torques on J_2 due only to the motion of J_1 ; c. final free-body diagram for J_2

$$(J_1 s^2 + D_1 s + K)\theta_1(s) - K\theta_2(s) = T(s)$$

$$-K\theta_1(s) + (J_2 s^2 + D_2 s + K)\theta_2(s) = 0$$

from which the required transfer function is found to be $\frac{\theta_2(s)}{T(s)} = \frac{K}{\Delta}$

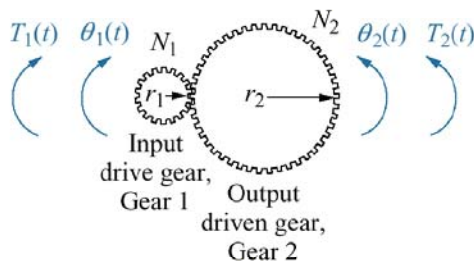
$$\Delta = \begin{vmatrix} (J_1 s^2 + D_1 s + K) & -K \\ -K & (J_2 s^2 + D_2 s + K) \end{vmatrix}$$

well-known form

$$\begin{aligned} \left[\begin{array}{c} \text{Sum of} \\ \text{impedances} \\ \text{connected} \\ \text{to the motion} \\ \text{at } \theta_1 \end{array} \right] \theta_1(s) - \left[\begin{array}{c} \text{Sum of} \\ \text{impedances} \\ \text{between} \\ \theta_1 \text{ and } \theta_2 \end{array} \right] \theta_2(s) &= \left[\begin{array}{c} \text{Sum of} \\ \text{applied torques} \\ \text{at } \theta_1 \end{array} \right] \\ - \left[\begin{array}{c} \text{Sum of} \\ \text{impedances} \\ \text{between} \\ \theta_1 \text{ and } \theta_2 \end{array} \right] \theta_1(s) + \left[\begin{array}{c} \text{Sum of} \\ \text{impedances} \\ \text{connected} \\ \text{to the motion} \\ \text{at } \theta_2 \end{array} \right] \theta_2(s) &= \left[\begin{array}{c} \text{Sum of} \\ \text{applied torques} \\ \text{at } \theta_2 \end{array} \right] \end{aligned}$$

Transfer Functions for Systems with Gears:

Gears provide mechanical advantage to rotational systems. Anyone who has ridden a 10-speed bicycle knows the effect of gearing. Going uphill, you shift to provide more torque and less speed. On the straightaway, you shift to obtain more speed and less torque. Thus, gears allow you to match the drive system and the load—a **trade-off between speed and torque**.



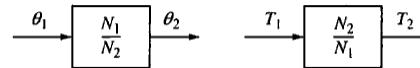
The distance traveled along each gear's circumference is the same.

$$r_1\theta_1 = r_2\theta_2$$

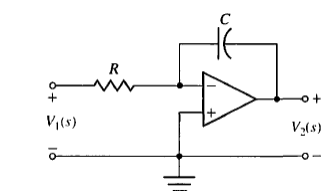
$$\frac{\theta_2}{\theta_1} = \frac{r_1}{r_2} = \frac{N_1}{N_2}$$

Assume the gears are lossless, that is they do not absorb or store energy, the energy into Gear 1 equals the energy out of Gear 2

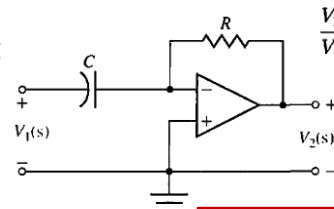
$$T_1\theta_1 = T_2\theta_2 \quad \rightarrow \quad \frac{T_2}{T_1} = \frac{\theta_1}{\theta_2} = \frac{N_2}{N_1}$$



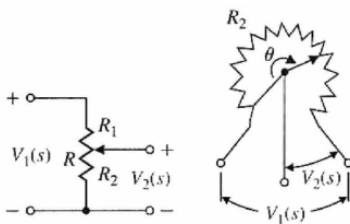
Transfer Functions of Dynamic Elements and Networks (Examples)



$$\frac{V_2(s)}{V_1(s)} = -\frac{1}{RCs}$$



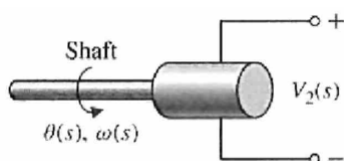
$$\frac{V_2(s)}{V_1(s)} = -RCs$$



$$\frac{V_2(s)}{V_1(s)} = \frac{R_2}{R_1 + R_2}$$

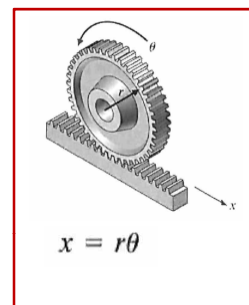
$$\frac{R_2}{R_1 + R_2} = \frac{\theta}{\theta_{\max}}$$

Tachometer, velocity sensor



$$V_2(s) = K_t\omega(s) = K_t s\theta(s)$$

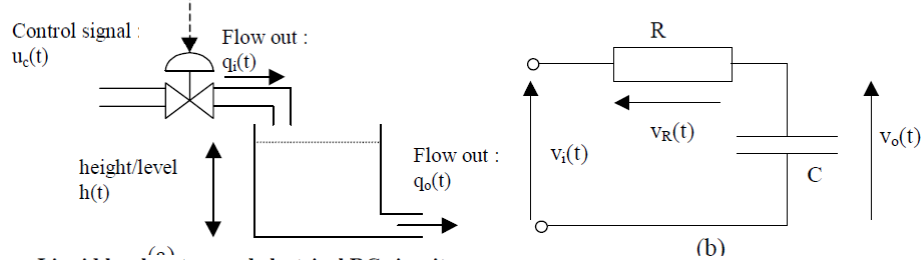
$$K_t = \text{constant}$$



Modelling LIQUID-LEVEL SYSTEMS:

Resistance and Capacitance of Liquid-Level Systems. Consider the flow

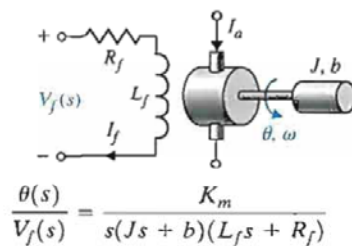
through a short pipe connecting two tanks. The resistance R for liquid flow in such a pipe or restriction is defined as the change in the level difference (the difference of the liquid levels of the two tanks) necessary to cause a unit change in flow rate; that is,



Liquid level system and electrical RC circuit

System	Liquid level system	Electrical RC circuit
input signal :	flow $q_i(t)$	input voltage $v_i(t)$
output signal :	height $h(t)$	output voltage $v_o(t)$
Differential equation:	$RA \frac{dh}{dt} + h(t) = Rq_i(t)$	$RC \frac{dv_o}{dt} + v_o(t) = v_i(t)$
General differential equation	$\tau \frac{dy}{dt} + y(t) = Ku(t)$	$\tau \frac{dy}{dt} + y(t) = Ku(t)$

DC motor, field-controlled, rotational actuator



DC motor, armature-controlled, rotational actuator

