

EEM322 : Lecture 3b

Time domain modelling**State Variable Models or State-space approach****Solving for the State Variable System Equations***State space to Transfer function**Transfer function to State space conversion***Durum Denklemlerinden Transfer Fonksiyonunun Elde Edilmesi:**

Bir sistemin durum değişkenlerinin farklı seçimleri durum denklemlerinin farklı kanonik formlarda yazılabilmesini sağlar. **Yani bir sistemin çok sayıda durum değişkeni modeli elde edilebilir.** Fakat transfer fonksiyonu gösterimi tektir.

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{u}$$

$$\mathbf{y} = \mathbf{C}\mathbf{x} + \mathbf{D}\mathbf{u}$$

Bu denklemlerin her iki tarafına Laplace dönüşümü uygulanırsa aşağıdaki gibi azılabilir.

$$s\mathbf{X}(s) - \mathbf{x}(0) = \mathbf{A}\mathbf{X}(s) + \mathbf{B}\mathbf{U}(s)$$

$$\mathbf{Y}(s) = \mathbf{C}\mathbf{X}(s) + \mathbf{D}\mathbf{U}(s)$$

Durum denklemi $\mathbf{X}(s)$ 'e göre düzenlenirse aşağıdaki eşitlik elde edilir.

$$\mathbf{X}(s) = (s\mathbf{I} - \mathbf{A})^{-1}\mathbf{x}(0) + (s\mathbf{I} - \mathbf{A})^{-1}\mathbf{B}\mathbf{U}(s)$$

Bu denklemden $\mathbf{X}(s)$ çıkış denkleminde yerine konduğunda aşağıdaki eşitlik elde edilir.

$$\mathbf{Y}(s) = \mathbf{C}(s\mathbf{I} - \mathbf{A})^{-1}\mathbf{x}(0) + \mathbf{C}(s\mathbf{I} - \mathbf{A})^{-1}\mathbf{B}\mathbf{U}(s) + \mathbf{D}\mathbf{U}(s)$$

Transfer fonksiyonunun tanımı gereği durum değişkenlerinin başlangıç koşullarının yerine $\mathbf{x}(0) = 0$ şeklinde yazıldığında aynı çıkış denklemi aşağıdaki gibi yazılabilir.

$$\mathbf{Y}(s) = [\mathbf{C}(s\mathbf{I} - \mathbf{A})^{-1}\mathbf{B} + \mathbf{D}]\mathbf{U}(s)$$

Böylece durum denklemleri verilen tek girişli tek çıkışlı bir sistemin transfer fonksiyonu

$$G(s) = \frac{Y(s)}{U(s)} = \mathbf{C}(s\mathbf{I} - \mathbf{A})^{-1}\mathbf{B} + D \quad (s\mathbf{I} - \mathbf{A})^{-1} = \frac{\text{Adj}(s\mathbf{I} - \mathbf{A})}{|s\mathbf{I} - \mathbf{A}|}$$

Yukarıdaki denklem şu şekilde yazılabilir.

$$G(s) = \mathbf{C} \frac{\text{Adj}(s\mathbf{I} - \mathbf{A})}{|s\mathbf{I} - \mathbf{A}|} \mathbf{B} + D$$

$$G(s) = \frac{\mathbf{C} \text{Adj}(s\mathbf{I} - \mathbf{A})\mathbf{B} + |s\mathbf{I} - \mathbf{A}|D}{|s\mathbf{I} - \mathbf{A}|}$$

Transfer fonksiyonunun paydası karakteristik denklem olarak tanımlanır

Örnek: Durum uzayı modeli aşağıda verilen sistemin transfer fonksiyonunu elde ediniz.

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u, \quad y = \begin{bmatrix} 3 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \Rightarrow G(s) = \frac{Y(s)}{U(s)} = ?$$

Örnek: Durum uzayı modeli aşağıda verilen sistemin transfer fonksiyonunu elde ediniz. (D=0)

Çözüm:

$$|s\mathbf{I} - \mathbf{A}| = \begin{vmatrix} s & -1 \\ 2 & s+3 \end{vmatrix} = s(s+3) + 2 = s^2 + 3s + 2, \quad \text{Adj}(s\mathbf{I} - \mathbf{A}) = \begin{bmatrix} s+3 & -2 \\ 1 & s \end{bmatrix}^T = \begin{bmatrix} s+3 & 1 \\ -2 & s \end{bmatrix}$$

$$G(s) = \mathbf{C}(s\mathbf{I} - \mathbf{A})^{-1}\mathbf{B} + D = \mathbf{C} \frac{\text{Adj}(s\mathbf{I} - \mathbf{A})}{|s\mathbf{I} - \mathbf{A}|} \mathbf{B} + D = \begin{bmatrix} 3 & 1 \end{bmatrix} \frac{1}{s^2 + 3s + 2} \begin{bmatrix} s+3 & 1 \\ -2 & s \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \frac{s+3}{s^2 + 3s + 2}$$

olarak elde edilir.

Example: Converting from State Space to a Transfer Function

$$\mathbf{x} = \begin{bmatrix} -4 & -1.5 \\ 4 & 0 \end{bmatrix} \mathbf{x} + \begin{bmatrix} 2 \\ 0 \end{bmatrix} u(t)$$

$$y = \begin{bmatrix} 1.5 & 0.625 \end{bmatrix} \mathbf{x}$$

ANSWER:

$$G(s) = \frac{3s + 5}{s^2 + 4s + 6}$$

Example:

$$\dot{\mathbf{x}} = \begin{bmatrix} -4 & -1.5 \\ 4 & 0 \end{bmatrix} \mathbf{x} + \begin{bmatrix} 2 \\ 0 \end{bmatrix} u(t)$$

$$y = [1.5 \ 0.625] \mathbf{x}$$

$$s\mathbf{I} - \mathbf{A} = s \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} -4 & -1.5 \\ 4 & 0 \end{bmatrix} = \begin{bmatrix} s+4 & 1.5 \\ -4 & s \end{bmatrix}$$

$$\det(s\mathbf{I} - \mathbf{A}) = \begin{vmatrix} s+4 & 1.5 \\ -4 & s \end{vmatrix} = (s+4)s - 1.5(-4) = s^2 + 4s + 6$$

$$\text{adj}(s\mathbf{I} - \mathbf{A}) = \begin{bmatrix} s & -(-4) \\ -1.5 & s+4 \end{bmatrix}^T = \begin{bmatrix} s & -1.5 \\ 4 & s+4 \end{bmatrix}$$

$$\begin{aligned} \mathbf{c}\{\text{adj}(s\mathbf{I} - \mathbf{A})\}\mathbf{b} &= [1.5 \ 0.625] \begin{bmatrix} s & -1.5 \\ 4 & s+4 \end{bmatrix} \begin{bmatrix} 2 \\ 0 \end{bmatrix} \\ &= [1.5 \ 0.625] \begin{bmatrix} 2s \\ 8 \end{bmatrix} = 3s + 5 \end{aligned} \quad G(s) = \frac{3s + 5}{s^2 + 4s + 6}$$

Example: Obtain the transfer function representation for a system represented by matrices,

$$\mathbf{A} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -10 & -2 & -3 \end{bmatrix}; \quad \mathbf{b} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}; \quad \mathbf{c} = [2 \ 1 \ 0].$$

$$[s\mathbf{I} - \mathbf{A}] = \begin{bmatrix} s & -1 & 0 \\ 0 & s & -1 \\ 10 & 2 & s+3 \end{bmatrix}$$

$$\det[s\mathbf{I} - \mathbf{A}] = s(s^2 + 3s + 2) + 10 = s^3 + 3s^2 + 2s + 10$$

$$\begin{aligned} \text{adj}[s\mathbf{I} - \mathbf{A}] &= \begin{bmatrix} s^2 + 3s + 2 & -10 & -10s \\ s + 3 & s^2 + 3s & -2s - 10 \\ 1 & s & s^2 \end{bmatrix}^T \\ &= \begin{bmatrix} s^2 + 3s + 2 & s + 3 & 1 \\ -10 & s^2 + 3s & s \\ -10s & -2s - 10 & s^2 \end{bmatrix} \end{aligned}$$

$$\mathbf{c}[\text{adj}(s\mathbf{I} - \mathbf{A})]\mathbf{b} = [2 \ 1 \ 0] \begin{bmatrix} 1 \\ s \\ s^2 \end{bmatrix} = s + 2 \quad G(s) = \frac{s + 2}{s^3 + 3s^2 + 2s + 10}$$

Example: State-Space Representation to Transfer Function

$$T(s) = \frac{Y(s)}{U(s)} = \mathbf{C}(s\mathbf{I} - \mathbf{A})^{-1} \mathbf{B} + \mathbf{D}$$

$$\dot{\mathbf{x}} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & -2 & -3 \end{bmatrix} \mathbf{x} + \begin{bmatrix} 10 \\ 0 \\ 0 \end{bmatrix} u$$

$$y = [1 \ 0 \ 0] \mathbf{x}$$

$$(s\mathbf{I} - \mathbf{A}) = \begin{bmatrix} s & 0 & 0 \\ 0 & s & 0 \\ 0 & 0 & s \end{bmatrix} - \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & -2 & -3 \end{bmatrix} = \begin{bmatrix} s & -1 & 0 \\ 0 & s & -1 \\ 1 & 2 & s+3 \end{bmatrix}$$

$$(s\mathbf{I} - \mathbf{A})^{-1} = \frac{\text{adj}(s\mathbf{I} - \mathbf{A})}{\det(s\mathbf{I} - \mathbf{A})} = \frac{\begin{bmatrix} (s^2 + 3s + 2) & s + 3 & 1 \\ -1 & s(s + 3) & s \\ -s & -(2s + 1) & s^2 \end{bmatrix}}{s^3 + 3s^2 + 2s + 1}$$

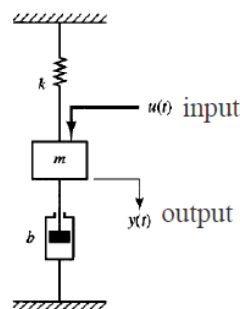
$$\mathbf{B} = \begin{bmatrix} 10 \\ 0 \\ 0 \end{bmatrix}$$

$$\mathbf{C} = [1 \ 0 \ 0]$$

$$\mathbf{D} = 0$$

$$T(s) = \frac{10(s^2 + 3s + 2)}{s^3 + 3s^2 + 2s + 1}$$

Example: Obtain the transfer function $G(s)$ of a linear single-input-single-output mechanical system whose state-space equations are known.



$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \underbrace{\begin{bmatrix} 0 & 1 \\ -\frac{k}{m} & -\frac{b}{m} \end{bmatrix}}_{\mathbf{A}} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \underbrace{\begin{bmatrix} 0 \\ \frac{1}{m} \end{bmatrix}}_{\mathbf{B}} u$$

$$y = \underbrace{[1 \ 0]}_{\mathbf{C}} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \quad \mathbf{D} = 0$$

$$G(s) = \mathbf{C}(s\mathbf{I} - \mathbf{A})^{-1} \mathbf{B} + \mathbf{D}$$

$$G(s) = [1 \ 0] \frac{1}{s^2 + \frac{b}{m}s + \frac{k}{m}} \begin{bmatrix} s + \frac{b}{m} & 1 \\ -\frac{k}{m} & s \end{bmatrix} \begin{bmatrix} 0 \\ \frac{1}{m} \end{bmatrix} = \frac{1}{ms^2 + bs + k}$$

Converting from State Space to a Transfer Function: Example:

$$\dot{\mathbf{x}} = \begin{bmatrix} -4 & -1.5 \\ 4 & 0 \end{bmatrix} \mathbf{x} + \begin{bmatrix} 2 \\ 0 \end{bmatrix} u(t)$$

$$y = [1.5 \quad 0.625] \mathbf{x}$$

ANSWER: $G(s) = \frac{3s + 5}{s^2 + 4s + 6}$

Converting from State Space to a Transfer Function:

Given the state and output equations:

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}u$$

Take the Laplace transform assuming zero initial conditions

$$\mathbf{y} = \mathbf{C}\mathbf{x} + \mathbf{D}u$$

$$s\mathbf{X}(s) = \mathbf{A}\mathbf{X}(s) + \mathbf{B}U(s)$$

$$\mathbf{Y}(s) = \mathbf{C}\mathbf{X}(s) + \mathbf{D}U(s)$$

Solving for X(s) $(s\mathbf{I} - \mathbf{A})\mathbf{X}(s) = \mathbf{B}U(s) \longrightarrow \mathbf{X}(s) = (s\mathbf{I} - \mathbf{A})^{-1}\mathbf{B}U(s)$

where I is the identity matrix.

$$\mathbf{Y}(s) = \mathbf{C}\mathbf{X}(s) + \mathbf{D}U(s)$$

$$\mathbf{Y}(s) = \mathbf{C}(s\mathbf{I} - \mathbf{A})^{-1}\mathbf{B}U(s) + \mathbf{D}U(s) = [\mathbf{C}(s\mathbf{I} - \mathbf{A})^{-1}\mathbf{B} + \mathbf{D}]U(s)$$

We call the matrix

$$[\mathbf{C}(s\mathbf{I} - \mathbf{A})^{-1}\mathbf{B} + \mathbf{D}]$$

the transfer function matrix, since it relates

the output vector, $\mathbf{Y}(s)$, to the input vector, $\mathbf{U}(s)$. However, if $U(s) = U(s)$ and $\mathbf{Y}(s) = Y(s)$ are scalars, we can find the transfer function

$$T(s) = \frac{Y(s)}{U(s)} = \mathbf{C}(s\mathbf{I} - \mathbf{A})^{-1}\mathbf{B} + \mathbf{D}$$

characteristic equation *Definition.* $\det(A - \lambda I) = 0$ is called the **characteristic equation** of the matrix A .

Eigenvalues λ of A are roots of the characteristic equation. Associated eigenvectors of A are nonzero solutions of the equation $(A - \lambda I)\mathbf{x} = \mathbf{0}$.

the poles of the system are the roots of the equation $\det[sI - A] = 0$. This expression is called the characteristic equation of the matrix A and its roots the eigenvalues of A . Thus, the poles of a system are the same as the eigenvalues of its state matrix.

The poles of the system $\frac{Y(s)}{U(s)}$ are, therefore,

$$(sI - A)^{-1} = \frac{\text{adj}(sI - A)}{\det(sI - A)} \quad \det(sI - A) = |sI - A| = 0$$

which is the eigen values of A .

Converting a Transfer Function to State Space: to apply this representation to transfer functions.

Consider the differential equation $\frac{d^n y}{dt^n} + a_{n-1} \frac{d^{n-1} y}{dt^{n-1}} + \dots + a_1 \frac{dy}{dt} + a_0 y = b_0 u$

Choose the output, $y(t)$, and its $(n - 1)$ derivatives as the state variables. This choice is called the **phase-variable choice**. Choosing the state variables, x_i , we get

$\begin{aligned} x_1 &= y \\ x_2 &= \frac{dy}{dt} \\ x_3 &= \frac{d^2 y}{dt^2} \\ &\vdots \\ x_n &= \frac{d^{n-1} y}{dt^{n-1}} \end{aligned}$	→	$\begin{aligned} \dot{x}_1 &= \frac{dy}{dt} \\ \dot{x}_2 &= \frac{d^2 y}{dt^2} \\ \dot{x}_3 &= \frac{d^3 y}{dt^3} \\ &\vdots \\ \dot{x}_n &= \frac{d^n y}{dt^n} \end{aligned}$	→	$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= x_3 \\ &\vdots \\ \dot{x}_{n-1} &= x_n \\ \dot{x}_n &= -a_0 x_1 - a_1 x_2 - \dots - a_{n-1} x_n + b_0 u \end{aligned}$
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differentiating both sides yields

These equations are the phase-variable form of the state equations. This form is easily recognized by the unique pattern of 1's and 0's and the negative of the coefficients of the differential equation written in reverse order in the last row of the system

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \vdots \\ \dot{x}_{n-1} \\ \dot{x}_n \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 1 \\ -a_0 & -a_1 & -a_2 & -a_3 & -a_4 & -a_5 & \cdots & -a_{n-1} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_{n-1} \\ x_n \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 0 \\ b_0 \end{bmatrix} u$$

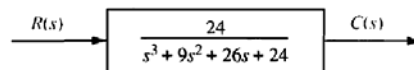
$$\dot{\mathbf{x}} = \mathbf{Ax} + \mathbf{bu}$$

$$y = [1 \ 0 \ 0 \ \cdots \ 0] \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_{n-1} \\ x_n \end{bmatrix}$$

In summary, then, to convert a transfer function into state equations in phase variable form, we first convert the transfer function to a differential equation by cross-multiplying and taking the inverse Laplace transform, assuming zero initial conditions. Then we represent the differential equation in state space in phase variable form. An example illustrates the process.

Converting a Transfer Function with Constant Term in Numerator:

PROBLEM: Find the state-space representation in phase-variable form for the transfer function shown in Figure



Step 1 Find the associated differential equation. Since

cross-multiplying yields
$$\frac{C(s)}{R(s)} = \frac{24}{(s^3 + 9s^2 + 26s + 24)}$$

The corresponding differential equation is found by taking the inverse Laplace transform, assuming zero initial conditions

$$(s^3 + 9s^2 + 26s + 24)C(s) = 24R(s)$$

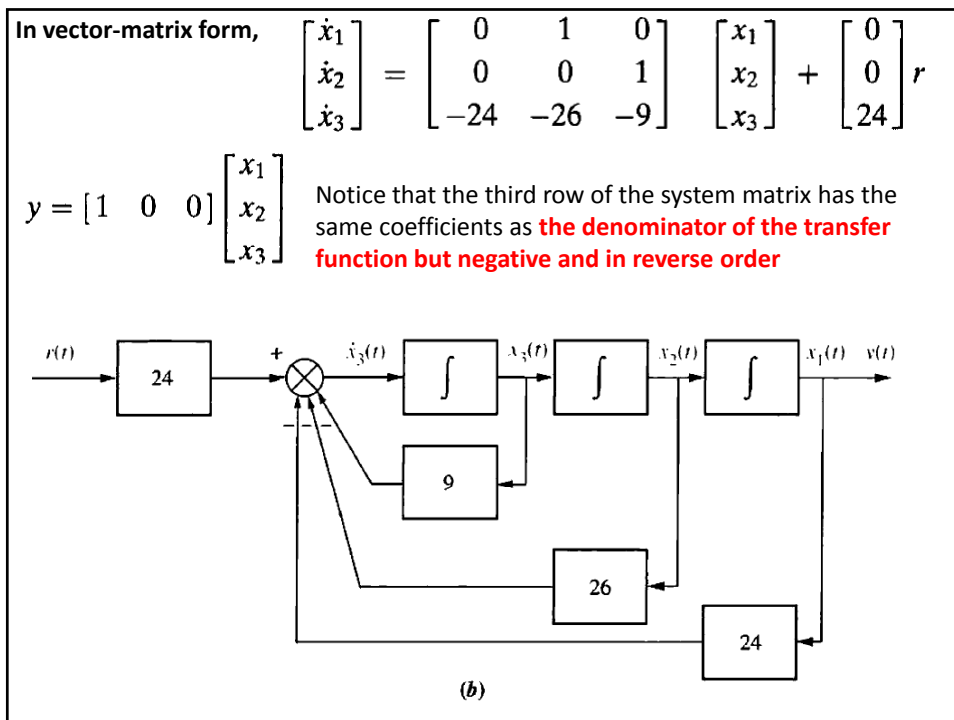
Step 2 Select the state variables.

Choosing the state variables as successive derivatives, we get

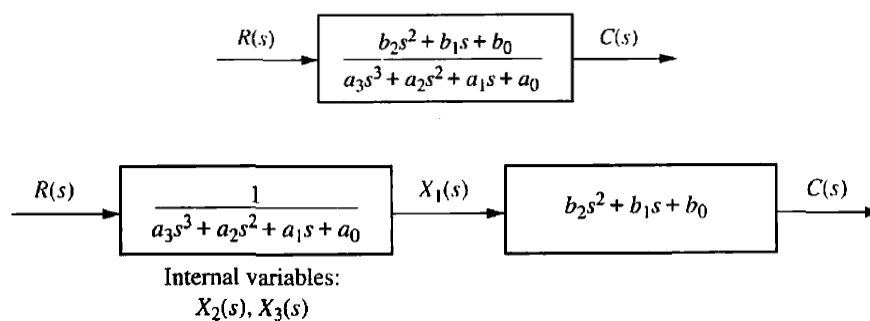
$$\ddot{c} + 9\dot{c} + 26c + 24c = 24r$$

Differentiating both sides

$$\begin{array}{ll} x_1 = c & \dot{x}_1 = x_2 \\ x_2 = \dot{c} & \dot{x}_2 = x_3 \\ x_3 = \dot{\dot{c}} & \dot{x}_3 = -24x_1 - 26x_2 - 9x_3 + 24r \\ & y = c = x_1 \end{array}$$

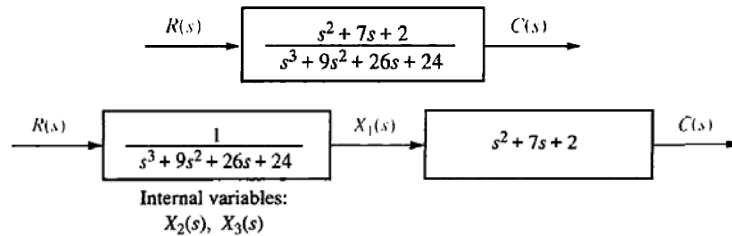


If a transfer function has a polynomial in s in the numerator that is of order less than the polynomial in the denominator, as shown in Figure, the numerator and denominator can be handled separately. First separate the transfer function into two cascaded transfer functions, as shown in Figure; the first is the denominator, and the second is just the numerator. The first transfer function with just the denominator is converted to the phase-variable representation in state space as demonstrated in the last example. Hence, phase variable x_1 is the output, and the rest of the phase variables are the internal variables of the first block, as shown in . The second transfer function with just the numerator yields



Converting a Transfer Function with Polynomial in Numerator

PROBLEM: Find the state-space representation of the transfer function shown in



Step 1 Separate the system into two cascaded blocks, as shown in Figure. The first block contains the denominator and the second block contains the numerator.

Step 2: Find the state equations for the block containing the denominator. We notice that the first block's numerator is $1/24$ that of last Example. Thus, the state equations are the same except that this system's input matrix is $1/24$ that of last Example.

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -24 & -26 & -9 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} r$$

Introduce the effect of the block with the numerator. The second block of Figure, where $b_2 = 1$, $b_1 = 7$, and $b_0 = 2$, states that

Taking the inverse Laplace transform with zero initial conditions $C(s) = (b_2 s^2 + b_1 s + b_0) X_1(s) = (s^2 + 7s + 2) X_1(s)$

$$c = \ddot{x}_1 + 7\dot{x}_1 + 2x_1 \quad y = c(t) = b_2 x_3 + b_1 x_2 + b_0 x_1 = x_3 + x_2 + 2x_1$$

$$y = [b_0 \quad b_1 \quad b_2] \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = [2 \quad 7 \quad 1] \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

$$x_1 = x_1$$

$$\dot{x}_1 = x_2$$

$$\ddot{x}_1 = x_3$$

