

EEM322 : Lecture 3C

Time domain modelling

State Variable Models or State-space approach**Durum Geçiş Denklemi ve Durum geçiş Matrisi****Durum Geçiş Denklemi ve Durum geçiş Matrisi**

Durum geçiş denklemi, doğrusal homojen olmayan durum denkleminin çözümü olarak tanımlanır. Doğrusal zamanla değişmeyen bir sistemin durum uzayı modeli aşağıdaki gibidir.

$$\dot{x}(t) = Ax(t) + Bu(t)$$

$$y(t) = Cx(t) + Du(t)$$

Durum denklemi diferansiyel denklem çözme yöntemleri ya da Laplace dönüşüm yöntemiyle çözülebilir. Laplace dönüşümü ile çözüm yöntemi aşağıda verilmiştir.

$$sX(s) - x(0) = AX(s) + BU(s) \quad \rightarrow \quad (sI - A)X(s) = x(0) + BU(s)$$

$$Y(s) = CX(s) + DU(s)$$

$$X(s) = (sI - A)^{-1}x(0) + (sI - A)^{-1}BU(s)$$

$$Y(s) = C(sI - A)^{-1}x(0) + \underbrace{[C(sI - A)^{-1}B + D]}_{\text{Transfer Fonksiyonu}}U(s)$$

Durum denkleminin çözümü:

$$\begin{aligned}
 x(t) &= \underbrace{L^{-1}[(sI - A)^{-1}]x(0)}_{\text{Doğal çözüm}} + \underbrace{L^{-1}\{(sI - A)^{-1}BU(s)\}}_{\text{Zorlanmış çözüm}} \\
 &= \underbrace{L^{-1}[(sI - A)^{-1}]x(0)}_{\text{Doğal çözüm}} + \underbrace{L^{-1}\{(sI - A)^{-1}\}B * u(t)}_{\text{Zorlanmış çözüm}} \\
 x(t) &= \underbrace{e^{At}x(0)}_{\text{Doğal çözüm}} + \underbrace{\int_0^t e^{A(t-\tau)}Bu(\tau)d\tau}_{\text{Zorlanmış çözüm}}, \quad t \geq 0
 \end{aligned}$$

Çıkış denklemi:

$$\begin{aligned}
 y(t) &= \underbrace{L^{-1}[C(sI - A)^{-1}]x(0)}_{\text{Doğal çözüm}} + \underbrace{L^{-1}\{[C(sI - A)^{-1}B + D]U(s)\}}_{\text{Zorlanmış çözüm}} \\
 &= \underbrace{CL^{-1}[(sI - A)^{-1}]x(0)}_{\text{Doğal çözüm}} + \underbrace{(CL^{-1}\{(sI - A)^{-1}\}B + D) * u(t)}_{\text{Zorlanmış çözüm}} \\
 y(t) &= \underbrace{Ce^{At}x(0)}_{\text{Doğal çözüm}} + \underbrace{\int_0^t Ce^{A(t-\tau)}Bu(\tau)d\tau + Du(t)}_{\text{Zorlanmış çözüm}}, \quad t \geq 0
 \end{aligned}$$

Durum geçiş matrisi :

$$e^{At} = L^{-1}\{(sI - A)^{-1}\}$$

şeklinde ters Laplace

dönüşümüyle hesaplanabilir.

Örnek: Durum denklemi $\begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(t)$ olarak verilen sistemde girişin $t \geq 0$ için $u(t) = 1$ olması halinde, e^{At} durum geçiş matrisini ve $x(t)$ durum denkleminin zaman domeni çözümünü hesaplayınız.

$$A = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \quad \text{katsayı matrisleridir.}$$

$$(sI - A)^{-1} = \begin{bmatrix} s & 0 \\ 0 & s \end{bmatrix} - \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix} = \begin{bmatrix} s & -1 \\ 2 & s+3 \end{bmatrix} \Rightarrow (sI - A)^{-1} = \frac{1}{s^2 + 3s + 2} \begin{bmatrix} s+3 & 1 \\ -2 & s \end{bmatrix}$$

A 'nın durum geçiş matrisi e^{At}

$$e^{At} = L^{-1}[(sI - A)^{-1}] = \begin{bmatrix} 2e^{-t} - e^{-2t} & e^{-t} - e^{-2t} \\ -2e^{-t} + 2e^{-2t} & -e^{-t} + 2e^{-2t} \end{bmatrix}$$

Durum geçiş denklemi ise

$$t \geq 0 \text{ için, } x(t) = e^{At}x(0) + \int_0^t e^{A(t-\tau)}Bu(\tau)d\tau$$

Durum geiş denklemini ise

$$x(t) = \begin{bmatrix} 2e^{-t} - e^{-2t} & e^{-t} - e^{-2t} \\ -2e^{-t} + 2e^{-2t} & -e^{-t} + 2e^{-2t} \end{bmatrix} x(0) + \int_0^t \begin{bmatrix} 2e^{-(t-\tau)} - e^{-2(t-\tau)} & e^{-(t-\tau)} - e^{-2(t-\tau)} \\ -2e^{-(t-\tau)} + e^{-2(t-\tau)} & -e^{-(t-\tau)} + 2e^{-2(t-\tau)} \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} d\tau$$

$$x(t) = \begin{bmatrix} 2e^{-t} - e^{-2t} & e^{-t} - e^{-2t} \\ -2e^{-t} + 2e^{-2t} & -e^{-t} + 2e^{-2t} \end{bmatrix} x(0) + \begin{bmatrix} 0.5 - e^{-t} + 0.5e^{-2t} \\ e^{-t} - e^{-2t} \end{bmatrix}, \quad t \geq 0$$

olarak elde edilir. Durum geiş denklemindeki ikinci terimi (zorlanmış özümü) elde etmenin diğeri bir yolu da ters Laplace dönüşümü yardımıyla $L^{-1}\{(sI - A)^{-1}BU(s)\}$ ifadesini hesaplamaktır. Burada zorlanmış özüm

$$L^{-1}[(sI - A)^{-1}]BU(s) = L^{-1}\left(\frac{1}{s^2 + 3s + 2} \begin{bmatrix} s+3 & 1 \\ -2 & s \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} \frac{1}{s}\right)$$

$$= L^{-1}\left(\frac{1}{s^2 + 3s + 2} \begin{bmatrix} 1 \\ s \\ 1 \end{bmatrix}\right) = \begin{bmatrix} 0.5 - e^{-t} + 0.5e^{-2t} \\ e^{-t} - e^{-2t} \end{bmatrix}, \quad t \geq 0$$

Properties of the state transition matrix (t): Durum geiş matrisi e^{At} Veya $\Phi(t)$

$$1. \Phi(t_1 + t_2) = \Phi(t_1)\Phi(t_2) \Leftrightarrow e^{A(t_1+t_2)} = e^{At_1}e^{At_2}$$

$$2. \frac{d}{dt}\Phi(t) = A\Phi(t) \Leftrightarrow \frac{d}{dt}e^{At} = Ae^{At} \Leftrightarrow \frac{d}{dt}e^{at} = ae^{at}$$

$$3. \Phi(0) = e^{A0} = I$$

$$4. \frac{d}{dt}\Phi(t)|_{t=0} = \frac{d}{dt}e^{At}|_{t=0} = A\Phi(t)|_{t=0} = A$$

The **matrix exponential function** exists and is defined as: ;; (Similar to Taylor series expansion of the exponential

$$e^{At} = \exp(\underline{A} \cdot t) = \underline{I} + \underline{A} \cdot t + \frac{\underline{A}^2 \cdot t^2}{2!} + \cdots + \frac{\underline{A}^k \cdot t^k}{k!} + \cdots$$

Example: Find the state transition matrix for the next state equation,

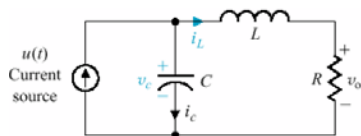
$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & -a \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u$$

Recalling the matrix inversion for 2×2 matrices (inverse matrix)

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \frac{1}{ad-bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}^T = \frac{1}{ad-bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

$$\begin{aligned} \Phi(t) &= \mathcal{L}^{-1} \{ (sI - A)^{-1} \} \\ &= \mathcal{L}^{-1} \left\{ (sI - \begin{bmatrix} 0 & 1 \\ 0 & -a \end{bmatrix})^{-1} \right\} = \mathcal{L}^{-1} \left\{ \frac{1}{s(s+a)} \begin{bmatrix} s+a & 1 \\ 0 & s \end{bmatrix} \right\} \\ &= \mathcal{L}^{-1} \left\{ \begin{bmatrix} \frac{1}{s} & \frac{1}{s(s+a)} \\ 0 & \frac{1}{s+a} \end{bmatrix} \right\} \\ &= \begin{bmatrix} 1 & \frac{1}{a}(1 - e^{-at}) \\ 0 & e^{-at} \end{bmatrix} \end{aligned}$$

Example for transition matrix



$$i_c = C \cdot \frac{dv_c}{dt} = u - i_L$$

$$v_c = L \cdot \frac{di_L}{dt} + R \cdot i_L$$

$$v_o = R \cdot i_L$$

$$\begin{bmatrix} \frac{dx_1}{dt} \\ \frac{dx_2}{dt} \end{bmatrix} = \begin{bmatrix} 0 & -\frac{1}{C} \\ \frac{1}{L} & -\frac{R}{L} \end{bmatrix} \cdot \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} \frac{1}{C} \\ 0 \end{bmatrix} \cdot u$$

$$y_1 = \begin{bmatrix} 0 & R \end{bmatrix} \cdot \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$\underline{\underline{\Phi(s)}} = \underline{\underline{[sI - A]^{-1}}} = \begin{bmatrix} s & \frac{1}{C} \\ -\frac{1}{L} & s + \frac{R}{L} \end{bmatrix}^{-1}$$

$$\underline{\underline{\Phi(s)}} = \frac{\begin{bmatrix} s + \frac{R}{L} & -\frac{1}{C} \\ \frac{1}{L} & s \end{bmatrix}}{\left(s + \frac{R}{L}\right) \cdot s + \frac{1}{L} \cdot \frac{1}{C}} = \frac{\begin{bmatrix} s + \frac{R}{L} & -\frac{1}{C} \\ \frac{1}{L} & s \end{bmatrix}}{s^2 + \frac{R}{L} \cdot s + \frac{1}{LC}}$$

the inverse of a 2x2 matrix ... $\text{adj}(A) \det(A)$

If: $R = 3, L = 1,$
and $C = 0.5$

$$\begin{bmatrix} \frac{dx_1}{dt} \\ \frac{dx_2}{dt} \end{bmatrix} = \begin{bmatrix} 0 & -\frac{1}{C} \\ \frac{1}{L} & -\frac{R}{L} \end{bmatrix} \cdot \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} \frac{1}{C} \\ 0 \end{bmatrix} \cdot u \quad \mathbf{A} = \begin{bmatrix} 0 & -2 \\ 1 & -3 \end{bmatrix}$$

$$\Phi(s) = [s\mathbf{I} - \mathbf{A}]^{-1} \quad \mathbf{A} = \begin{bmatrix} 0 & 2 \\ 1 & -3 \end{bmatrix} \quad [s\mathbf{I} - \mathbf{A}] = \begin{bmatrix} s & 2 \\ -1 & (s+3) \end{bmatrix}$$

$$\Phi(s) = [s\mathbf{I} - \mathbf{A}]^{-1} = \frac{1}{\det(s\mathbf{I} - \mathbf{A})} \begin{bmatrix} (s+3) & -2 \\ 1 & s \end{bmatrix}$$

$$\det(s\mathbf{I} - \mathbf{A}) = s(s+3) + 2 = s^2 + 3s + 2 = (s+1)(s+2)$$

$$\Phi(s) = \begin{bmatrix} (s+3)/(s+1)(s+2) & -2/(s+1)(s+2) \\ 1/(s+1)(s+2) & s/(s+1)(s+2) \end{bmatrix}$$

$$\underline{\underline{\Phi}}(t) = \begin{bmatrix} (2e^{-t} - e^{-2t}) & (-2e^{-t} + 2e^{-2t}) \\ (e^{-t} - e^{-2t}) & (-e^{-t} + 2e^{-2t}) \end{bmatrix}$$

Initial Condition Response

$$\underline{\underline{x}}(t) = \begin{bmatrix} (2e^{-t} - e^{-2t}) & (-2e^{-t} + 2e^{-2t}) \\ (e^{-t} - e^{-2t}) & (-e^{-t} + 2e^{-2t}) \end{bmatrix} \cdot \underline{\underline{x}}(0)$$

$$\underline{\underline{x}}(t) = \underline{\underline{\Phi}}(t) \cdot \underline{\underline{x}}(0)$$

Solving for the states in X

$$\underline{\underline{X}}(s) = \underline{\underline{\Phi}}(s) \cdot \underline{\underline{x}}(0) + \underline{\underline{\Phi}}(s) \cdot \underline{\underline{B}} \cdot \underline{\underline{U}}(s)$$

$$\underline{\underline{X}}(s) = \underbrace{\begin{bmatrix} s + \frac{R}{L} & -\frac{1}{C} \\ \frac{1}{L} & s \end{bmatrix}}_{s^2 + \frac{R}{L}s + \frac{1}{LC}} \cdot \underline{\underline{x}}(0) + \underbrace{\begin{bmatrix} \left(s + \frac{R}{L}\right) \cdot \frac{1}{C} \\ \frac{1}{LC} \end{bmatrix}}_{s^2 + \frac{R}{L}s + \frac{1}{LC}} \cdot \underline{\underline{U}}(s)$$

Solving for the transfer function

If: $R = 3$, $L = 1$,
and $C = 0.5$

$$y_1 = \begin{bmatrix} 0 & R \end{bmatrix} \cdot \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \longrightarrow$$

$$\underline{\underline{G}}(s) = \underline{\underline{C}} \cdot \underline{\underline{\Phi}}(s) \cdot \underline{\underline{B}} = \begin{bmatrix} 0 & R \end{bmatrix} \cdot \frac{\begin{bmatrix} s + \frac{R}{L} & -\frac{1}{C} \\ \frac{1}{L} & s \end{bmatrix}}{s^2 + \frac{R}{L} \cdot s + \frac{1}{LC}} \cdot \begin{bmatrix} \frac{1}{C} \\ 0 \end{bmatrix}$$

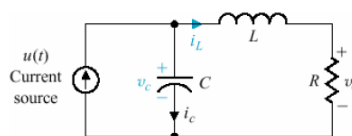
$$\underline{\underline{G}}(s) = \begin{bmatrix} 0 & R \end{bmatrix} \cdot \frac{\begin{bmatrix} \left(s + \frac{R}{L}\right) \cdot \frac{1}{C} \\ \frac{1}{LC} \end{bmatrix}}{s^2 + \frac{R}{L} \cdot s + \frac{1}{LC}} = \frac{\frac{R}{LC}}{s^2 + \frac{R}{L} \cdot s + \frac{1}{LC}}$$

$$\underline{\underline{\Phi}}(s) = \frac{\begin{bmatrix} s + \frac{R}{L} & -\frac{1}{C} \\ \frac{1}{L} & s \end{bmatrix}}{s^2 + \frac{R}{L} \cdot s + \frac{1}{LC}} = \frac{\begin{bmatrix} s+3 & -2 \\ 1 & s \end{bmatrix}}{s^2 + 3 \cdot s + 2} = \frac{\begin{bmatrix} s+3 & -2 \\ 1 & s \end{bmatrix}}{(s+1) \cdot (s+2)}$$

$$\underline{\underline{G}}(s) = \frac{\frac{R}{LC}}{s^2 + \frac{R}{L} \cdot s + \frac{1}{LC}} = \frac{\frac{3}{0.5}}{s^2 + 3 \cdot s + 2} = \frac{6}{(s+2) \cdot (s+1)}$$

Making a Signal Flow Graph from the State Models

For some physical implementations it may be easier to write the state equations than it is to define a single transfer function!



$$\begin{bmatrix} \frac{dv_c}{dt} \\ \frac{di_L}{dt} \end{bmatrix} = \begin{bmatrix} 0 & -\frac{1}{C} \\ \frac{1}{L} & -\frac{R}{L} \end{bmatrix} \cdot \begin{bmatrix} v_c \\ i_L \end{bmatrix} + \begin{bmatrix} \frac{1}{C} \\ 0 \end{bmatrix} \cdot u$$

$$\begin{bmatrix} \frac{dx_1}{dt} \\ \frac{dx_2}{dt} \end{bmatrix} = \begin{bmatrix} 0 & -\frac{1}{C} \\ \frac{1}{L} & -\frac{R}{L} \end{bmatrix} \cdot \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} \frac{1}{C} \\ 0 \end{bmatrix} \cdot u$$

$$y_1 = \begin{bmatrix} 0 & R \end{bmatrix} \cdot \begin{bmatrix} v_c \\ i_L \end{bmatrix}$$

$$y_1 = \begin{bmatrix} 0 & R \end{bmatrix} \cdot \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

