

EEM322 : Lecture 3

Time domain modelling

State Variable Models or State-space approach

Solving for the State Variable System Equations

Making a Signal Flow Graph from the State Models

State space to Transfer function

Transfer function to State space conversion

Analysis of Dynamic Systems: frequency domain (classical approach): or Time domain (Modern approach)

Two approaches are available for the analysis and design of feedback control systems. The first, which we began to study in Chapter 2, is known as the classical, or frequency-domain, technique.

Advantage: A major advantage of frequency-domain techniques is that they rapidly provide **stability and transient response information**. Thus, we can immediately see the effects of varying system parameters until an acceptable design is met.

Disadvantage:

The primary disadvantage of the classical approach is its limited applicability: **It can be applied only to linear, time-invariant systems** or systems that can be approximated as such.

State Variable Models or State-space approach (tim-based approach)

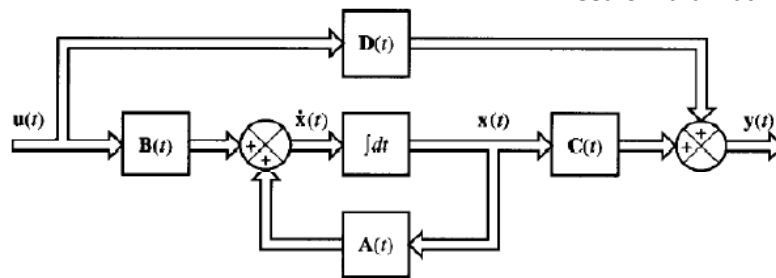
Time-domain techniques can be utilized for **Linear/nonlinear, Time-invariant/time-varying**, and **multivariable** systems. A time-varying control system is a system for which one or more of the parameters of the system may vary as a function of time

Block diagram of the linearized system is illustrated in the below figure

$$\dot{\mathbf{x}}(t) = \mathbf{A}(t)\mathbf{x}(t) + \mathbf{B}(t)\mathbf{u}(t)$$

$$\mathbf{y}(t) = \mathbf{C}(t)\mathbf{x}(t) + \mathbf{D}(t)\mathbf{u}(t)$$

$\mathbf{x}(t)$: State vector
 $\mathbf{y}(t)$: Output vector
 $\mathbf{u}(t)$: Input vector
 $\mathbf{A}(t)$: State matrix
 $\mathbf{B}(t)$: Input matrix = System matrix
 $\mathbf{C}(t)$: Output matrix
 $\mathbf{D}(t)$: Direct transmission matrix = feedforward matrix



The state of a system is a set of variables such that the knowledge of these variables and the input functions will, with the equations describing the dynamics, provide the future state and output of the system.

State: State of a dynamic system is the smallest set of variables (called state variables) such that knowledge of these variables at $t = t_0$, together with the knowledge of the input for $t \geq t_0$, completely determines the behavior of the system for any time $t \geq t_0$.

State Variables: State variables of a dynamic system are the variables making up the smallest set of variables that determine the state of the dynamic system.

If at least n variables are needed to completely describe the behavior of a dynamic system, then such n variables are a set of state variables.

State variables need not be physically measurable or observable quantities. However, if possible, measurable quantities should be selected for state variables.

State Vector: If n state variables are needed to completely describe the state of a given system, then these n state variables can be considered the n components of a vector $\mathbf{x}$. Such a vector is called a state vector.

state equations.

For an n th-order system, we write n simultaneous, first-order differential equations in terms of the state variables. We call this system of simultaneous differential equations **state equations**.

Note : The minimum number of state variables required to describe a system equals the order of the differential equation. Thus, a second-order system requires a minimum of two state variables to describe it

Note : State variables must be linearly independent; that is, no state variable can be written as a linear combination of the other state variables, **Note** that the concept of state is by no means limited to physical systems. It is applicable to biological systems, economic systems, social systems, and others.

Note : A state-space representation is not unique, since a different choice of state variables leads to a different representation of the same system.

Note : state variables need not be physically measurable or observable quantities.

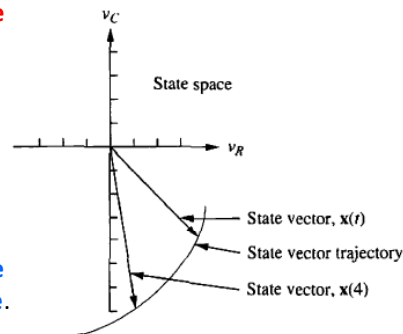
Variables that do not represent physical quantities and those that are neither measurable nor observable can be chosen as state variables. Such freedom in choosing state variables is an advantage of the state-space methods. Practically, however, it is convenient to choose easily measurable quantities for the state variables, if this is possible at all, because optimal control laws will require the feedback of all state variables with suitable weighting.

Graphic representation of state space and a state vector. State space. The n -dimensional space whose axes are the state variables. This is illustrated in Figure

State vector: A vector whose elements are the state variables.

STATE TRAJECTORY. State trajectory is defined as the path produced in the state space by the state vector $\mathbf{x}(t)$ as it changes with the passage of time.

State space and state trajectory in the two-dimensional case are referred to as the phase plane and phase trajectory, respectively

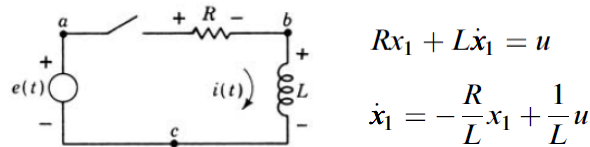


The selection of the state variables:

The **selection** of the state variables for the physical-variable method is based upon the **energy-storage elements** of the system.

Example 1: Series RL Circuit . **Only one energy-storage element**, the inductor, is present in this circuit; thus there is only one state variable. The state variable is x_1 . The equation desired is one that contains the first derivative of the state variable. Letting $u=e$, the loop equation can be rewritten as

There is only one state equation because this is a first order system, $n=1$.

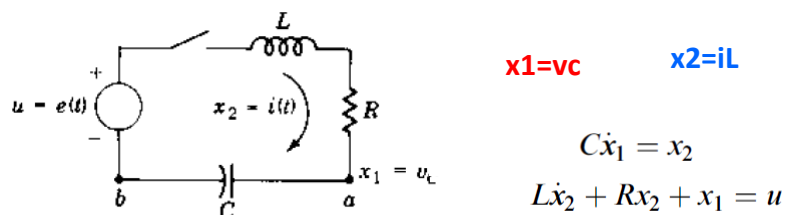


Example 2: Series RLC Circuit. This circuit contains **two energy-storage** elements, the inductor and capacitor. The two assigned state variables are identified as **$x_1=vc$ (the voltage across the capacitor)** and **$x_2=i$ (the current in the inductor)**. Thus two state equations are required.

A loop or branch equation is written to obtain an equation containing the derivative of the current in the inductor. A node equation is written to obtain an equation containing the derivative of the capacitor voltage. The number of loop equations that must be written is equal to the number of state variables representing currents in inductors. The number of equations involving node voltages that must be written is equal to the number of state variables representing voltages across capacitors.

Example 2:

The node equation for node a and the loop equation are, respectively



Rearranging terms to the standard state equation format yields

$$\dot{x}_1 = \frac{1}{C}x_2$$

$$\dot{x}_2 = -\frac{1}{L}x_1 - \frac{R}{L}x_2 + \frac{1}{L}u$$

These Equations represent the state equations of the system containing two independent state variables. Note that they are first-order linear differential equations and are $n=2$ in number. They are the minimum number of state equations required to represent the system's future performance. **in matrix notation as**

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & \frac{1}{C} \\ -\frac{1}{L} & -\frac{R}{L} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ \frac{1}{L} \end{bmatrix} [u]$$

$$\dot{\mathbf{x}} = \mathbf{Ax} + \mathbf{bu}$$

$$\dot{\mathbf{x}} = \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix}$$

an $n \times 1$ column vector

$$\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} = \begin{bmatrix} 0 & \frac{1}{C} \\ -\frac{1}{L} & -\frac{R}{L} \end{bmatrix}$$

an $n \times n$ plant coefficient matrix

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

an $n \times 1$ state vector

$$\mathbf{b} = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix} = \begin{bmatrix} 0 \\ \frac{1}{L} \end{bmatrix}$$

an $n \times 1$ control matrix

If the output quantity $y(t)$ is the voltage across the capacitor VC, then

$$\mathbf{y}(t) = \mathbf{c}^T \mathbf{x} + \mathbf{d}u = [1 \quad 0] \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + [0][u]$$

$$\mathbf{c}^T = [1 \quad 0] \quad \text{a } 1 \times 2 \text{ row vector}$$

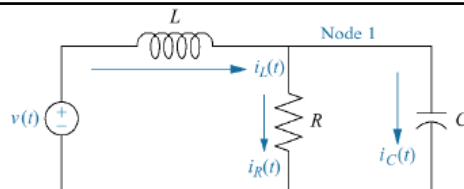
$$\mathbf{y} = [y] \quad \text{a one-dimensional output vector}$$

$$\mathbf{d} = 0$$

Book: LINEAR CONTROL SYSTEM ANALYSIS AND DESIGN WITH MATLAB

Example-: State-space Representation

Step 1: Label all the branch currents in the network.



Step 2: Select the state variables (energy-storage elements: L and C) and write derivative equations.

$$C \frac{dv_C}{dt} = i_C, \quad L \frac{di_L}{dt} = v_L$$

Step 3: Express non-state variables (right-hand side: i_C and v_L) as a linear combinations of the state variables (differentiated variables: v_C and i_L) and the input, $v(t)$.

$$i_C = -i_R + i_L = -\frac{1}{R}v_C + i_L \quad v_L = -v_C + v(t) \quad \text{Step 4: Obtain state equations: (by substituting the values and rearranging)}$$

$$\Rightarrow \frac{dv_C}{dt} = -\frac{1}{RC}v_C + \frac{1}{C}i_L \quad L \frac{di_L}{dt} = -v_C + v(t) \quad \begin{bmatrix} \dot{v}_C \\ \dot{i}_L \end{bmatrix} = \begin{bmatrix} -\frac{1}{RC} & \frac{1}{C} \\ -\frac{1}{L} & 0 \end{bmatrix} \begin{bmatrix} v_C \\ i_L \end{bmatrix} + \begin{bmatrix} 0 \\ \frac{1}{L} \end{bmatrix} v(t)$$

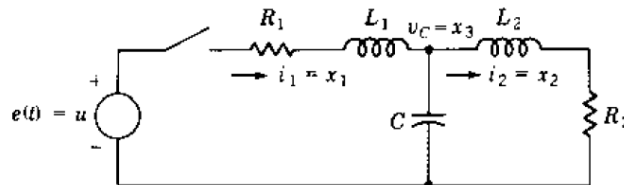
Step 5: Find the output equation

$$i_R = \frac{1}{R}v_C \quad i_R = \begin{bmatrix} \frac{1}{R} & 0 \end{bmatrix} \begin{bmatrix} v_C \\ i_L \end{bmatrix}$$

Example 3. Obtain the state equation for the circuit of Fig., where i_2 is considered to be the output of this system. The assigned state variables are $x_1=i_1$, $x_2=i_2$, and $x_3=v_C$. Thus, two loop and one node equations are written

Loop
Loop
Node

$$\begin{aligned} R_1 x_1 + L_1 \dot{x}_1 + x_3 &= u \\ -x_3 + L_2 \dot{x}_2 + R_2 x_2 &= 0 \\ -x_1 + x_2 + C \dot{x}_3 &= 0 \end{aligned}$$



The three state variables are independent,

$$\dot{\mathbf{x}} = \begin{bmatrix} -\frac{R_1}{L_1} & 0 & -\frac{1}{L_1} \\ 0 & -\frac{R_2}{L_2} & \frac{1}{L_2} \\ \frac{1}{C} & -\frac{1}{C} & 0 \end{bmatrix} \mathbf{x} + \begin{bmatrix} \frac{1}{L_1} \\ 0 \\ 0 \end{bmatrix} \mathbf{u}$$

$\mathbf{C} = l \times n$ output matrix

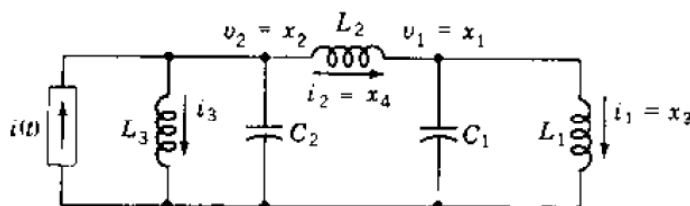
$\mathbf{D} = l \times m$ feedforward matrix

$\mathbf{u} = m$ -dimensional control vector

$\mathbf{y} = l$ -dimensional output vector

$$\mathbf{y} = [0 \quad 1 \quad 0] \mathbf{x}$$

Example 4. Obtain the state equations for the circuit of Fig. The output is the voltage v_1 . The input or control variable is a current source $i(t)$. The assigned state variables are i_1, i_2, i_3, v_1 , and v_2 . Three loop equations and two node equations are written:



$$\begin{aligned} v_1 &= L_1 \frac{di_1}{dt} \\ v_2 &= L_2 \frac{di_2}{dt} + v_1 \\ v_2 &= L_3 \frac{di_3}{dt} \end{aligned}$$

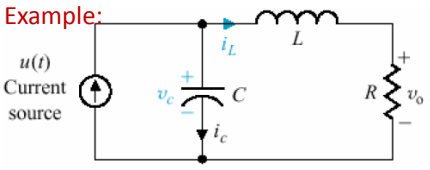
$$L_3 \dot{i}_3 = L_2 \dot{i}_2 + L_1 \dot{i}_1 + K$$

$$\dot{\mathbf{x}} = \begin{bmatrix} 0 & 0 & -\frac{1}{C_1} & \frac{1}{C_1} \\ 0 & 0 & -\frac{L_1}{L_3 C_2} & -\frac{L_2 + L_3}{L_3 C_2} \\ \frac{1}{L_1} & 0 & 0 & 0 \\ -\frac{1}{L_2} & \frac{1}{L_2} & 0 & 0 \end{bmatrix} \mathbf{x} + \begin{bmatrix} 0 \\ \frac{1}{C_2} \\ 0 \\ 0 \end{bmatrix} \mathbf{u} = \mathbf{u}$$

$$\begin{aligned} i_2 &= C_1 \frac{dv_1}{dt} + i_1 \\ i &= i_3 + C_2 \frac{dv_2}{dt} + i_2 \end{aligned}$$

$$\mathbf{y} = [1 \quad 0 \quad 0 \quad 0] \mathbf{x}$$

Example:



$$v_c = L \cdot \frac{di_L}{dt} + R \cdot i_L$$

$$i_c = C \cdot \frac{dv_c}{dt} = u - i_L$$

$$v_o = R \cdot i_L$$

Let $x_1 = v_c$ and $x_2 = i_L$ and $y_1 = v_o$

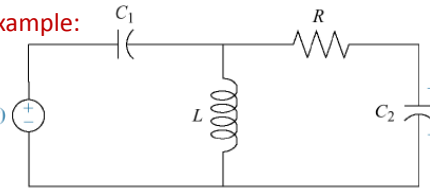
$$\begin{bmatrix} \frac{dx_1}{dt} \\ \frac{dx_2}{dt} \end{bmatrix} = \begin{bmatrix} 0 & -\frac{1}{C} \\ \frac{1}{L} & -\frac{R}{L} \end{bmatrix} \cdot \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} \frac{1}{C} \\ 0 \end{bmatrix} \cdot u$$

$$\frac{dx_1}{dt} = -\frac{1}{C} \cdot x_2 + \frac{1}{C} \cdot u$$

$$\frac{dx_2}{dt} = +\frac{1}{L} \cdot x_1 - \frac{R}{L} \cdot x_2$$

$$y_1 = R \cdot x_2 \Rightarrow y_1 = \begin{bmatrix} 0 & R \end{bmatrix} \cdot \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

Example:



$$\dot{x} = \begin{bmatrix} 1/C_1 & 1/C_1 & -1/C_1 \\ -1/L & 0 & 0 \\ 1/C_2 & 0 & -1/C_2 \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} v_i(t)$$

$$y = \begin{bmatrix} 0 & 0 & 1 \end{bmatrix} x$$

For a multiple-input multiple-output (MIMO) system, with m inputs and l outputs, these equations become

$\mathbf{A} = n \times n$ plant matrix

$\mathbf{B} = n \times m$ control matrix

$\mathbf{C} = l \times n$ output matrix

$\mathbf{D} = l \times m$ feedforward matrix

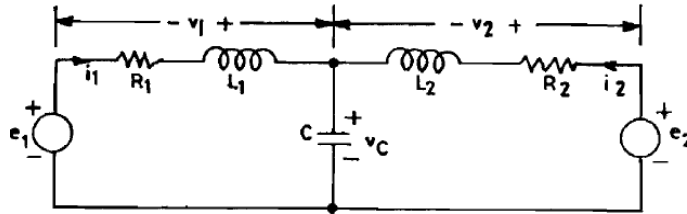
$\mathbf{u} = m$ -dimensional control vector

$\mathbf{y} = l$ -dimensional output vector

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{u}$$

$$\mathbf{y} = \mathbf{C}\mathbf{x} + \mathbf{D}\mathbf{u}$$

Example: In the circuit shown in Fig., the inputs are e_1 and e_2 and the outputs are V_1 , V_2 and V_c . Obtain a state variable model for this system.



As there are three independent energy storage elements, L_1 , L_2 and C , three state variables will be required for a correct representation of the system. There are three outputs and two inputs.

$$i_{L1} = i_1 = x_1; \quad i_{L2} = i_2 = x_2; \quad \text{and} \quad v_c = x_3.$$

$$i_1 R_1 + L_1 \frac{di_1}{dt} + v_c = e_1 \quad i_2 R_2 + L_2 \frac{di_2}{dt} + v_c = e_2.$$

$$\dot{x}_1 = \frac{di_1}{dt} = -\frac{R_1}{L_1} i_1 - \frac{1}{L_1} v_c + \frac{1}{L_1} e_1 = -\frac{R_1}{L_1} x_1 - \frac{1}{L_1} x_3 + \frac{1}{L_1} e_1$$

$$\dot{x}_2 = \frac{di_2}{dt} = -\frac{R_2}{L_2} i_2 - \frac{1}{L_2} v_c + \frac{1}{L_2} e_2 = -\frac{R_2}{L_2} x_2 - \frac{1}{L_2} x_3 + \frac{1}{L_2} e_2.$$

For the third state equation, we apply Kirchhoff's current law to the central node to get

$$C \frac{dv_c}{dt} = (i_1 + i_2) \quad i_2 = x_2; \quad x_3 = \frac{1}{C} x_1 + \frac{1}{C} x_2.$$

$$i_1 = x_1$$

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} -\frac{R_1}{L_1} & 0 & -\frac{1}{L_1} \\ 0 & -\frac{R_2}{L_2} & -\frac{1}{L_2} \\ \frac{1}{C} & \frac{1}{C} & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} \frac{1}{L_1} & 0 \\ 0 & \frac{1}{L_2} \\ 0 & 0 \end{bmatrix} \begin{bmatrix} e_1 \\ e_2 \end{bmatrix}$$

The outputs, as linear combinations of state variables, are given by the relations,

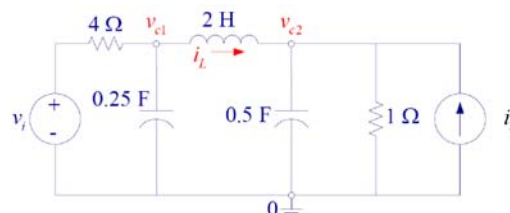
$$y_1 = v_1 = e_1 - v_c; \quad y_2 = v_2 = e_2 - v_c; \quad \text{and} \quad y_3 = v_c.$$

$$y_1 = -x_3 + e_1; \quad y_2 = -x_3 + e_2; \quad \text{and} \quad y_3 = x_3.$$

In the standard form, the output equations are,

$$\begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} 0 & 0 & -1 \\ 0 & 0 & -1 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} e_1 \\ e_2 \end{bmatrix}.$$

Example : Write the state equation for the network shown in Figure. Define the state variables as current through the inductor and voltage across the capacitors. Write two node equations containing capacitors and a loop equation containing the inductor. The state variables will be v_{c1} , v_{c2} , and i_L . Define the state variables Current through the inductor and voltage across the capacitors.

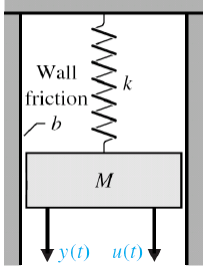


$$0.25 \frac{dv_{c1}}{dt} + i_L + \frac{v_{c1} - v_i}{4} = 0 \Rightarrow \dot{v}_{c1} = -v_{c1} - 4i_L + v_i$$

$$0.5 \frac{dv_{c2}}{dt} - i_L + v_{c2} - i_s = 0 \Rightarrow \dot{v}_{c2} = -2i_L + 2v_{c2} + 2i_s$$

$$2 \frac{di_L}{dt} + v_{c2} - v_{c1} = 0 \Rightarrow \dot{i}_L = 0.5v_{c1} - 0.5v_{c2}$$

$$\begin{bmatrix} \dot{v}_{c1} \\ \dot{v}_{c2} \\ \dot{i}_L \end{bmatrix} = \begin{bmatrix} -1 & 0 & -4 \\ 0 & -2 & 2 \\ 0.5 & -0.5 & 0 \end{bmatrix} \begin{bmatrix} v_{c1} \\ v_{c2} \\ i_L \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 2 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} v_i \\ i_s \end{bmatrix}$$



Example : State Variable Models

A spring-mass damper system

$$M \frac{d^2 y}{dt^2} + b \frac{dy}{dt} + ky = u(t)$$

$$M \cdot s^2 \cdot Y(s) + b \cdot s \cdot Y(s) + k \cdot Y(s) = U(s)$$

Define a set of state variables as x_1 , x_2 , position and velocity, where

$$x_1 = y(t)$$

$$x_2 = \frac{dy(t)}{dt}$$

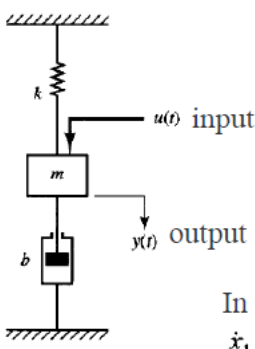
Substituting this into the system equation $M \frac{dx_2}{dt} + bx_2 + kx_1 = u(t)$

Then we can describe the system using two equations having two state variables:

$$\begin{bmatrix} \frac{dx_1}{dt} \\ \frac{dx_2}{dt} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -\frac{k}{M} & -\frac{b}{M} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} \cdot u(t)$$

$$y = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + 0 \cdot u(t)$$

Example: State-space modeling of a linear single-input-single-output mechanical system



$u(t)$: external force applied to the system
 $y(t)$: displacement of mass m , measured from the equilibrium position in the absence of $u(t)$.

$m\ddot{y} + b\dot{y} + ky = u$

System equation

$x_1(t) = y(t)$
 $x_2(t) = \dot{y}(t)$

}

State variables

In view of system equation and state variables

$\dot{x}_1 = x_2$
 $\dot{x}_2 = \frac{1}{m}(-ky - b\dot{y}) + \frac{1}{m}u$

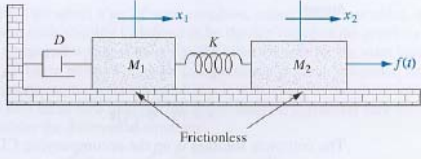
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$\dot{x}_1 = x_2$
 $\dot{x}_2 = -\frac{k}{m}x_1 - \frac{b}{m}x_2 + \frac{1}{m}u$

$y = x_1$

Output equation

Two first-order differential eqns (State eqns)



The sum of all forces at M_1 is zero.

$$M_1 \frac{d^2 x_1}{dt^2} + D \frac{dx_1}{dt} + K(x_1 - x_2) = 0$$

Force $f(t)$ balances the rest of all forces.

$$K(-x_1 + x_2) + M_2 \frac{d^2 x_2}{dt^2} = f(t)$$

$$\begin{aligned} \frac{dx_1}{dt} &= v_1 \\ \frac{dv_1}{dt} &= -\frac{K}{M_1} x_1 - \frac{D}{M_1} v_1 + \frac{K}{M_1} x_2 \\ \frac{dx_2}{dt} &= v_2 \\ \frac{dv_2}{dt} &= \frac{K}{M_2} x_1 - \frac{K}{M_2} x_2 + \frac{1}{M_2} f(t) \end{aligned}$$

Writing in vector-matrix form,

$$\begin{bmatrix} \dot{x}_1 \\ \dot{v}_1 \\ \dot{x}_2 \\ \dot{v}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -K/M_1 & -D/M_1 & K/M_1 & 0 \\ 0 & 0 & 0 & 1 \\ K/M_2 & 0 & -K/M_2 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ v_1 \\ x_2 \\ v_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1/M_2 \end{bmatrix} f(t)$$