

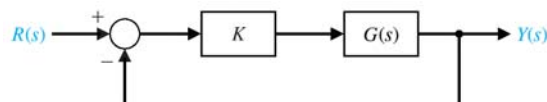
EEM322L8:
Root locus (kök yer eğrisi)

Four primarily graphical methods are available to analysis a system in frequency domain:

1. The Root-Locus Method
2. Bode- Plot Representations
3. Nyquist Diagrams
4. Nichols Charts

What is Root Locus ? **VARIATION OF CLOSED-LOOP SYSTEM POLES**

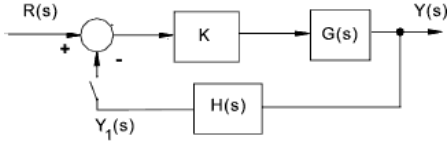
The characteristic equation of the closed-loop system is



$$1 + K G(s) = 0$$

The root locus is essentially the trajectories of roots of the characteristic equation as the parameter K is varied from 0 to infinity.

The **open-loop** transfer function between the forcing input $R(s)$ and the measured output $Y_1(s)$ = $T_1(s) = K \cdot G(s)H(s)$



The **closed-loop** transfer function = $T(s) = \frac{KG(s)}{1 + KG(s)H(s)}$

The control system problem:

- ✗ Poles of open-loop transfer function:
 - i. Easy to obtain via inspection of the denominator
 - ii. Not affected by changes in system gain
- ✗ Poles of closed-loop transfer function:
 - i. More difficult to obtain
 - ii. Affected by changes in system gain

The Root Locus is a graphical method for depicting location of the closed-loop poles as a system parameter is varied. It is applicable for first and second-order systems, but also to higher order systems.

- ✗ Provides a **qualitative** description of systems **performance**, and this technique can be used for analysis of **order higher systems**, Also provide a graphical representation of **system's stability**. Ranges of stability, Ranges of instability,

Geribeslemeli kontrol sistemleri tasarımında açık döngü sistemin analiz edilmesi, kapalı döngü sistemin nasıl davranacağı hakkında bilgi edinilebilmesi açısından çok önemlidir.

Yöntemlerden bir tanesi sistem için **kök yer eğrisinin** oluşturulması ve yorumlanmasıdır.

Tanım: Kök yer eğrisi sistem parametrelerinin değişimi ile Sistem kapalı döngü köklerinin s düzlemindeki yerini gösteren grafiktir

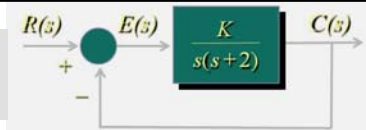
Kapalı döngü sistemlerin geçici rejim cevap karakteristikleri kapalı döngü kutuplarının yerlerine bağlıdır. Eğer sistem değişken kazanca sahipse, kapalı döngü sistemin kutupları seçilen kazanca göre değişir. Dolayısıyla kontrol tasarımcısının döngü kazancı değiştikçe kapalı döngü sistemin kutuplarının nasıl hareket ettiğini bilmesi önemlidir.

Amaç: **İstenilen sistem cevabını elde edebilmek için uygun kutuplar seçmek ve dolayısıyla bu kutuplar için sistem kazancını belirlemektir**

Example

The closed-loop
TF is

$$\frac{C(s)}{R(s)} = \frac{K}{s(s+2)+K}$$



Consider the characteristic roots as $K = 0 \rightarrow \infty$. • The characteristic equation is

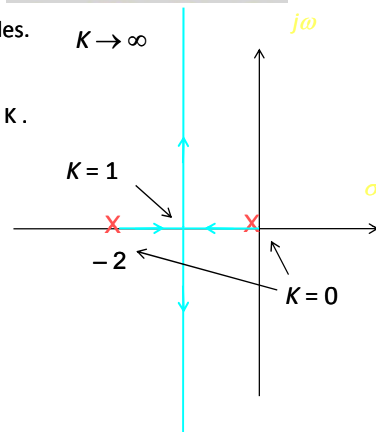
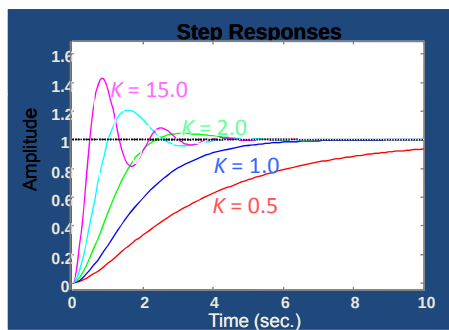
$$s = -1 \pm \sqrt{1-K}$$

$$s^2 + 2s + K = 0$$

For $K = 0$ the closed-loop poles are at the open-loop poles.

For $0 < K < 1$ the closed-loop poles are on the real axis.

For $K > 1$ the closed-loop poles are complex, with a real value of -1 and an imaginary value increasing with gain K .



Example : - The open loop TF. of a unity feedback system is given,

Determine the location of its closed loop poles for increasing values of K . Also determine the limiting value of K beyond which the closed loop system with unity feedback becomes unstable. **All the three poles of the plant, $S_1 = -1$, $S_2 = -2$ and $S_3 = -3$, are negative. So the uncontrolled system (without feedback) is stable.**

Open loop $G(s) = \frac{K}{(s+1)(s+2)(s+3)} = \frac{K}{s^3 + 6s^2 + 11s + 6}$

Closed loop $\frac{C(s)}{R(s)} = \frac{G(s)}{1+G(s)} = \frac{K}{s^3 + 6s^2 + 11s + 6 + K}$

characteristic equation $1 + G(s) = 0 \quad s^3 + 6s^2 + 11s + 6 + K = 0.$

Example:

$$s^3 + 6s^2 + 11s + 6 + K = 0.$$

(1) $K = 0$: The closed loop characteristic equation becomes the plant characteristic equation. Hence, the closed loop poles will be the same as the open loop poles. That is,

$$s_1 = -1, s_2 = -2, s_3 = -3.$$

(2) $K = 6$:

$$s_1, s_2 = -1 \pm j\sqrt{2} \text{ and } s_3 = -4.$$

(3) $K = 24$:

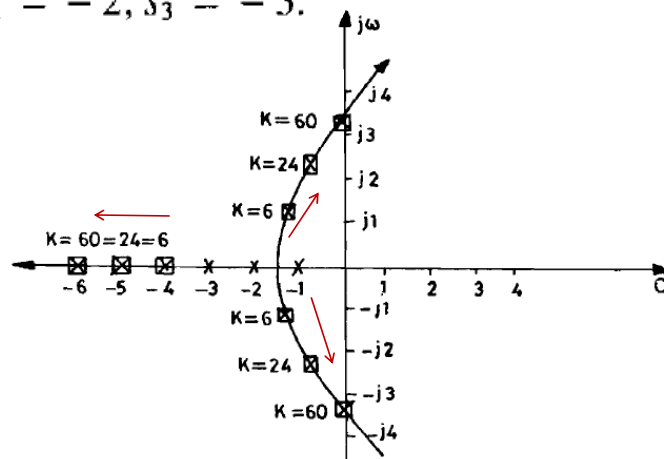
$$s_1, s_2 = -0.5 \pm j\sqrt{23}/2 \text{ and } s_3 = -5.$$

(4) $K = 60$:

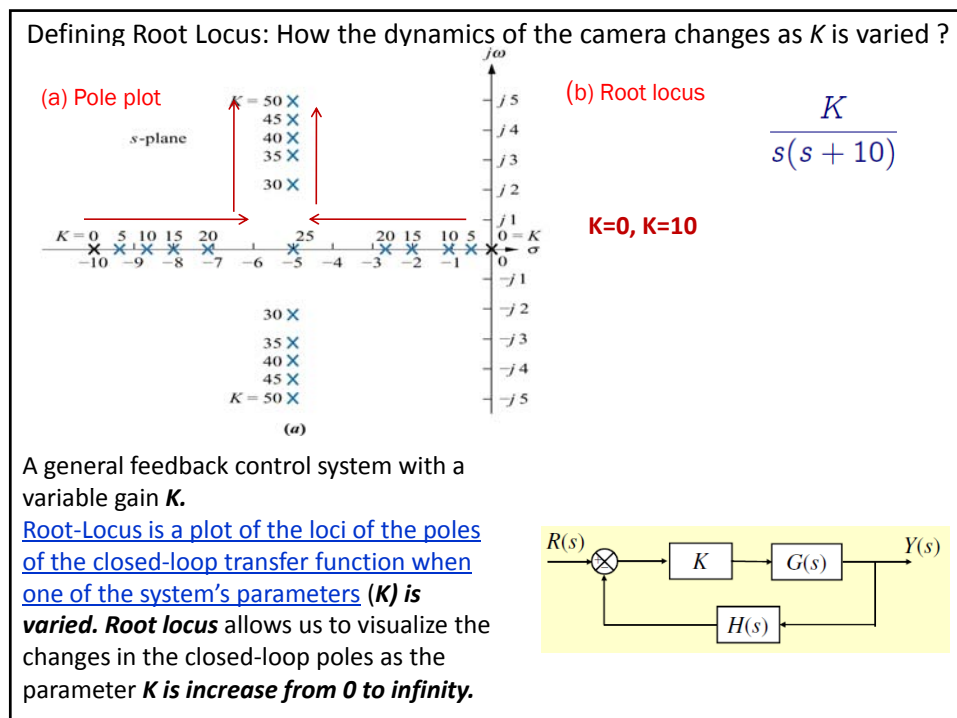
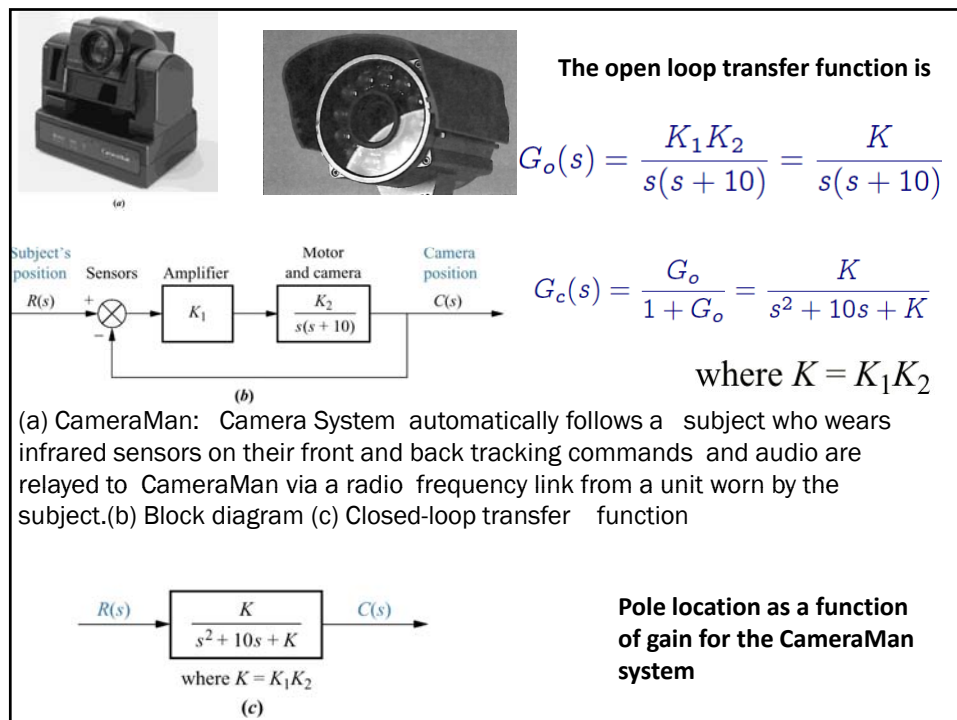
$$s_1, s_2 = \pm j\sqrt{11} \text{ and } s_3 = -6.$$

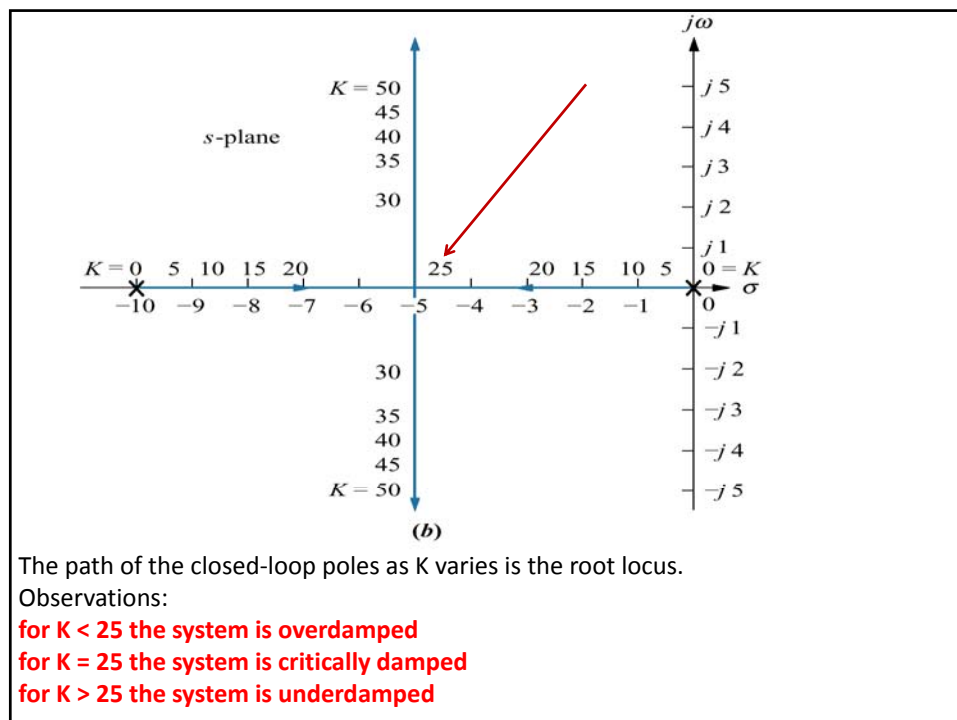
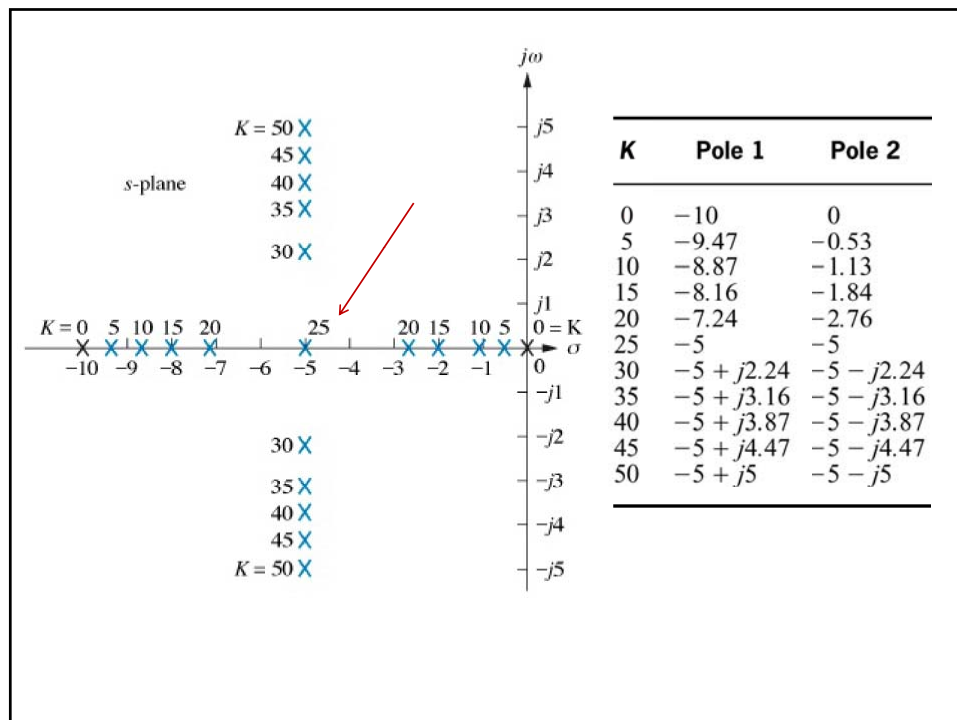
The locus of the closed loop poles for increasing values of K is shown in Fig. Diagram, showing the movement of the roots of the closed loop characteristic Equation. This figure shows that for values of **gain K greater than 60, the closed loop poles will enter the r.h.s. of the s -plane. The system will then become unstable.**

$$s_1 = -1, s_2 = -2, s_3 = -3.$$



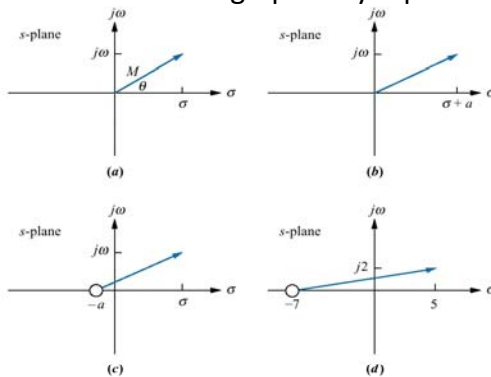
Locus of Closed Loop Poles (Root Loci)





Complex Numbers and Their Representation as Vectors

Any complex number, $\sigma + j\omega$, described in Cartesian coordinates can be graphically represented by a vector.



- (a) $s = \sigma + j\omega$
- (b) $F(s) = s + a$
- (c) Alternate representation of $F(s) = s + a$
- (d) $F(s) = s + 7 \mid_{s \rightarrow 5 + j2}$

Complex Numbers as Vector

- It can be concluded that $(s + a)$ is a complex number and can be represented by a vector drawn from the zero of the function to the point s .
- The complex number can also be represented in polar form with magnitude M and angle θ .

$$M \angle \theta$$

Complex Numbers as Vector

- Assume a function,

$$F(s) = \frac{\prod_{i=1}^m (s + z_i)}{\prod_{j=1}^n (s + p_j)} = \frac{\prod \text{numerator's complex factors}}{\prod \text{denominator's complex factors}}$$

Where,

$\prod$: Product; m : Number of zeros; n : Number of poles

- Each factor in the numerator and denominator is a complex number that can be represented as a vector.
- The function defines the complex arithmetic to be performed to evaluate $F(s)$ at any point, s .

Since each complex factor can be thought of as a vector, the magnitude M , of $F(s)$ at any point s is,

$$F(s) = \frac{\prod_{i=1}^m |(s + z_i)|}{\prod_{j=1}^n |(s + p_j)|} = \frac{\prod \text{zero lengths}}{\prod \text{pole lengths}}$$

Where,

- $|(s + z_i)|$ is magnitude of the vector
- drawn from the zero of $F(s)$ at $-z_i$ to the point s .
- $|(s + p_j)|$ is magnitude of the vector drawn from the pole of $F(s)$ at $-p_j$ to the point s .

The magnitude and phase of $F(s)$ are as follows

$$|F(s)| = \frac{r_{z1} r_{z2} \cdots}{r_{p1} r_{p2} \cdots} \quad \theta = \sum_{i=1}^m \angle(s + z_i) - \sum_{j=1}^n \angle(s + p_j)$$

$$= \frac{|s + z_1| |s + z_2| \cdots}{|s + p_1| |s + p_2| \cdots} = \sum \text{zero angles} - \sum \text{pole angles}$$

$$\angle F(s) = \theta_{z1} + \theta_{z2} + \cdots - \theta_{p1} - \theta_{p2} - \cdots$$

$$= \angle(s + z_1) + \angle(s + z_2) + \cdots - \angle(s + p_1) - \angle(s + p_2) - \cdots$$

Example: Find $F(s)$ at the point $s = -3 + j4$

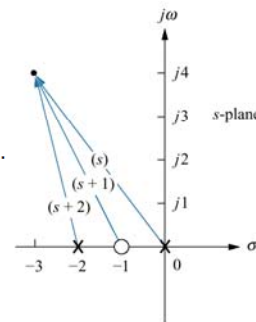
$$F(s) = \frac{(s+1)}{s(s+2)}$$

The magnitude and phase of the zero is (zero at -1) $\sqrt{20} \angle 116.6^\circ$.

The pole at the origin evaluates to (at the origin) $5 \angle 126.9^\circ$.

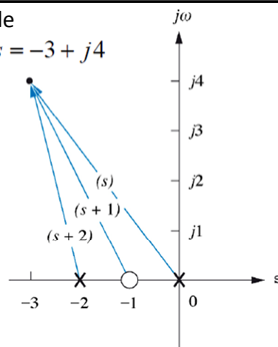
The pole at -2 evaluates to (at -2) $\sqrt{17} \angle 104.0^\circ$.

$$|F(s)| \angle F(s) = \frac{\sqrt{20}}{5\sqrt{17}} \angle [116.9^\circ - 129.9^\circ - 104.0^\circ]$$



Example

$$s = -3 + j4$$



$$\text{II: } \theta = \sum \text{zero angles} - \sum \text{pole angles}$$

$$\theta = \sum_{i=1}^m \angle(s + z_i) - \sum_{j=1}^n \angle(s + p_j)$$

$$\text{Angle from zero: } \theta_z = 180^\circ - \tan^{-1}\left(\frac{4}{2}\right) = 116.6^\circ$$

$$\text{Angle from pole: } \theta_{p1} = 180^\circ - \tan^{-1}\left(\frac{4}{3}\right) = 126.87^\circ$$

$$\theta_{p2} = 180^\circ - \tan^{-1}\left(\frac{4}{1}\right) = 104.04^\circ$$

$$\angle F(s) = 116.6^\circ - 126.87^\circ - 104.04^\circ = -114.31^\circ$$

Example

$$F(s) = \frac{(s+2)(s+4)}{s(s+3)(s+6)}$$

$$-0.0339 - j0.0899 = 0.096 \angle -110.7^\circ$$

$$F(s) = \frac{(s+2)(s+4)}{s(s+3)(s+6)}$$

find $F(s)$ at the point $s = -7 + j9$ the following ways:

- Directly substituting the point into $F(s)$
- Calculating the result using vectors

- Example: Find $F(s)$ at the point $s = -7 + j9$ using the following ways.

$$F(s) = \frac{(s+2)(s+4)}{s(s+3)(s+6)}$$

- Directly substituting the point into $F(s)$.
- Calculating the result using vectors.

- Answer:

$$0.096 \angle -110.7^\circ$$

❑ Complicated Function

$$F(s) = \frac{\prod_{i=1}^m (s + z_i)}{\prod_{j=1}^n (s + p_j)} = \frac{\prod \text{numerator's complex factors}}{\prod \text{denominator's complex factors}}$$

❑ Magnitude

$$M = \frac{\prod \text{zero lengths}}{\prod \text{pole lengths}} = \frac{\prod_{i=1}^m |(s + z_i)|}{\prod_{j=1}^n |(s + p_j)|} \quad \text{M = magnitude of } F(s) \text{ at any point, } s.$$

❑ Angle

$$\theta = \sum \text{zero angles} - \sum \text{pole angles}$$

$$\theta = \sum_{i=1}^m \angle(s + z_i) - \sum_{j=1}^n \angle(s + p_j) \quad \theta = \text{angle of } F(s) \text{ at any point, } s.$$

Properties of the root locus:

when the characteristic polynomial in the denominator becomes zero.

$$T(s) = \frac{KG(s)}{1 + KG(s)H(s)}$$

$$KG(s)H(s) = -1 = 1 \angle (2k+1)180^\circ, \quad \text{where } k = 0, \pm 1, \pm 2, \pm 3, \dots$$

The closed loop poles have to satisfy the following conditions

$$|KG(s)H(s)| = 1$$

$$\angle KG(s)H(s) = (2k+1)180^\circ$$

K that satisfies the first condition is given by

$$K = \frac{1}{|G(s)||H(s)|}$$

$$KG(s)H(s) = -1 = 1 \angle (2k+1)180^\circ, \quad \text{where } k = 0, \pm 1, \pm 2, \pm 3, \dots$$

Equation above implies that if a value of s is substituted into the function $KG(s)H(s)$, a complex number results. If the angle of the complex number is an odd multiple of 180° , that value of s is a system pole for some particular value of K . What value of K ?

We have just found that a pole of the closed-loop system causes the angle of $KG(s)H(s)$, or simply $G(s)H(s)$ since K is a scalar, to be an odd multiple of 180° . Furthermore, the magnitude of $KG(s)H(s)$ must be unity, implying that the value of K is the reciprocal of the magnitude of $G(s)H(s)$ when the pole value is substituted for s .

Example:

closed-loop poles exist at -9.47 and -0.53

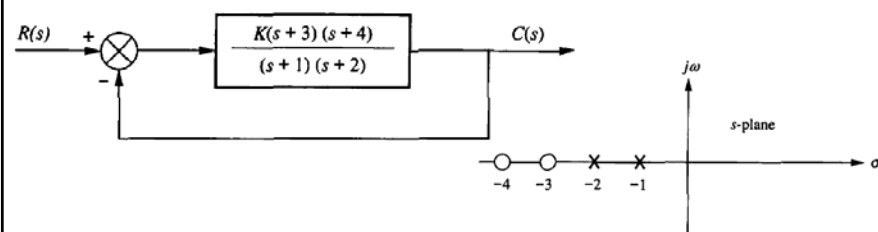
when the gain is 5

Substituting the pole at -9.47 for s and 5 for K yields $KG(s)H(s) = -1$. The student can repeat the exercise for other points in Table 8.1 and show that each case yields $KG(s)H(s) = -1$.

For this system the open-loop transfer function is $KG(s)H(s) = \frac{K(s+3)(s+4)}{(s+1)(s+2)}$

The closed-loop transfer function

$$T(s) = \frac{K(s+3)(s+4)}{(1+K)s^2 + (3+7K)s + (2+12K)}$$



Vector representation of $G(s)$ from Figure a) at $-2 + j3$

Consider the point $-2 + j3$. If this point is a closed-loop pole for some value of gain, then the angles of the zeros minus the angles of the poles must equal an odd multiple of 180° .

$$K = \frac{1}{|G(s)H(s)|} = \frac{1}{M} = \frac{\prod \text{pole lengths}}{\prod \text{zero lengths}}$$

$$\theta_1 + \theta_2 - \theta_3 - \theta_4 = 56.31^\circ + 71.57^\circ - 90^\circ - 108.43^\circ = -70.55^\circ$$

$-2 + j3$ is not a closed loop pole

with the point $-2 + j3$ replaced by $-2 + j(\sqrt{2}/2)$, the gain, K , is

$$K = \frac{L_3 L_4}{L_1 L_2} = \frac{\frac{\sqrt{2}}{2} (1.22)}{(2.12)(1.22)} = 0.33$$

the point $-2 + j(\sqrt{2}/2)$ is a point on the root locus for a gain of 0.33.

