

**EEM322 : Lecture 9**  
**Frequency Domain analysis**  
**Frequency Response**  
**Techniques**  
**Bode diagrams**

**FREQUENCY DOMAIN RESPONSE**

**The performance of a control system is normally measured in terms of its time-domain characteristics.** It may equally be measured in frequency-domain, since there are **correlations between the frequency-domain and time-domain** performances in a linear system, so that the time-domain properties of the system can be predicted based on the frequency-domain analysis. Besides, **the frequency domain approach provides an alternative perspective to the complex analysis** and design problems of linear control systems in addition to the time-domain studies. The frequency-domain analysis of a linear control system assumes that the input is sinusoidal,

**Four primarily graphical methods** are available to analysis a system in frequency domain:

- |                           |                               |
|---------------------------|-------------------------------|
| 1. The Root-Locus Method; | 2. Bode- Plot Representations |
| 3. Nyquist Diagrams;      | 4. Nichols Charts             |

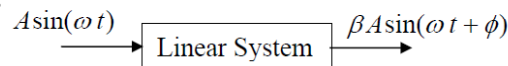
**The frequency response of a system is defined as the steady-state response of the system to a sinusoidal input signal.** The sinusoid is a unique input signal, and the resulting output signal for a linear system, as well as signals throughout the system, is sinusoidal in the steady state; it differs from the input waveform only in amplitude and phase angle

Frequency response methods, developed by **Nyquist and Bode in the 1930s**, are older than the root locus method, which was discovered by Evans in 1948 (*Nyquist, 1932; Bode, 1945*)

### Frequency Domain Analysis

The frequency response of a system is defined as the **steady-state response of the system to a sinusoidal (harmonic) input**.

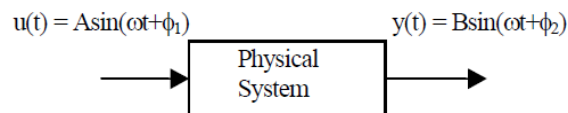
**For linear systems, the resulting output is itself harmonic**; it differs from the input in **amplitude and phase only**. This may be illustrated in a block diagram as follows:



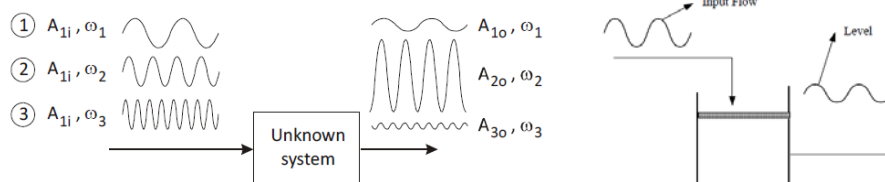
Here,  $\beta$  represents the multiplication factor for the magnitude, and  $\phi$  represents the relative **phase shift** between the input and the output. If  $\beta > 1$ , the system **amplifies the input**, and if  $\beta < 1$ , the system **attenuates the input**.

**System phase shift:**

$$\phi_2 = \phi_1 + \text{Phase shift}$$

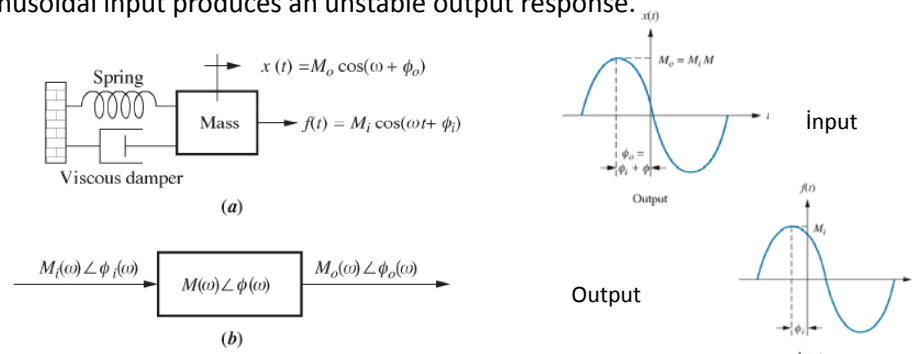


$$\text{System Gain} = \frac{\text{Amplitude of } y(t)}{\text{Amplitude of } u(t)} = \frac{B}{A}$$



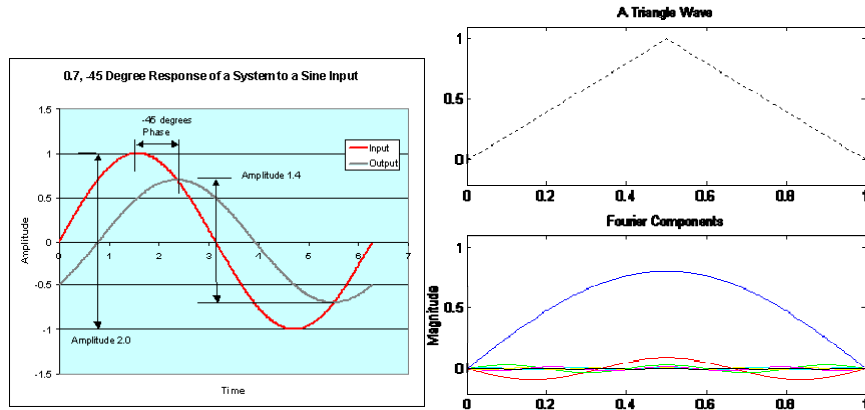
For a **linear system**, the steady-state response will exhibit the same frequency  $\omega$  as the input, but will vary in magnitude and phase.

An **unstable process** does not have a “frequency response” because a sinusoidal input produces an unstable output response.



### Neden Sinüs Sinyali?

Genelde bir devre veya sistemin testi için kullanılan test fonksiyonu sinüs sinyalidir. Bunun nedeni her yerde ortak olarak kullanılması değildir. Aslında ortak da değildir. Örneğin, Amerika'da 60 Hz diğer bazı yerlerde ise 50 Hz olarak kullanılırlar. Sinüs sinyalinin önemli olmasının nedeni Fourier Teorisidir. Bir sinyal bir devrenin girişine uygulandığında çözümlemesinin kolay yapılabilmesi için genlik ve frekansının türevleri olan sinüs sinyallerinin toplamıyla ifade edilebilir. Açıklama için basit bir örnek aşağıda verilmiştir.



**Complex Numbers :** A complex number is defined like this:

$$z = a + jb$$

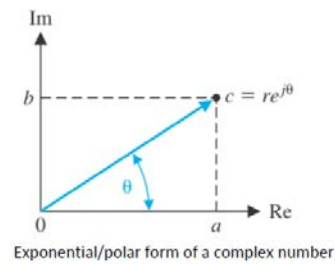
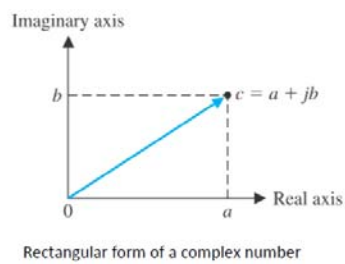
Where  $a$  is called the real part of and  $b$  is called the imaginary part of

$$\text{Re}(z) = a, \text{Im}(z) = b$$

$$j = \sqrt{-1}$$

You may also imagine numbers on exponential/polar form:

$$z = re^{j\theta} \quad r = |z| = \sqrt{a^2 + b^2} \quad \theta = \text{atan} \frac{b}{a}$$



**Decibels**

Note that the quantities  $\left| \frac{V_2}{V_1} \right|$ ,  $\left| \frac{I_2}{I_1} \right|$ , and  $\left| \frac{P_2}{P_1} \right|$  are unitless quantities.

However, when scaled logs of the quantities are taken, the unit of **decibels** (dB), is assigned.

$$\begin{aligned} 20 \log_{10} \left| \frac{V_2}{V_1} \right| &= \text{log-magnitude (LM) of the voltage gain in dB} \\ 20 \log_{10} \left| \frac{I_2}{I_1} \right| &= \text{log-magnitude (LM) of the current gain in dB} \\ 10 \log_{10} \left| \frac{P_2}{P_1} \right| &= \text{log-magnitude (LM) of the power gain in dB} \end{aligned}$$

In terms of dB (decibels)  $\text{Gain}_{dB} = 20 \log_{10} \text{Gain}$

$$20 \log(K \times 10) = 20 \log K + 20$$

$$20 \log(K \times 10^n) = 20 \log K + 20n$$

$$20 \log K = -20 \log \frac{1}{K}$$

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**FREQUENCY RESPONSE FROM TRANSFER FUNCTIONS**

by knowing the **transfer function  $G(s)$  of a linear system, the magnitude characteristics,  $G(j\omega)$ , and the phase characteristics,  $\angle G(j\omega)$ , completely describe the steady state performance for sinusoidal input.** the amplitude and phase characteristics of a closed-loop system can be used to predict the time-domain transient performance characteristics such as overshoot, rise time and the steady-state performance

**Frequency Response of Closed-Loop Systems**

**For a** The closed loop transfer function of system

$$M(s) = \frac{Y(s)}{R(s)} = \frac{G(s)}{1 + G(s)H(s)}$$

Under the sinusoidal steady state,  $s = j\omega$

$$M(j\omega) = \frac{Y(j\omega)}{R(j\omega)} = \frac{G(j\omega)}{1 + G(j\omega)H(j\omega)}$$

In terms of its magnitude and phase, the sinusoidal steady-state transfer function  $M(j\omega)$  may be expressed as:

The magnitude of  $M(j\omega)$  is

$$M(j\omega) = |M(j\omega)| \angle M(j\omega) \quad |M(j\omega)| = \left| \frac{G(j\omega)}{1 + G(j\omega)H(j\omega)} \right| = \frac{|G(j\omega)|}{|1 + G(j\omega)H(j\omega)|}$$

$$M(j\omega) = \text{Re}[M(j\omega)] + j\text{Im}[M(j\omega)]$$

$$\angle M(j\omega) = \phi_M(j\omega) = \angle G(j\omega) - \angle [1 + G(j\omega)H(j\omega)]$$

**The Laplace transform** of the output of a linear single-input single-output system with transfer function  $G(s)$  can be expressed in terms of the input from the definition of transfer function as:  $Y(s) = G(s)R(s)$  (1)

We know, in general that  $s = \sigma + j\omega$ . However, for sinusoidal steady-state analysis, we shall replace  $s$  by its imaginary component  $j\omega$  only, since in the steady state, the contribution of the real part will disappear for a stable system. So with  $s = j\omega$ , the Equation (1) becomes  $Y(j\omega) = G(j\omega)R(j\omega)$  (2)

By writing the functions  $G(j\omega)$  and  $R(j\omega)$  in terms of their amplitudes and phases, respectively, as  $Y(j\omega) = |G(j\omega)| \angle G(j\omega)$  (3) and  $R(j\omega) = |R(j\omega)| \angle R(j\omega)$

Amplitude relation between input and output of the linear system

$$|Y(j\omega)| = |G(j\omega)| |R(j\omega)|$$

and the phase relation  $\angle Y(j\omega) = \angle G(j\omega) + \angle R(j\omega)$

**For :** input with amplitude  $R$  and frequency  $\omega_0$  given by  $r(t) = R \sin \omega_0 t$

**The output is**  $y(t) = Y \sin(\omega_0 t + \phi)$ , these relation can be defined as:

the amplitude of the output sinusoid is

$$Y = R |G(j\omega_0)|$$

and the phase of the output is

$$\phi = \angle G(j\omega_0)$$

### The shortcut method consists of the following steps:

**Step 1.** Set  $s=j\omega$  in  $G(s)$  to obtain

**Step 2.** Rationalize  $G(j\omega)$ ; We want to express it in the form.

$$G(j\omega) = R + jI$$

where  $R$  and  $I$  are functions of  $\omega$ . Simplify  $G(j\omega)$  by multiplying the numerator and denominator by the complex conjugate of the denominator.

**Step 3.** The amplitude ratio and phase angle of  $G(s)$  are given by:

$\Rightarrow$

$$\begin{aligned} AR &= \sqrt{R^2 + I^2} \\ \phi &= \tan^{-1}(R/I) \end{aligned}$$

**Example :** Find the frequency response of a first-order system, with

$$G(s) = \frac{1}{\tau s + 1} \quad (13-16)$$

**Solution**

First, substitute  $s = j\omega$  in the transfer function

$$G(j\omega) = \frac{1}{\tau j\omega + 1} = \frac{1}{j\omega\tau + 1} \quad (13-17)$$

Then multiply both numerator and denominator by the complex conjugate of the denominator, that is,

$$-j\omega\tau + 1$$

$$\begin{aligned} G(j\omega) &= \frac{-j\omega\tau + 1}{(j\omega\tau + 1)(-j\omega\tau + 1)} = \frac{-j\omega\tau + 1}{\omega^2\tau^2 + 1} \\ &= \frac{1}{\omega^2\tau^2 + 1} + j \frac{(-\omega\tau)}{\omega^2\tau^2 + 1} = R + jI \end{aligned} \quad (13-18)$$

**Example :** Consider the transfer function:

$$G(s) = \frac{3}{s + 2}$$

The frequency response of this system is obtained by setting  $s = j\omega$ , such that

$$G(j\omega) = \frac{3}{j\omega + 2} \quad |G(j\omega)| = \frac{3}{(\omega^2 + 4)^{1/2}} \quad \text{and} \quad \angle G(j\omega) = -\tan^{-1} \left[ \frac{\omega}{2} \right]$$

**Example:** Consider the system shown in Figure .The transfer function  $G(s)$  is

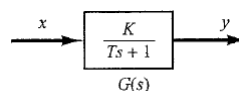
$$G(s) = \frac{K}{Ts + 1}$$

For the sinusoidal input  $x(t) = X \sin \omega t$ , the steady-state output  $y(t)$  can be found as follows: Substituting  $j\omega$  for  $s$  in  $G(s)$  yields

The amplitude ratio

$$G(j\omega) = \frac{K}{jT\omega + 1}$$

$$|G(j\omega)| = \frac{K}{\sqrt{1 + T^2\omega^2}}$$



while the phase angle

$$\phi = \angle G(j\omega) = -\tan^{-1} T\omega$$

The output

$$y_{ss}(t) = \frac{XK}{\sqrt{1 + T^2\omega^2}} \sin(\omega t - \tan^{-1} T\omega)$$

**Example:** Consider the system shown in Figure .

The transfer function  $G(s)$  is

$$G(s) = \frac{s + \frac{1}{T_1}}{s + \frac{1}{T_2}}$$

For the sinusoidal input  $x(t) = X \sin \omega t$ , the steady-state output  $y(t)$  can be found as follows: Substituting  $j\omega$  for  $s$  in  $G(s)$  yields

$$G(j\omega) = \frac{j\omega + \frac{1}{T_1}}{j\omega + \frac{1}{T_2}} = \frac{T_2(1 + T_1j\omega)}{T_1(1 + T_2j\omega)} \quad |G(j\omega)| = \frac{T_2\sqrt{1 + T_1^2\omega^2}}{T_1\sqrt{1 + T_2^2\omega^2}}$$

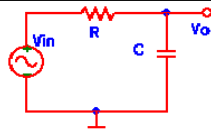
while the phase angle

$$\phi = \angle G(j\omega) = \tan^{-1} T_1 \omega - \tan^{-1} T_2 \omega$$

output is

$$y_{ss}(t) = \frac{XT_2\sqrt{1 + T_1^2\omega^2}}{T_1\sqrt{1 + T_2^2\omega^2}} \sin(\omega t + \tan^{-1} T_1 \omega - \tan^{-1} T_2 \omega)$$

Example:



$$H(s) = \frac{V_o(s)}{V_i(s)} = \frac{1}{1 + sRC}$$

Consider the case where  $R=1$  and  $C=0.1$ . In that case:

$$H(s) = \frac{V_o(s)}{V_i(s)} = \frac{1}{1 + 0.1 \cdot s} = \frac{1}{1 + j \cdot 0.1 \cdot \omega}$$

$$V_o(s) = V_i(s) \cdot H(s) = V_i(s) \cdot \frac{1}{1 + 0.1 \cdot s} = V_i(j\omega) \cdot \frac{1}{1 + j \cdot 0.1 \cdot \omega}$$

If we have a phasor representation for the input and the transfer function, the multiplication is simple (multiply magnitudes and add phases). Finding the output becomes easy. Let's look at some examples:

$$V_i = 3 \sin(30 \cdot t) = 3 \angle 0^\circ$$

The output is just the product of the input and the transfer function (evaluated as phasors)

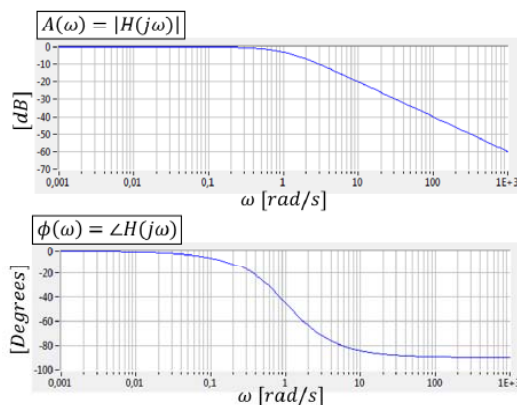
$$\frac{1}{1 + j \cdot 0.1 \cdot \omega} = \frac{1}{1 + j \cdot 0.1 \cdot 30} = \frac{1}{1 + 3j} = (0.1 - 0.3j) = 0.316 \angle -71.6^\circ$$

$$V_o = V_i \cdot H = (3 \angle 0^\circ) \cdot (0.316 \angle -71.6^\circ) = 0.95 \angle -71.6^\circ = 0.95 \sin(30t - 71.6^\circ)$$

### The Bode diagram

When an input signal, whose magnitude varies sinusoidally with respect to time, is applied to a linear system which is stable, the output of the system will itself be a sinusoid in the steady-state, although its magnitude will be some multiple of the input signal and there may be a phase shift. If the magnitude of the input signal is held constant and a different frequency of signal is chosen, then another output magnitude and corresponding phase shift will be found. For a constant magnitude input signal the resultant output magnitude or gain can therefore be found for a wide range of frequency values and a plot made of the different values, a similar procedure being carried out to form a separate plot of the corresponding phase shifts. A system transfer function can therefore be characterized in terms of its frequency response considered in the form of two plots, one of gain-v-frequency and the other of phase-v-frequency, noting the employment of gain, which is not dependent on input signal magnitude, rather than output magnitude which is.

**(Bode plot:)** The Bode plot is a **graphical representation of the frequency response** in the semilog Cartesian co-ordinate system. It comprises of two plots,  
 (i) **Gain plot: The gain (commonly plotted in units of dB) of the system versus frequency (semi-log scale)**  
 (ii) **a Phase plot: The phase shift induced by the system versus frequency (axis is linear and usually has units of degrees)**

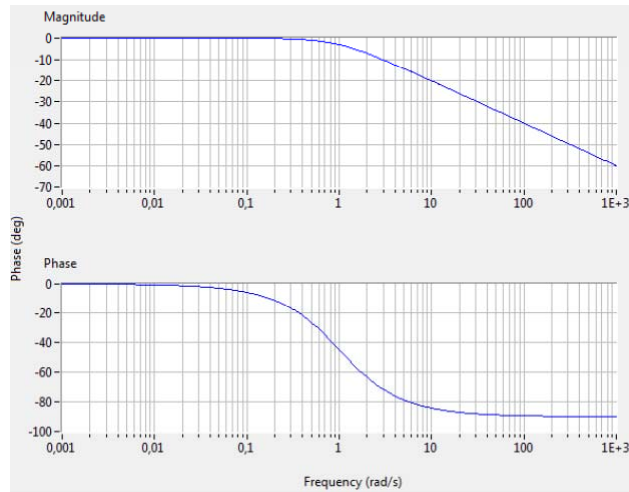


The  $A(\omega)$ -axis is in decibel (dB), where the decibel value of  $x$  is calculated as:  $x[\text{dB}] = 20\log_{10}x$

The  $\phi(\omega)$ -axis is in degrees (not radians!). We use logarithmic scale on the  $x$ -axes.

Example:

$$H(s) = \frac{y(s)}{u(s)} = \frac{1}{s + 1}$$



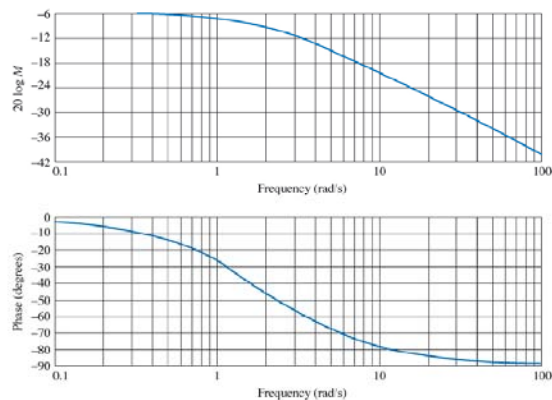
### Magnitude and phase plots (Bode plots):

We are often interested in displaying the frequency response over a very broad range of  $\omega$ . The magnitude itself will vary greatly in this range. For this reason we plot the magnitude in decibels and the frequency on a logarithmic scale. Recall that decibels are dened as follows for units of amplitude:  **$\text{dB}(x) = 20 \log_{10}(x)$**

For the phase plot we will display the phase directly on the vertical axis.

The following are the  
magnitude and phase  
plots for

$$G(s) = 1/(s + 2),$$



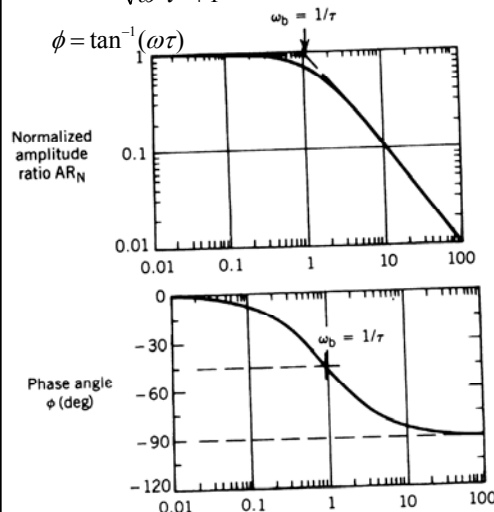
**First-order system**

– Process  $G(s) = \frac{K}{\tau s + 1}$

– Frequency response

$$AR_N = \frac{1}{\sqrt{\omega^2 \tau^2 + 1}}$$

$$\phi = \tan^{-1}(\omega \tau)$$



Bode diagram for a first-order process.

• **Properties**

– At low frequency, as  $\omega \rightarrow 0$

$$AR_N \approx 1 \quad (AR \approx K)$$

$$\phi \approx 0$$

– At high frequency, as  $\omega \rightarrow \infty$

$$AR_N \approx 1/\omega \tau, \quad \phi \approx -90^\circ$$

**Note)**  $\log AR_N = \log 1 - \log \omega \tau$   
 $= -\log \omega \tau$

▮ slope of asymptote is -1

• **Break frequency ( or corner)**

$$\omega = \omega_b = 1/\tau$$

$$AR_N(\omega = \omega_b) = \frac{1}{\sqrt{1+1}} = 0.707$$

$$\phi(\omega = \omega_b) = \tan^{-1}(1) = 45^\circ$$

• **Procedure** for constructing Bode diagram of first-order system.

**Step 1.** Locate the **break frequency**  $\omega_b = 1/\tau$ . AR at this point is  $0.707 K$ .

**Step 2.** Draw a horizontal asymptote ( $AR = K$ ) for low-frequency region,  $\omega \leq \omega_b$ .

**Step 3.** Draw the high-frequency asymptote with slope -1, intersecting with  $AR = K$  at  $\omega = \omega_b$ .

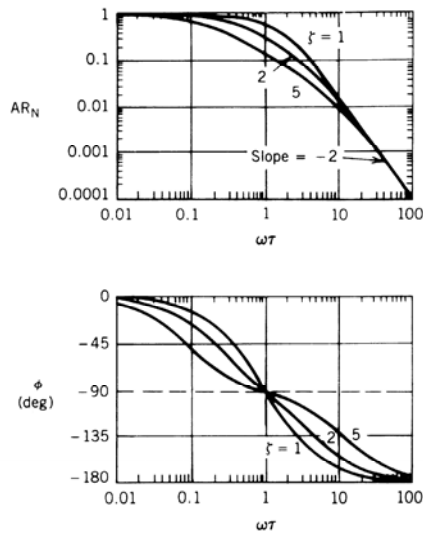
**Step 4.** Sketch the actual frequency response, using the shape shown in Figure 14.2.

**Step 5.** At  $\omega = \omega_b$ ,  $\phi = -45^\circ$ . Locate this point and draw horizontal asymptotes at  $\phi = 0^\circ$  and  $-90^\circ$ . Sketch the phase angle curve using Figure 14.2 as a pattern or employing the data points from Table 14.1.

Table 14.1. Phase Angles as a Function of  $\omega \tau$ , First-order System

| $\omega \tau$ (radians) | $\phi$ (°) |
|-------------------------|------------|
| 0.01                    | -0.6       |
| 0.1                     | -5.7       |
| 0.5                     | -26.6      |
| 1.0                     | -45.0      |
| 2.0                     | -63.4      |
| 10.0                    | -84.3      |
| 100.0                   | -89.4      |

### Second-order system



Bode diagram for critically damped and overdamped second-order processes

– Process  $G(s) = \frac{K}{\tau^2 s^2 + 2\zeta\tau s + 1}$

– Frequency response

$$AR = \frac{K}{\sqrt{(1 - \omega^2 \tau^2)^2 + (2\omega\tau\zeta)^2}}$$

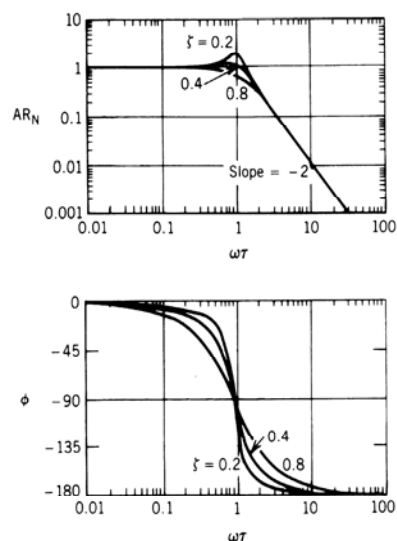
$$\phi = \tan^{-1} \left[ \frac{-2\omega\tau\zeta}{1 - \omega^2 \tau^2} \right]$$

– At low frequency, as  $\omega \rightarrow 0$   
 $AR \approx K$ ,  $\phi \approx 0$

– At high frequency, as  $\omega \rightarrow \infty$   
 $AR_N \approx 1/(\omega\tau)^2$ ,  $\phi \approx -180^\circ$

**Note)**  $\log AR_N = \log 1 - 2 \log \omega\tau$   
 $= -2 \log \omega\tau$

▣ slope of asymptote is -2



Bode diagram for underdamped second-order processes

▣ **For overdamped systems**, the normalized amplitude ratio attenuated ( $\hat{A}/KA < 1$ ) for all  $\omega$ .

▣ **For underdamped systems**, the amplitude ratio plot exhibits a **maximum** ( $0 < \zeta < \sqrt{2}/2$ ) at the **resonant frequency**.

$$\frac{d(AR_N)}{d\omega} = 0 \text{ at } \omega = \omega_r$$

$$\omega_r^2 = \frac{1 - 2\zeta^2}{\tau^2} \Rightarrow \omega_r = \frac{\sqrt{1 - 2\zeta^2}}{\tau}$$

To exist  $\omega_r$ ,  $1 - 2\zeta^2 > 0 \Rightarrow 0 < \zeta < \sqrt{2}/2$

$$(AR_N)_{\max} = \frac{1}{2\zeta\sqrt{1 - \zeta^2}}$$

❖ Feedback controllers are tuned to give a slight amount of resonant so as to speed up the controlled system response.

•**Procedure** for constructing AR plot of overdamped second-order system.

**Step 1.** Draw the low-frequency asymptote at  $K$ .

**Step 2.** At  $\omega_{b1} = 1/\tau_1$  (is the larger time constant) the slope of the asymptote changes to -1.

**Step 3.** At  $\omega_{b2} = 1/\tau_2$  the slope of the asymptote changes to -2.

**Step 4.** Calculate several points on the actual curve and sketch it using the asymptotic representations as guidelines.

•**Procedure** for constructing AR plot of underdamped second-order system.

**Step 1.** Draw the low-frequency asymptote at  $K$ .

**Step 2.** At  $\omega_{b3} = 1/\tau$  the slope of the asymptote changes to -2.

**Step 4.** Calculate several points on the actual curve and sketch it using the asymptotic representations as guidelines. The maximum value in AR will be observed only if  $0 < \zeta < \sqrt{2}/2$

### Frequency-Domain Specifications

In the design of linear control systems using the frequency-domain methods, the performance of the system can be properly assessed in terms of some specifications.

**(a) Resonance Peak  $M_r$ :** This is the maximum value of the amplitude  $M$  and it indicates a measure of **relative stability**. The large value of  $M_r$  is associated with **large overshoot in transient input response**. Recommended value of  $M_r$  is in the range of 1.1 to 1.5 for a closed loop system.

**(b) Resonant Frequency  $\omega_r$ :** The resonant frequency  $\omega_r$  is defined as the frequency at which the peak amplitude  $M_r$  occurs.

**(c) Bandwidth:** The bandwidth (BW) is defined as the frequency at which the magnitude of  $|M(j\omega)|$  drops to 70.7 percent of its **zero-frequency value**, or 3 dB down from the zero frequency value. In general, the bandwidth of a control system gives a measure of the transient response properties, in that **a large bandwidth corresponds to a faster rise time**, since higher frequency signals are more easily passed on to the outputs. Conversely, if the bandwidth is small, signals of relatively low frequencies only can pass to the output which makes the time response sluggish.

**(d) Cutoff Rate:** Sometimes it may be necessary to specify the cutoff rate of the frequency response, which is the slope of  $M(j\omega)$  at the high frequency end of its spectrum. Apparently, two systems can have the same bandwidth, but the cutoff rates of the frequency responses may be different.

The Bode diagram of a typical second order system is shown in the diagram below. Using the magnitude plot,

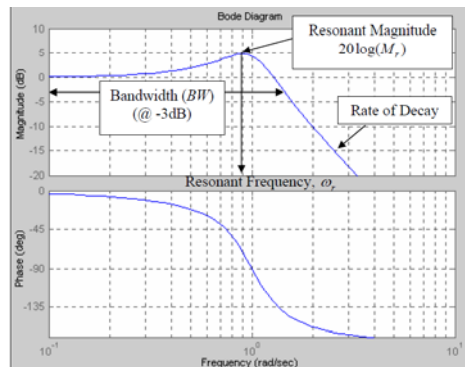
- the resonant frequency ( $\omega_r$ )
- the resonant magnitude ( $M_r$ ),
- the bandwidth (BW), and the rate of decay of the system can be identified.

### Bandwidth (BW)

The bandwidth (BW) is defined as the frequency at which the system is 3dB down from its low frequency value.

Note that -3dB represents an amplitude of 0.71, so the system responds at 71% of its low frequency value.

Using the magnitude plot, the resonant frequency ( $\omega_r$ ) the resonant magnitude ( $M_r$ ), the bandwidth (BW), and the rate of decay of the system can be identified.



If a system is second order and has a damping ratio  $\zeta < 0.707$ , the resonant frequency and resonant magnitude may be estimated using the equations,

Resonant Frequency  $\omega_r = \omega_n \sqrt{1 - 2\zeta^2}$

the resonant magnitude  $M_r = (2\zeta \sqrt{1 - \zeta^2})^{-1}$

These characteristics in the frequency-domain correlate with behavior of the system in the time-domain. The resonant magnitude  $M_r$  gives an indication of the relative stability.

Large values of  $M_r$  are indicative of low damping, suggesting oscillatory response with large overshoots. Systems with large bandwidths have faster response than systems with small bandwidths; however, they may be more sensitive to noise.

Sensitivity to noise is determined by a combination of the bandwidth and the rate of decay of the magnitude at high frequencies.

• **Amplitude ratio AR** ; AR

**14.3.5 Pure integrator**

$$G(s) = \frac{1}{s}, \quad G(j\omega) = \frac{1}{j\omega} = -j \frac{1}{\omega}$$

$$AR = \frac{\hat{A}}{A} = \frac{K}{\sqrt{\omega^2 \tau^2 + 1}}$$

$$AR = |G(j\omega)| = \sqrt{0^2 + \left(-\frac{1}{\omega}\right)^2} = \frac{1}{\omega}$$

• **Phase shift (or angle)**;  $\phi$

$$\phi = \angle G(j\omega) = \tan^{-1}\left(-\frac{1/\omega}{0}\right) = -90^\circ$$

$$\phi = -\tan^{-1}(\omega\tau)$$

**14.3.6 Pure differentiator**

$$G(s) = s, \quad G(j\omega) = j\omega$$

• **Normalized amplitude ratio AR<sub>N</sub>**

$$AR = \omega$$

$$AR_N = \frac{1}{\sqrt{\omega^2 \tau^2 + 1}}$$

$$\phi = \angle G(j\omega) = \tan^{-1}\left(\frac{\omega}{0}\right) = 90^\circ$$

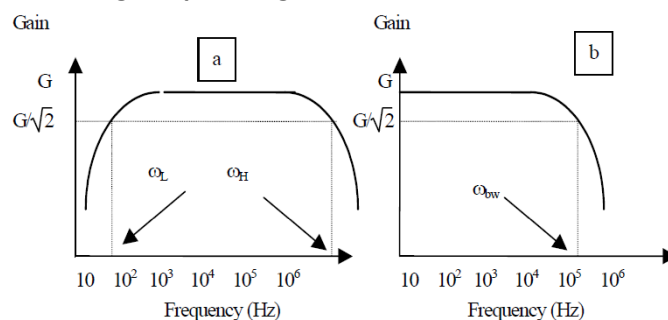
**Special Frequency Points**

• **Gain Crossover frequency**  $\omega_{gco}$  = frequency at which the gain crosses 0 dB

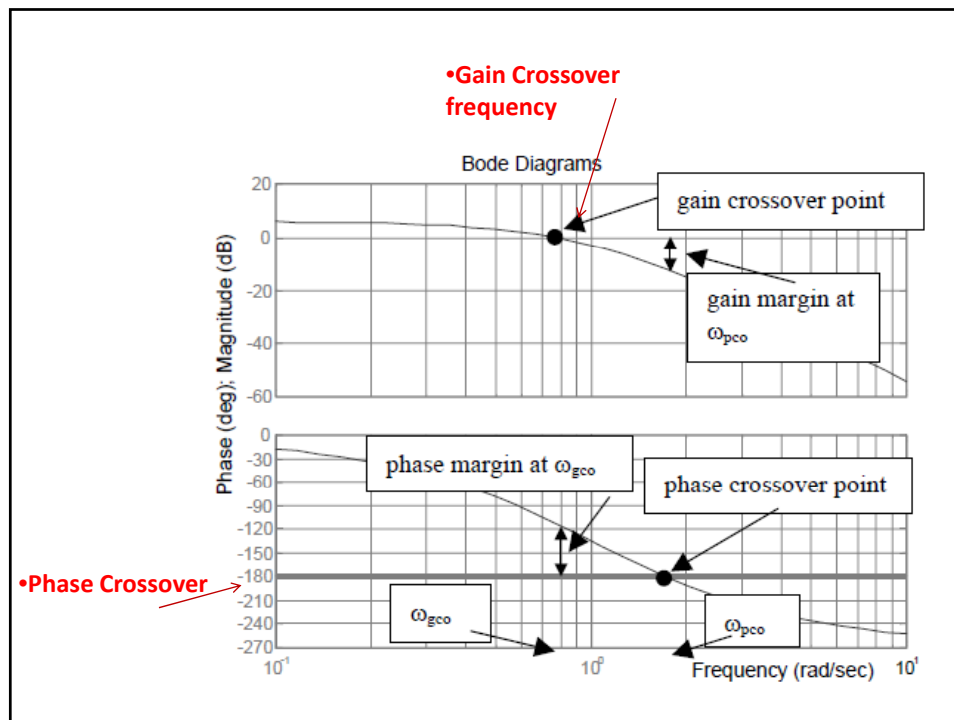
• **Phase Crossover**

**frequency**  $\omega_{pco}$  = frequency at which the phase crosses '-180°'

**Bandwidth** The signal bandwidth is defined for (i) those signals with a midband range or peak, Figure



(a) (ii) those signals with a constant d.c. gain, Figure (b). It is defined as the range of frequencies over which the gain is greater than  $0.707 G$ . Since we use often use log scales for gain, we note that the value of **0.707 = -3dB**.



**Minimum Phase Systems:** The loop transfer function  $GH(s)$  of a minimum phase system has no zeros or poles in the right half of the s-plane.

If a system has poles or zeros in the right half plane, it is referred to as a **non-minimum phase system**.

Gain and Phase Margins and the Bode Diagram

**Gain and phase margins of a minimum phase system are determined by plotting the Bode diagram of the loop transfer function  $GH(s)$ .** For example, consider the Bode diagram of the loop transfer function

$$GM = 0\text{dB} - G(\omega_{pco}).$$

**Phase margin (PM) is the additional phase lag required to make the phase angle  $-180$  (deg) at the frequency where the magnitude of the system crosses the zero-dB line.**  $PM = \phi(\omega_{gco}) - (-180 \text{ degree})$ .

The **gain and phase margins determine the stability of minimum phase systems. A minimum phase system is stable if both margins are positive, and unstable if they are negative.**

o Systems with a **higher degree of stability have larger margins and less stable systems** have smaller margins. The Bode diagram above represents a **stable closed-loop system**.

If the PM is negative the system is unstable.; If the GM is negative the system is unstable

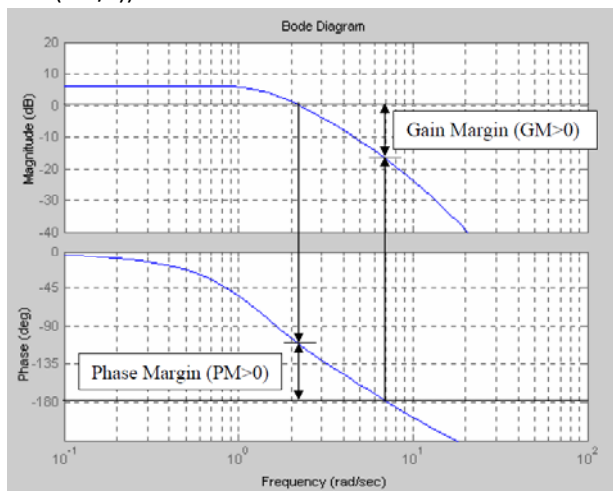
**Gain and phase margins of a minimum phase system are determined by plotting the Bode diagram of the loop transfer function  $GH(s)$ . For example, consider the Bode diagram of the loop transfer function.**

Example:

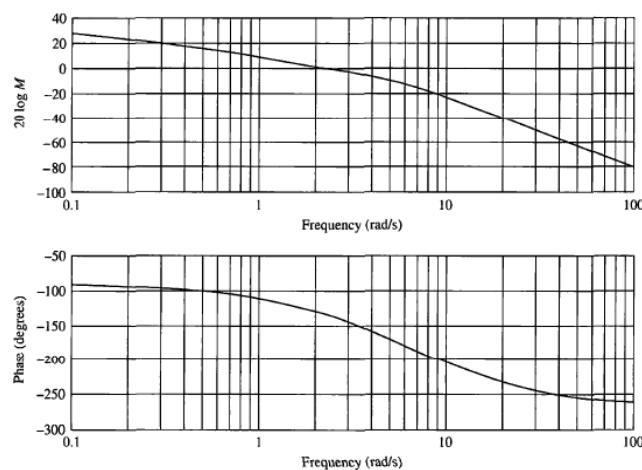
the **phase margin is**

$PM = +68$  (deg) (measured at 2.2 (rad/s)), and the **gain margin is  $GM = +17.4$  dB** (measured at 7.15 (rad/s))

$$GH(s) = \frac{100(s+2)}{(s+5)(s+10)(s^2+2s+2)}$$



**Find the gain margin, phase margin, zero dB frequency, 180° frequency,**



From the Bode plots: **Gain margin = 14.96 dB; phase margin = 49.570; 0 dB frequency = 2.152 rad/s; 180° frequency = 6.325 rad/s;**