

Power and RMS Values of Fourier Series

ECEN 2260 Supplementary Notes

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These notes treat the flow of energy in systems containing nonsinusoidal waveforms. Average power, rms values, and power factor are expressed in terms of the Fourier series of the voltage and current waveforms.

1. Average power

Let us consider the transmission of energy from a source to a load, through a given surface as in Fig. 1. In the network of Fig. 1, the voltage waveform $v(t)$ (not necessarily sinusoidal) is given by the source, and the current waveform is determined by the response of the load. In the more general case in which the source output impedance is significant, then $v(t)$ and $i(t)$ both depend on the characteristics of the source and load.

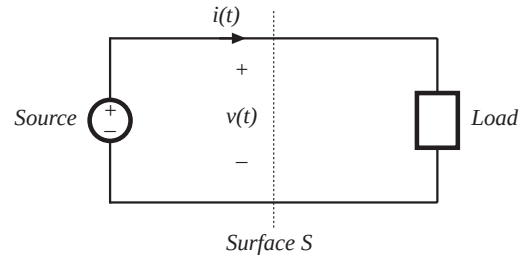


Fig. 1. Observe the transmission of energy through surface S.

If $v(t)$ and $i(t)$ are periodic, then they may be expressed as Fourier series:

$$v(t) = V_0 + \sum_{n=1}^{\infty} V_n \cos(n\omega t - \phi_n) \quad (1)$$

$$i(t) = I_0 + \sum_{n=1}^{\infty} I_n \cos(n\omega t - \theta_n)$$

where the period of the ac line voltage waveform is defined as $T = 2\pi/\omega$. In general, the instantaneous power $p(t) = v(t) i(t)$ can assume both positive and negative values at various points during the ac line cycle. Energy then flows in both directions between the source and load. It is of interest to determine the net energy transmitted to the load over one cycle, or

$$W_{\text{cycle}} = \int_0^T v(t) i(t) dt \quad (2)$$

This is directly related to the average power as follows:

$$P_{\text{av}} = \frac{W_{\text{cycle}}}{T} = \frac{1}{T} \int_0^T v(t) i(t) dt \quad (3)$$

Let us investigate the relationship between the harmonic content of the voltage and current waveforms, and the average power. Substitution of the Fourier series, Eq. (1), into Eq. (3) yields

$$P_{av} = \frac{1}{T} \int_0^T \left(V_0 + \sum_{n=1}^{\infty} V_n \cos(n\omega t - \phi_n) \right) \left(I_0 + \sum_{n=1}^{\infty} I_n \cos(n\omega t - \theta_n) \right) dt \quad (4)$$

To evaluate this integral, we must multiply out the infinite series. It can be shown that the integrals of cross-product terms are zero, and the only contributions to the integral comes from the products of voltage and current harmonics of the same frequency:

$$\int_0^T \left(V_n \cos(n\omega t - \phi_n) \right) \left(I_m \cos(m\omega t - \theta_m) \right) dt = \begin{cases} 0 & \text{if } n \neq m \\ \frac{V_n I_n}{2} \cos(\phi_n - \theta_n) & \text{if } n = m \end{cases} \quad (5)$$

The average power is therefore

$$P_{av} = V_0 I_0 + \sum_{n=1}^{\infty} \frac{V_n I_n}{2} \cos(\phi_n - \theta_n) \quad (6)$$

So net energy is transmitted to the load only when the Fourier series of $v(t)$ and $i(t)$ contain terms at the same frequency. For example, if $v(t)$ and $i(t)$ both contain third harmonic, then

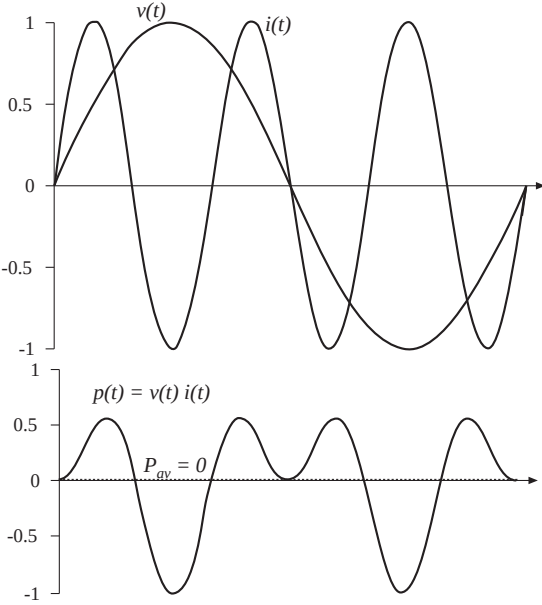


Fig. 2. Voltage, current, and instantaneous power waveforms, example 1. The voltage contains only fundamental, and the current contains only third harmonic. The average power is zero.

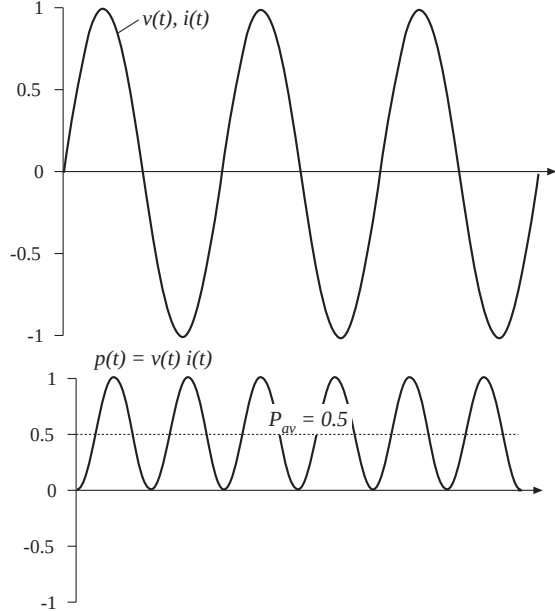


Fig. 3. Voltage, current, and instantaneous power waveforms, example 2. The voltage and current each contain only third harmonic, and are in phase. Net energy is transmitted at the third harmonic frequency.

net energy is transmitted at the third harmonic frequency, with average power equal to

$$\frac{V_3 I_3}{2} \cos(\phi_3 - \theta_3) \quad (7)$$

Here, $V_3 I_3 / 2$ is equal to the rms volt-amperes of the third harmonic current and voltage. The $\cos(\phi_3 - \theta_3)$ term is a displacement term which accounts for the phase difference between the third harmonic voltage and current.

Some examples of power flow in systems containing harmonics are illustrated in Figs. 2 to 4. In example 1, Fig. 2, the voltage contains fundamental only, while the current contains third harmonic only. It can be seen that the instantaneous power waveform $p(t)$ has a zero average value, and hence P_{av} is zero. Energy circulates between the source and load, but over one cycle the net energy transferred to the load is zero. In example 2, Fig. 3, the voltage and current each contain only third harmonic. The average power is given by Eq. (7) in this case.

In example 3, Fig. 4, the voltage waveform contains fundamental, third harmonic, and fifth harmonic, while the current contains fundamental, fifth harmonic, and seventh harmonic, as follows:

$$\begin{aligned} v(t) &= 1.2 \cos(\omega t) + 0.33 \cos(3\omega t) + 0.2 \cos(5\omega t) \\ i(t) &= 0.6 \cos(\omega t + 30^\circ) + 0.1 \cos(5\omega t + 45^\circ) + 0.1 \cos(7\omega t + 60^\circ) \end{aligned} \quad (8)$$

Average power is transmitted at the fundamental and fifth harmonic frequencies, since only these frequencies are present in both waveforms. The average power is found by evaluation of Eq. (6); all terms are zero except for the fundamental and fifth harmonic terms, as follows:

$$P_{av} = \frac{(1.2)(0.6)}{2} \cos(30^\circ) + \frac{(0.2)(0.1)}{2} \cos(45^\circ) = 0.32 \quad (9)$$

The instantaneous power and its average are illustrated in Fig. 4(b).

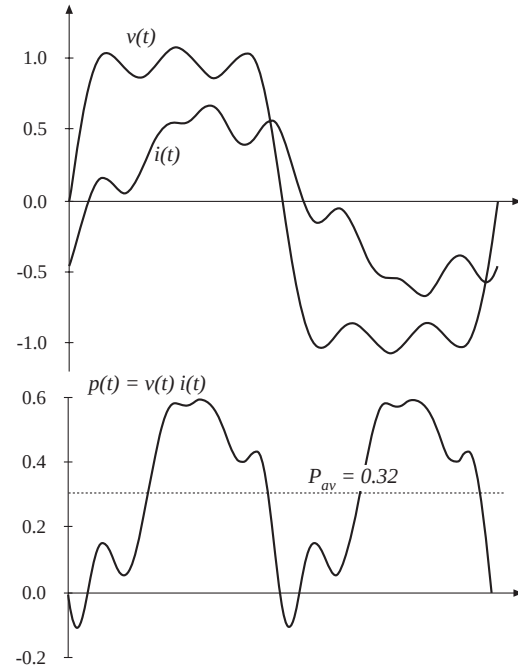


Fig. 15.4. Voltage, current, and instantaneous power waveforms, example 3. The voltage contains fundamental, third, and fifth harmonics. The current contains fundamental, fifth, and seventh harmonics. Net energy is transmitted at the fundamental and fifth harmonic frequencies.

2. Root-mean-square (rms) value of a waveform

The rms value of a periodic waveform $v(t)$ with period T is defined as

$$(\text{rms value}) = \sqrt{\frac{1}{T} \int_0^T v^2(t) dt} \quad (10)$$

The rms value can also be expressed in terms of the Fourier components. Insertion of Eq. (1) into Eq. (10), and simplification using Eq. (5), yields

$$(\text{rms value}) = \sqrt{V_0^2 + \sum_{n=1}^{\infty} \frac{V_n^2}{2}} \quad (11)$$

Again, the integrals of the cross-product terms are zero. This expression holds when the waveform is a current:

$$(\text{rms current}) = \sqrt{I_0^2 + \sum_{n=1}^{\infty} \frac{I_n^2}{2}} \quad (12)$$

Thus, the presence of harmonics in a waveform always increases its rms value. In particular, in the case where the voltage $v(t)$ contains only fundamental while the current $i(t)$ contains harmonics, then the harmonics increase the rms value of the current while leaving the average power unchanged. This is undesirable, because the harmonics do not lead to net delivery of energy to the load, yet they increase the $I_{rms}^2 R$ losses in the system.

3. Power factor

Power factor is a figure of merit which measures how effectively energy is transmitted between a source and load network. It is measured at a given surface as in Fig. 15.1, and is defined as

$$\text{power factor} = \frac{(\text{average power})}{(\text{rms voltage})(\text{rms current})} \quad (13)$$

The power factor always has a value between zero and one. The ideal case, unity power factor, occurs for a load which obeys Ohm's Law. In this case, the voltage and current waveforms have the same shape, contain the same harmonic spectrum, and are in phase. For a given average power throughput, the rms current and voltage are minimized at maximum (unity) power factor, i.e., with a linear resistive load. In the case where the voltage contains no harmonics but the load is nonlinear and contains dynamics, then the power factor can be expressed as a product of two terms, one resulting from the phase shift of the fundamental component of the current, and the other resulting from the current harmonics.

3.1. Linear resistive load, nonsinusoidal voltage

In this case, the current harmonics are in phase with, and proportional to, the voltage harmonics. As a result, all harmonics result in the net transfer of energy to the load. The current harmonic magnitudes and phases are

$$I_n = V_n / R \quad (14)$$

$$\theta_n = \varphi_n \quad \text{so } \cos(\theta_n - \varphi_n) = 1 \quad (15)$$

The rms voltage is again

$$(rms \text{ voltage}) = \sqrt{V_0^2 + \sum_{n=1}^{\infty} \frac{V_n^2}{2}} \quad (16)$$

and the rms current is

$$\begin{aligned} (rms \text{ current}) &= \sqrt{I_0^2 + \sum_{n=1}^{\infty} \frac{I_n^2}{2}} = \sqrt{\frac{V_0^2}{R^2} + \sum_{n=1}^{\infty} \frac{V_n^2}{2R^2}} \\ &= \frac{1}{R} (rms \text{ voltage}) \end{aligned} \quad (17)$$

By use of Eq. (6), the average power is

$$\begin{aligned} P_{av} &= V_0 I_0 + \sum_{n=1}^{\infty} \frac{V_n I_n}{2} \cos(\phi_n - \theta_n) \\ &= \frac{V_0^2}{R} + \sum_{n=1}^{\infty} \frac{V_n^2}{2R} \\ &= \frac{1}{R} (rms \text{ voltage})^2 \end{aligned} \quad (18)$$

Insertion of Eqs. (17) and (18) into Eq. (13) then shows that the power factor is unity. Thus, if the load is linear and purely resistive, then the power factor is unity regardless of the harmonic content of $v(t)$. The harmonic content of the load current waveform $i(t)$ is identical to that of $v(t)$, and all harmonics result in the transfer of energy to the load.

3.2. Nonlinear dynamical load, sinusoidal voltage

If the voltage $v(t)$ contains no dc component or harmonics, so that $V_0 = V_2 = V_3 = \dots = 0$, then harmonics in $i(t)$ do not result in transmission of net energy to the load. The average power expression, Eq. (6), becomes

$$P_{av} = \frac{V_1 I_1}{2} \cos(\phi_1 - \theta_1) \quad (19)$$

However, the harmonics in $i(t)$ do affect the value of the rms current:

$$(rms \text{ current}) = \sqrt{I_0^2 + \sum_{n=1}^{\infty} \frac{I_n^2}{2}} \quad (20)$$

Hence, as in Example 1 (Fig. 2), harmonics cause the load to draw more rms current from the source, but not more average power. Increasing the current harmonics does not cause more energy to be transferred to the load, but does cause additional losses in series resistive elements R_{series} .

Also, the presence of load dynamics and reactive elements, which cause the phase of the fundamental components of the voltage and current to differ ($\theta_1 \neq \phi_1$) also reduces the power factor. The $\cos(\phi_1 - \theta_1)$ term in the average power Eq. (19) becomes less than unity. However, the rms value of the current, Eq. (20), does not depend on the phase. So shifting the phase of $i(t)$ with respect to $v(t)$ reduces the average power without changing the rms voltage or current, and hence the power factor is reduced.

By substituting Eqs. (19) and (20) into (13), the power factor for the sinusoidal voltage case can be written

$$\begin{aligned} (\text{power factor}) &= \left(\frac{\frac{I_1}{\sqrt{2}}}{\sqrt{I_0^2 + \sum_{n=1}^{\infty} \frac{I_n^2}{2}}} \right) (\cos(\phi_1 - \theta_1)) \\ &= (\text{distortion factor}) (\text{displacement factor}) \end{aligned} \quad (21)$$

So when the voltage contains no harmonics, then the power factor can be written as the product of two terms. The first, called the distortion factor, is the ratio of the rms fundamental component of the current to the total rms value of the current

$$(\text{distortion factor}) = \left(\frac{\frac{I_1}{\sqrt{2}}}{\sqrt{I_0^2 + \sum_{n=1}^{\infty} \frac{I_n^2}{2}}} \right) = \frac{(\text{rms fundamental current})}{(\text{rms current})} \quad (22)$$

The second term of Eq. (21) is called the displacement factor, and is the cosine of the angle between the fundamental components of the voltage and current waveforms.

The Total Harmonic Distortion (THD) is defined as the ratio of the rms value of the waveform not including the fundamental, to the rms fundamental magnitude. When no dc is present, this can be written:

$$(\text{THD}) = \frac{\sqrt{\sum_{n=2}^{\infty} I_n^2}}{I_1} \quad (23)$$

The total harmonic distortion and the distortion factor are closely related. Comparison of Eqs. (22) and (23), with $I_o = 0$, leads to

$$(\text{distortion factor}) = \frac{1}{\sqrt{1 + (\text{THD})^2}} \quad (24)$$

This equation is plotted in Fig. 5. The distortion factor of a waveform with a moderate amount of distortion is quite close to unity. For example, if the waveform contains third harmonic whose magnitude is ten percent of the fundamental, the distortion factor is 99.5%. Increasing the third harmonic to twenty percent decreases the distortion factor to 98%, and a thirty-three percent harmonic magnitude yields a distortion factor of 95%. So the power factor is not significantly degraded by the presence of harmonics unless the harmonics are quite large in magnitude.

An example of a case in which the distortion factor is much less than unity is the conventional peak detection rectifier of Fig. 6. In this circuit, the ac line current consists of short-duration current pulses occurring at the peak of the voltage waveform. The fundamental component of the line current is essentially in phase with the voltage, and the displacement factor is close to unity. However, the low-order current harmonics are quite large, close in magnitude to that of the fundamental—a typical current spectrum is given in Fig. 7. The displacement factor of peak detection rectifiers is usually in the range 55% to 65%. The resulting power factor is similar in value.

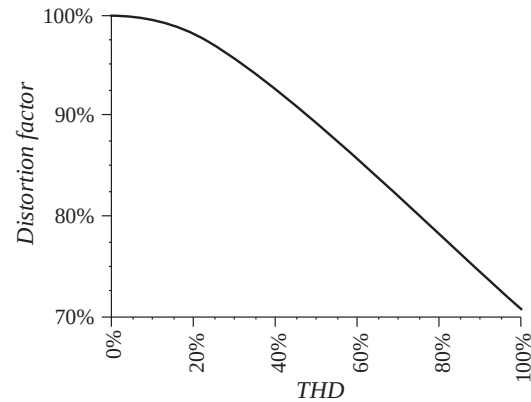


Fig. 5. Distortion factor vs. total harmonic distortion.

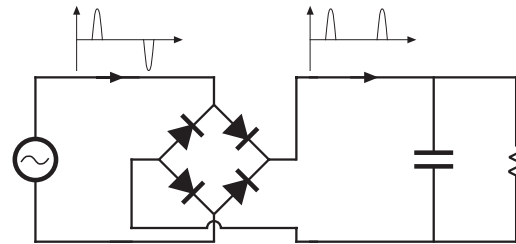


Fig. 6. Conventional peak detection rectifier.

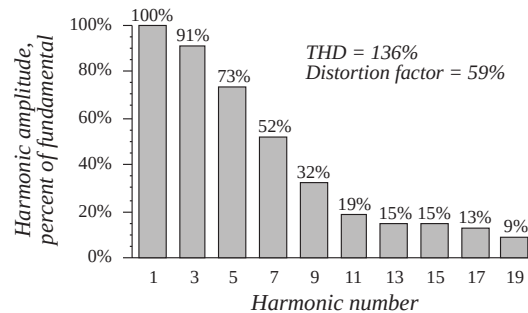


Fig. 7. Typical ac line current spectrum of a peak detection rectifier. Harmonics 1-19 are shown.

4. Power phasors in sinusoidal systems

The *apparent power* is defined as the product of the rms voltage and rms current. Apparent power is easily measured—it is simply the product of the readings of a voltmeter and ammeter placed in the circuit at the given surface. Many power system elements, such as transformers, must be rated according to the apparent power which they are able to supply. The unit of apparent power is the volt-ampere, or VA. The power factor, defined in Eq. (15), is the ratio of average power to apparent power.

In the case of sinusoidal voltage and current waveforms, we can additionally define the *complex power* $\mathbf{S}$ and the *reactive power* Q . If the sinusoidal voltage $v(t)$ and current $i(t)$ can be represented by the phasors $\mathbf{V}$ and $\mathbf{I}$, then the complex power is a phasor defined as

$$\mathbf{S} = \mathbf{V}\mathbf{I}^* = P + jQ \quad (25)$$

Here, $\mathbf{I}^*$ is the complex conjugate of $\mathbf{I}$, and j is the square root of -1 . The magnitude of $\mathbf{S}$, or $\|\mathbf{S}\|$, is equal to the apparent power, measured in VA. The real part of $\mathbf{S}$ is the average power P , having units of watts. The imaginary part of $\mathbf{S}$ is the reactive power Q , having units of reactive volt-amperes, or VAR's.

A phasor diagram illustrating $\mathbf{S}$, P , and Q , is given in Fig. 8. The angle $(\phi_1 - \theta_1)$ is the angle between the voltage phasor $\mathbf{V}$ and the current phasor $\mathbf{I}$. $(\phi_1 - \theta_1)$ is additionally the phase of the complex power $\mathbf{S}$. The power factor in the purely sinusoidal case is therefore

$$\text{power factor} = \frac{P}{\|\mathbf{S}\|} = \cos(\phi_1 - \theta_1) \quad (26)$$

It should be emphasized that this equation is valid only for systems in which the voltage and current are purely sinusoidal. The distortion factor of Eq. (22) then becomes unity, and the power factor is equal to the displacement factor as in Eq. (26).

The reactive power Q does not lead to net transmission of energy between the source and load. When reactive power is present, the rms current and apparent power are greater than the minimum amount necessary to transmit the average power P . In an inductor, the current lags the voltage by 90° , causing the displacement factor to be zero. The alternate storing and releasing of energy in an inductor leads to current flow and nonzero apparent power, but the average power P is zero. Just as resistors consume real (average) power P , inductors can be viewed as consumers of reactive power Q . In a capacitor, the current leads to voltage by 90° , again causing the displacement factor to be zero. Capacitors supply reactive power Q , and are commonly placed in the utility power distribution system near inductive loads. If the reactive power supplied by the capacitor is equal to the reactive power consumed by the inductor, then the net current (flowing from the source into the capacitor-inductive-load combination) will be in phase with the voltage, leading unity power factor and minimum rms current magnitude.

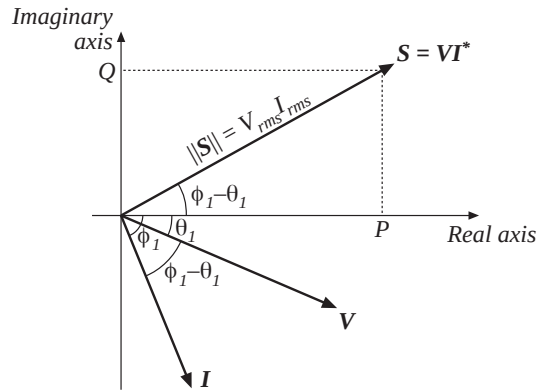


Fig. 8. Power phasor diagram, for a sinusoidal system, illustrating the voltage, current, and complex power phasors.

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