

DIFFERENTIAL EQUATIONS

- (1) If $A : \mathbb{R}^n \rightarrow \mathbb{R}^n$ is a linear operator and ϵ a real number, then

$$\det(I + \epsilon A) = 1 + \epsilon \operatorname{Tr} A + O(\epsilon^2)$$

as $\epsilon \rightarrow 0$.

- (2) If $A : \mathbb{R}^n \rightarrow \mathbb{R}^n$ is a linear operator, then

$$e^A = \lim_{n \rightarrow \infty} \left(I + \frac{A}{n}\right)^n$$

- (3) For any linear operator $A : \mathbb{R}^n \rightarrow \mathbb{R}^n$

$$\det e^A = e^{\operatorname{Tr} A}.$$

- (4) The flow of the linear equation

$$\dot{x} = Ax, \quad x \in \mathbb{R}^n$$

multiplies the volume of any figure by the factor $e^{t \operatorname{Tr} A}$.

- (5) Show that the flow generated by the Hamiltonian vector field

$$\dot{x} = H_y, \quad \dot{y} = -H_x, \quad (x, y) \in \mathbb{R}^n \times \mathbb{R}^n$$

is volume-preserving.

- (6) If $A \in L(\mathbb{R}^n)$ show

$$1) \text{ If } AB = BA \text{ then } Ae^B = e^B A.$$

$$2) \|e^{tA}\| \leq e^{|t|\|A\|}.$$

- (7) Prove that the eigenvalues of the operator e^A equal e^{λ_i} , where the λ_i are the eigenvalues of the operator A .

- (8) Show that the differential equation considered in a sufficiently small neighborhood of a nonsingular point x_0

$$\dot{x} = Ax, \quad x \in \mathbb{R}^n$$

is equivalent to the normal form

$$\dot{y}_i = 0, \quad i = 1, 2, \dots, n-1, \quad \dot{y}_n = 1.$$

- (9) Sketch the phase portraits of the planar system

$$\begin{aligned} \dot{x}_1 &= -x_2 + x_1(x_1^2 + x_2^2) \\ \dot{x}_2 &= x_1 + x_2(x_1^2 + x_2^2). \end{aligned}$$

Hint: Pass to polar coordinates.

- (10) Show that the non-linear change of coordinates

$$y_1 = x_1 + x_2^3, \quad y_2 = x_2 + x_2^2$$

satisfies the requirements of the flow box theorem for the system

$$\dot{x}_1 = -\frac{3x_2^2}{1+2x_2}, \quad \dot{x}_2 = \frac{1}{1+2x_2}$$

in the neighborhood of any point (x_1, x_2) with $x_2 \neq -1/2$. Find the trajectories of the system.

- (11) Find the flows of the following systems using a) Lie series b) solving differential equations:

$$a) \dot{x}_1 = x_1, \quad \dot{x}_2 = x_1 - x_2^2, \quad b) \dot{x}_1 = x_1 x_2, \quad \dot{x}_2 = x_2^2$$

- (12) Given the non-autonomous system

$$\dot{x} = \xi(x, t), \quad x(0) = x_0.$$

Introduce a new variable τ such that $\dot{\tau} = 1$, $\tau(0) = 0$ and bring it to the autonomous system

$$\dot{x} = \xi(x, \tau), \quad \dot{\tau} = 1, \quad x(0) = x_0, \quad \tau(0) = 0.$$

Apply to the initial value problem $\dot{x} = tx^2$, $x(0) = x_0$

- (13) Convert the second order equation

$$y'' - y + 2y^3 = 0, \quad y(0) = 1, \quad y'(0) = 0$$

to a system in $\mathbb{R}^2$ and use the Lie series to find its flow.

- (14) Show that $\bar{x}$ is a fixed point of the flow generated by a vector field $\mathbf{v}$ if and only if $\bar{x}$ is a singular point of $\mathbf{v}$.

- (15) Let the system $\dot{x} = A(t)x + b(t)$ on $(0, \infty)$ be given, where

$$A(t) = \begin{pmatrix} 0 & 1 \\ -\frac{2}{t^2} & \frac{2}{t} \end{pmatrix} \quad \text{and} \quad b(t) = \begin{pmatrix} t^4 \\ t^3 \end{pmatrix}.$$

Determine the solution for the initial value

$$x(2) = \begin{bmatrix} 1 \\ 4 \end{bmatrix}.$$

- (16) Show that a function $I : \mathbb{R}^n \rightarrow \mathbb{R}$ is invariant under a flow e^{tA} iff the Lie derivative of the function with respect to the vector field $\mathbf{v}$ generating the flow vanishes.

- (17) Show that the first integral of the system

$$\dot{x} = ax - by, \quad \dot{y} = bx + ay$$

is

$$I = (x^2 + y^2) \exp\left(-\frac{2b}{a} \arctan \frac{y}{x}\right).$$

Use its level curves to sketch the phase portraits of the system.

- (18) Let an operator $A : \mathbb{R}^n \rightarrow \mathbb{R}^n$ leave invariant a subspace $E \subset \mathbb{R}^n$ (that is, $Ax \in E$ for all $x \in E$). Show that e^A also leaves E invariant.

- (19) Show that the system of differential equations

$$\dot{x} = f\left(\frac{x}{t}, \frac{y}{t}\right), \quad \dot{y} = g\left(\frac{x}{t}, \frac{y}{t}\right)$$

has a first integral of the form

$$I = F\left(\frac{x}{t}, \frac{y}{t}\right) = \text{const.}$$

Determine a necessary and sufficient condition for the trajectories of this system to lie on spheres centered at the origin.

- (20) Determine if the linear system $\dot{x} = Ax$ has a saddle, node, focus or center at the origin and determine the stability of each node or focus:

$$a) \quad A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \quad b) \quad A = \begin{bmatrix} 1 & -1 \\ 2 & 3 \end{bmatrix}$$

$$c) \quad A = \begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix} \quad d) \quad A = \begin{bmatrix} \lambda & 2 \\ 1 & \lambda \end{bmatrix}$$

- (21) Solve the initial value problem $\dot{x} = Ax$, $x(0) = x_0$ with

$$A = \begin{bmatrix} 0 & -2 & 0 \\ 1 & 2 & 0 \\ 0 & 0 & -2 \end{bmatrix}.$$

- (22) Solve the initial value problem $\dot{x} = Ax$, $x(0) = x_0$ with the matrix

$$a) \quad A = \begin{bmatrix} 1 & -1 \\ 1 & 3 \end{bmatrix} \quad b) \quad A = \begin{bmatrix} 1 & 1 \\ 0 & -1 \end{bmatrix}.$$

- (23) Solve the initial value problem $\dot{x} = Ax$, $x(0) = x_0$ with the matrix

$$a) \quad \begin{bmatrix} 2 & 1 & 1 \\ 0 & 2 & 2 \\ 0 & 0 & 2 \end{bmatrix} \quad b) \quad A = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 2 & 0 \\ 1 & 0 & 2 \end{bmatrix}.$$

- (24) Find the Jordan Canonical forms for the following matrices

$$a) \quad \begin{bmatrix} 1 & 1 \\ 0 & -1 \end{bmatrix} \quad b) \quad \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} \quad c) \quad \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}.$$

- (25) Find the Jordan Canonical forms for the following matrices

$$a) \quad A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad b) \quad A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & -1 \end{bmatrix}.$$