

HOMEWORK PROBLEMS FOR PDEs: I

- (1) Solve $uu_x + u_y = 1$ with $u = 0$ when $y = x$. Something nasty happens if we replace the initial condition $u = 0$ by $u = 1$. What?
- (2) Show that the initial value problem

$$(y - x)u_x - (y + x)u_y = 0, \quad u(x, 0) = f(x), \quad x > 0$$

has no solution if $f(x)$ is an arbitrary function.

- (3) Given the initial value problem

$$yu_x + xu_y = cu, \quad u(x, x) = f(x), \quad x > 0$$

where c is a real constant, determine a condition on the function $f(x)$ so that this becomes a characteristic initial value problem. Supposing that $f(x)$ satisfies that condition, obtain a (nonunique) solution of the problem.

- (4)

$$\begin{cases} x \frac{\partial u}{\partial x} - (1 + x^4 + y^2) \frac{\partial u}{\partial y} = 2u \\ u|_{x=0} = 0 \end{cases}$$

$$\begin{cases} (2 + x^2 + y^2) \frac{\partial u}{\partial x} + y \frac{\partial u}{\partial y} = u \\ u|_{y=0} = 0 \end{cases}$$

$$\begin{cases} x \frac{\partial u}{\partial x} + (1 + 2u^4 + y^2) \frac{\partial u}{\partial y} = 3u \\ u|_{x=0} = 0 \end{cases}$$

- a) Does there exist an unbounded solution defined on the whole plane?
- b) Is the value of u bounded along the characteristics?
- c) Does each characteristic intersect the surface $y = x + u^2$?
- d) Does there exist a first integral of the equation of characteristics whose derivative with respect to y at the origin is equal to 1? Find the derivative of this derivative along the characteristic vector?

- (5) Does the Cauchy problem

$$x(x^2 + y^2) \frac{\partial u}{\partial x} + y^3 \frac{\partial u}{\partial y} = 0, \quad u|_{y=0} = 1$$

has a unique solution in the vicinity of the point $(x_0, 0)$ of the x axis? Is it unique there?

- (6) Find all solutions of the equation

$$y \frac{\partial u}{\partial x} - \sin x \frac{\partial u}{\partial y} = u^2$$

near the point $(0, 0)$?

- (7) Study the existence and uniqueness of the solution of the problem

$$yu_x = xu_y, \quad u|_{x=1} = \cos y$$

near the point $(1, y_0)$?

- (8) Does the Cauchy problem

$$\begin{cases} y \frac{\partial u}{\partial x} + \sin x \frac{\partial u}{\partial y} = y, \\ u|_{x=0} = y^4 \end{cases}$$

has a solution in the plane? Is it unique?

- (9) Does the Cauchy problem

$$u|_{y=x^2} = 1, \quad (\nabla u)^2 = 1$$

has a unique C^∞ solution in the domain $y \leq x^2$? In the domain $y \geq x^2$?

- (10) Solve the Dirichlet problem

$$\begin{aligned} \Delta u &= 0 && \text{for } x^2 + y^2 < 1; \\ u &= 1 && \text{for } x^2 + y^2 = 1, \quad y > 0; \\ u &= -1 && \text{for } x^2 + y^2 = 1, \quad y < 0. \end{aligned}$$

- (11) Show that

$$\mathbf{u}_y + \mathbf{A}\mathbf{u}_x = \mathbf{0},$$

where

$$\mathbf{A} = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix}$$

is a hyperbolic system and reduce it to the canonical form. Find its general solution and use this result to solve the IVP for $\mathbf{u}_y + \mathbf{A}\mathbf{u}_x = \mathbf{0}$ with the initial condition

$$\mathbf{u}(x, 0) = \begin{pmatrix} \sin x \\ 1 \\ e^x \end{pmatrix}.$$

- (12) Consider $u = u_x^2 + u_y^2$ with the initial condition $u(x, 0) = ax^2$. For what positive constants a is there a solution? Is it unique? Find all solutions?
- (13) The Hamilton-Jacobi equation $u_t + H(\nabla_x u) = 0$, where H depends only on

$$\nabla_x u = (u_{x_1}, \dots, u_{x_n}),$$

arises in geometric optics.

- (a) Verify that

$$u(x_1, \dots, x_n; \mathbf{a}, b) = a_1 x_1 + \dots + a_n x_n - tH(\mathbf{a}) + b$$

is a complete integral.

- (b) Use envelopes to generate a solution of
- $u_t + |\nabla u|^2 = 0$
- which is not linear in
- x
- and
- t
- .

- (14) Consider the initial value problem $u_{zz} = u^2 u_x + (u_{xy})^2$, $u(x, y, 0) = x - y$, $u_z(x, y, 0) = \sin x$. Find the values of u_{xz}, u_{yz}, u_{zz} when $z = 0$.
- (15) Is the heat equation $u_t = k u_{xx}$ in normal form for the Cauchy data on the x -axis? On the t -axis? What form would the Cauchy data take?
- (16) Find the solution of the initial value problem $u_{yy} = u_{xx} + u$, $u(x, 0) = e^x$, $u_y(x, 0) = 0$ in the form of power series expansion with respect to y (i.e., $\sum_0^\infty a_n(x)y^n$).

- (17) Consider the initial value problem $u_t = u_{xx}$, $u(x, 0) = g(x)$, where $g(x) = a_n x^n + \dots + a_0$ is a polynomial. Find a Taylor series solution about $(0, 0)$. Where does it converge?
- (18) Show that the Cauchy problem for the Laplace's equation

$$u_{xx} + u_{yy} = 0, \quad u(x, 0) = 0, \quad u_y(x, 0) = k^{-1} \sin kx,$$

where $k > 0$, is not well-posed in the sense of Hadamard.

- (19) Show that $K(x, y) = [2\pi i(x + iy)]^{-1}$ is a fundamental solution for the Cauchy-Riemann operator $L = \partial_x + i\partial_y$ on $\mathbb{R}^2$.
- (20) Let $Lu = u_{\xi\eta}$ with a jump discontinuity along $\xi = 0$.
- (a) Obtain the transmission condition for u to be a weak solution:

$$\int_{-\infty}^{\infty} [u^-(0, \eta) - u^+(0, \eta)] v_{\eta}(0, \eta) d\eta = 0.$$

(b) Perform integration by parts first with respect to $d\xi$ and then $d\eta$ to obtain the transmission condition.

(c) Why does this imply that the jump discontinuity $u^+ - u^-$ along γ must be constant?

- (21) Consider the first order equation $u_t + cu_x = 0$.
- (a) If $f \in C(\mathbb{R})$, show that $u(x, t) = f(x - ct)$ is a weak solution.
- (b) Can you find any discontinuous weak solutions?
- (c) Is there a transmission condition for a weak solution with jump discontinuity along the characteristic $x = ct$?
- (22) Using $\delta(\xi, \eta)$, show that each of the following functions is a fundamental solution of $L = \partial^2 / \partial \xi \partial \eta$:

$$F_1(\xi, \eta) = H(\xi)H(\eta), \quad F_2(\xi, \eta) = -H(\xi)H(-\eta),$$

$$F_3(\xi, \eta) = -H(-\xi)H(\eta), \quad F_4(\xi, \eta) = H(-\xi)H(-\eta).$$

(c) Use part (a) with the change of variables $\xi = x + ct$, $\eta = x - ct$ to obtain four distinct fundamental solutions for the one-dimensional wave operator, each having support in one of the wedges determined by the lines $x = \pm ct$.

- (23) For the operators of the form $\partial_t - L$, where L is a differential operator in $\mathbb{R}^n$ with constant coefficients, show that the fundamental solution of the initial value problem, $K(x, t)$ defined by

$$\begin{cases} K_t - LK = 0 & \text{as a distribution in } \mathbb{R}^n \text{ for each } t > 0 \\ K(x, 0) = \delta(x) & \text{as a distribution in } \mathbb{R}^n \end{cases}$$

coincides with the free-space fundamental solution, which satisfies

$$(\partial_t - L)K = \delta(x, t) = \delta(x)\delta(t),$$

provided we extend by zero to $t < 0$.

- (24) Find the solution of the initial value problem

$$\begin{cases} u_{tt} = u_{xx} + u_{yy} + u_{zz} \\ u(x, y, z, 0) = x^2 + y^2, \quad u_t(x, y, z, 0) = 0 \end{cases}$$

(a) by using Kirchhoff's formula, and (b) by using Hadamard's method.

- (25) Use Duhamel's principle to find the solution of the nonhomogeneous wave equation for three space dimensions $u_{tt} - c^2 \Delta u = f(x, t)$ with initial conditions $u(x, 0) = 0 = u_t(x, 0)$. What regularity in $f(x, t)$ is required for the solution to be C^2 ?